The Decomposition Method of Research into the Nonlinear Dynamical Systems’ Space of Parameters

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Abstract

The work suggests to use the approach oriented at precise analytical methods of research into nonlinear dynamical systems that have a complex structure of the state-space. This method allows to investigate the behavior of essential multivariable system through the dynamic behaviors of its basic components i.e. sub-systems (both linear and non-linear) which don’t compose any reserved system and have the more-lower order state-space. Such approach allows bringing out analytically strict proof of existence of the periodical movements both free and force. In the majority of cases the approach allows to fulfill “partition” of parametric-space into the areas of its qualitatively different dynamic behavior.

Keywords: multivariable dynamical system; system’s state-space; liner variable-space transform; decomposition of the system’s parametric-space; bilateral phase plane
**Introduction**

Rapid development of computer engineering technique ingeniously brought up the emergence of such computer engineering programs as Matlab, Matcad, Wonfram mathematica etc. widely used in scientific and engineering supplements. Nevertheless the results obtained by these methods can’t be reliable enough.

The main virtue of faithfully analytical methods remains the presentation of problem solution as mathematical expression. This helps to move to a deeper level of comprehending of phenomenon and get new info about it [12, 14, 15].

The suggested in this work method allows to a certain extent unite the virtues of both approaches giving the research the preliminary info obtained on the basis of exact analytical method about dynamical behavior of the system with the further solution of the problem applying other methods (for example, computer modeling).

**Method of investigation**

Basically the method under consideration reveals the decomposition of parametric-space of non-linear dynamical system. It all allows to investigate the behavior of essential multivariable system which is to be judged by the behavior of subsystems both linear and nonlinear. Subsystems have more-lower order of the state space, when subsystems don’t create close dynamical systems.

The suggested method allows to consider non-linear dynamical systems combined in the matrix format

\[
\begin{align*}
x(t) &= A \cdot x(t) + B \cdot \left[ N[y(t)] + \psi(t) \right], \\
y(t) &= C \cdot x(t) + D \cdot \left[ N[y(t)] + \psi(t) \right]; \\
x(0) &= x_0, \quad y(0) = y_0, \quad \psi(0) = \psi_0, \quad N[y(0)] = N_-(y_0), \\
y_0 &= C \cdot x_0 + D \cdot \left[ N_-(y_0) + \psi_0 \right].
\end{align*}
\]

Thus in (1) linear part there are: \(x(t)\) is the \(n\)-dimensional vector; \(y(t)\) is the \(m\)-dimensional vector; \(\psi(t)\) is the \(m\)-dimensional vector of external harmonic actions \(\psi_1(t) = \psi_{m_1} \cdot \text{Sin}(\omega_1 \cdot t + \varphi_1), \ldots, \psi_m(t) = \psi_{m_m} \cdot \text{Sin}(\omega_m \cdot t + \varphi_{m_m})\); \(A\) is the \(n \times n\) real-valued matrix; \(B\) is the \(m \times n\) real-valued matrix; \(C\) is the \(m \times n\) real-valued matrix; \(D\) is the \(m \times m\) real-valued diagonal matrix (when \(n \geq m\)); \(\cdot = d/dt\) - symbol of differentiation \((t\text{ - time})\). In non-linear part there are: \(N[y(t)]\) is the \(m\)-dimensional vector of non-linear functions \(n_1[y_1(t)], \ldots, n_m[y_m(t)]\); \(N_-\) - \(m\)-dimensional vector of past-history of the non-linear functions. In the first time ambiguous non-linearities are under consideration. They are widely used in scientific and engineering supplements. This is called by...
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such physical phenomena as “gap”, “reley”, “hysteresis”, “dry friction” etc.[4-8,12,16,17]. System (1) when \( \psi_0 = 0 \) is considered as autonomous. Rather often in systems matrix \( D = 0 \).

As a rule, the elements of matrix \( A \) usually are considered (can’t be liable to changes), and the elements of matrices \( B, C, D \) are parameters adjustment.

Taking liner variable-space transforms \[ 2 \] \[ g(t) = \Lambda \cdot g(t) + B_m \cdot [N[y(t)] + \psi(t)]; \]

\[ y(t) = C_m \cdot g(t) + D \cdot [N[y(t)] + \psi(t)]; \]

\[ g(0) = g_0, y(0) = y_0, \psi(0) = \psi_0, N[y(0)] = N_-(y_0), \]

\[ y_0 = C \cdot g_0 + D \cdot [N_-(y_0) + \psi_0] \]

where

\[ \Lambda = M^{-1} \cdot A \cdot M, B_m = M^{-1} \cdot B, C_m = C \cdot M. \]

After having used decomposition of the elements of matrices \( A, B_m, C_m, D \), system (3) may be considered as the linear and non-linear lower order subsystems. The subsystems must’n’t compose the close-loop systems. The decomposition has it disjunction of transformed system (3) into subsystems. The decomposition should be performed so that each subsystem could be totally-analytically investigated. Dynamic of initial system (1) thus is analyzed not for the whole parametric-space but only for parameters which belong to certain areas of their meanings – decompositions. The quantity of decompositions as well as their lay-out in the parametric-space is defined by unchangeable part of parameter system.

**Example**

Let us consider the special case of (1) when \( n = 3, m = 2, D = 0 \).

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
2 & -2 & 3 \\
1 & 1 & 1 \\
1 & 3 & -1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
+ 
\begin{bmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22} \\
 b_{31} & b_{32} \\
\end{bmatrix}
\begin{bmatrix}
 n_1(y_1) + \psi_1(t) \\
 n_2(y_2) + \psi_2(t) \\
\end{bmatrix};
\]

\[
y = 
\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}.
\]

(5)
In case when for (5) eigenvalues of matrix $A$ are negative the number of possible decompositions is 36. When among the negative eigenvalues one is positive (or zero), the number of decompositions is 24. When among the negative eigenvalues two are positive (or zero) the number of decompositions is 12. When all eigenvalues are positive (and, or zero) decompositions are not possible.

By the following transformation

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= \begin{bmatrix}
  -1 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & -14 & 1
\end{bmatrix}
\begin{bmatrix}
  g_1 \\
  g_2 \\
  g_3
\end{bmatrix},

\begin{bmatrix}
  g_1 \\
  g_2 \\
  g_3
\end{bmatrix}
= \begin{bmatrix}
  -1 & 5 & -1 \\
  2 & 6 & 1 \\
  6 & 15 & -15
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix},
$$

(6)

in accordance with (2) system (5) is driven to the form (Fig.1)

Fig. 1. Analytic structure of system (7)

$$
\begin{bmatrix}
  \dot{g}_1 \\
  \dot{g}_2 \\
  \dot{g}_3
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 \\
  0 & -2 & 0 \\
  0 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
  g_1 \\
  g_2 \\
  g_3
\end{bmatrix}
+ \begin{bmatrix}
  b_{m_1} & b_{m_2} \\
  b_{m_1} & b_{m_2} \\
  b_{m_1} & b_{m_2}
\end{bmatrix}
\begin{bmatrix}
  n_1(y_1) + \psi_1(t) \\
  n_2(y_2) + \psi_2(t)
\end{bmatrix},

(7)

\begin{bmatrix}
  y
\end{bmatrix}
= \begin{bmatrix}
  c_{m_1} & c_{m_2} & c_{m_3} \\
  c_{m_1} & c_{m_2} & c_{m_3}
\end{bmatrix}
\begin{bmatrix}
  g_1 \\
  g_2 \\
  g_3
\end{bmatrix}.
where in accordance with (4) the elements of matrices

\[
B_m = \frac{1}{30} \begin{bmatrix}
-5 \cdot (3 \cdot b_{11} - 5 \cdot b_{21} + 2 \cdot b_{31}) & -5 \cdot (3 \cdot b_{12} - 5 \cdot b_{22} + 2 \cdot b_{32}) \\
2 \cdot (b_{21} - b_{31}) & 2 \cdot (b_{22} - b_{32}) \\
3 \cdot (5 \cdot b_{11} + b_{21} + 4 \cdot b_{31}) & 3 \cdot (5 \cdot b_{12} + b_{22} + 4 \cdot b_{32})
\end{bmatrix},
\]

\[
C_m = \begin{bmatrix}
-c_{11} + c_{12} + c_{13} & 11 \cdot c_{11} + c_{12} - 14 \cdot c_{13} & c_{11} + c_{12} + c_{13} \\
-c_{21} + c_{22} + c_{23} & 11 \cdot c_{21} + c_{22} - 14 \cdot c_{23} & c_{21} + c_{22} + c_{23}
\end{bmatrix},
\]

For the system (7) maximum possible decompositions is 12 in the conditions of which the dynamical behavior of system can be judged by the three subsystems in which there are two – non-linear subsystems, other is – autonomic linear.

For the example under consideration within the 12 decompositions one can obtain two decompositions in which all three subsystems are independent from each other.

Thus, as a result of application of the suggested method, the consideration of 12 dimensional parametric-space is substituted by the consideration of 8 dimensional parametric-space. All periodical movements then are placed on phase plane into 3-dimensional state space.

Each of nonlinear subsystems can be totally-analytically investigated with the aim of obtaining this or that periodical motions occurred by external harmonic action. Linear subsystem is always stable ($\lambda_2 = -2$).

Choosing the meanings of the elements of matrices B and C satisfying the following conditions

\[
b_{11} = b_{21} = b_{31}, b_{22} = b_{32} = -b_{12}, 4c_{11} = 20c_{12} = 5c_{13}, 2c_{21} = 2c_{22} = -3c_{23},
\]

(8) for example

\[
B = \begin{bmatrix} 5 & -3 \\ 5 & 3 \\ 5 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{4} & -\frac{5}{12} & \frac{1}{6} \end{bmatrix},
\]

we’ll obtain one of two possible decompositions of the transformed system. (Fig. 2)

\[
B_m = \begin{bmatrix} 0 & b_{m_{12}} \\ 0 & 0 \\ b_{m_{31}} & 0 \end{bmatrix}, \quad C_m = \begin{bmatrix} 0 & 0 & c_{m_{13}} \\ c_{m_{21}} & 0 & 0 \end{bmatrix}.
\]
\[
\dot{y}_1 = \lambda_1 g_1 + b_{m_1} n_1(y_1) + \psi_1(t);
\]
\[
y_1 = c_{m_1} g_1; \\
\psi_1(t) = \psi_{m_1} \sin(\omega t + \varphi);
\]
\[
g_1(0) = g_{1a}, y_{1a} = c_{m_1} g_{30}, \\
n_1(y_1(0)) = n_1(y_{1a});
\]
\[
\dot{y}_2 = \lambda_2 g_2, g_2(0) = g_{2a}.
\]

**Fig. 2.** System (7) in conditions of decomposition: (8).

a) scalar form; b) structure form

For example \(n_1(y_1(t))\) models 3-th position relay with hysteresis (Fig. 3), \(n_2(y_2(t))\) - non-Coulomb dry friction (Fig. 4).

\[
\begin{align*}
\text{a) Geometrical interpretation of } n_1(y_1(t)) : & \quad y_1 \leq a_1, \text{ or } y_1 \geq b_1, \text{ and } n_1(y_1) = 0, \text{ then } n_1(y_1) = 0; \\
& \text{if } y_1 \leq -b_1, \text{ or } -b_1 \leq y_1 \leq a_1, \text{ and } n_1(y_1) = c, \text{ then } n_1(y_1) = c; \\
& \text{if } y_1 \leq -b_1, \text{ or } -b_1 \leq y_1 \leq -a_1, \text{ and } n_1(y_1) = -c, \text{ then } n_1(y_1) = -c,
\end{align*}
\]

where \(c, b_1, a_1, (b_1 \geq a_1)\) are parameters of non-linearity; \(n_2\) is past history of non-linearity.

**Fig. 3.** Non-linear function \(n_1[y_1(t)]\) - 3-th position relay with hysteresis

a) Geometrical interpretation of \(n_1[y_1(t)]\); b) Analytical description of \(n_1[y_1(t)]\)
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a) If $|n_2 - y_2| < a_2$, then $n_2 = n_{2-}$;
if $a \leq |n_2 - y_2| < b_2$ and $n_2 = n_{2-}$,
then $n_2 = n_{2-}$;
if $|n_2 - y_2| \geq a_2$ and $n_2 \neq n_{2-}$,
or $|n_2 - y_2| \geq b_2$,
then $n_2 = y_2 + a_2 \cdot \text{Sign}(n_{2-} - y_2)$.
where $b_2, a_2$ ($b_2 \geq a_2$) are parameters
of non-linearity; $n_{2-}$ is past history of non-linearity.

Fig. 4. Non-linear function $n_2[y_2(t)]$ considering non-Coulomb dry friction
a) Geometrical interpretation of $n_2[y_2(t)]$; b) Analytical description of

Thus, as a result of use the suggested method, the consideration of 12-
dimensional space of parameters $b_{11}, b_{12}, b_{21}, b_{22}, b_{31}, b_{32}, c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}$
is replaced with the consideration of 4-dimensional space of parameters
$b_{m12}, b_{m31}, c_{m13}, c_{m21}$. And all periodical motions are placed on 2-dimensional
plane into 3-dimensional space of states.

State space of subsystem with $n_1[y_1(t)]$ is 3-leaved bilateral phase plane, state
space of subsystem with $n_2[y_2(t)]$ is $\infty$-leaved bilateral phase plane. These sub-
systems have been investigated in the works [1, 4, 8].

Examples of disturbed periodical motions in state-space of subsystems are
shown on Fig. 5.

Example of disturbed periodical motion in state-space of initial system (5) in
the conditions of decomposition (8) is shown on fig. 6.

Fig. 5. Trajectories of disturbed periodical motions in state-space of subsystems:
a) in subsystem with $n_1[y_1(t)]$; b) in subsystem with $n_2[y_2(t)]$
Conclusion

The obtained results give a preliminary info about the structure of parametric space of the system (7) and existence of state space of these or other periodical solutions. When the number of decomposition is enough the method allows to qualitatively “see” the structure of parametric space of the external harmonic disturbance system.

Fig.6. Trajectory of disturbed periodical motions in state-space of system (5) in the conditions of decomposition (8)

By the present time the method was applied in respect of (1) split by decomposition to nonlinear first-order subsystems [5]. Nonetheless the authors don’t find any obstacles for spreading the method on non-linear two-order subsystems, also when matrix $A$ contains equal eigenvalues (zero one-too) and (or) complex conjugate eigenvalues (matrix $\Lambda = M^{-1} \cdot A \cdot M$ contains Jordan-boxes) [3, 9-11, 13].

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