Some Common Fixed Point Results of Mappings in 0-σ-Complete Metric-like Spaces via New Function

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Abstract

In this paper, using the context of 0-σ-complete metric-like spaces, some common fixed point results of maps that satisfy the generalized so-called \((F,\psi,\varphi)\)-weak contractive conditions are obtained. Our results generalize, extend, unify and complement many existing results in the literature. Examples are given to show the validity of our results.

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1 Introduction and preliminaries

Matthews [26], [27] generalized the concept of a metric space introducing partial metric spaces. Based on the notion of partial metric spaces, S. G. Matthews [26], [27], S. Oltra and O. Valero [32], D. Ilić et al. [21], [22], Z. Kadelburg et al. [23], Cristina Di Bari et al. [12], Hemant Kumar Nashine et al. [29], Radenović [35], V.Ć. Rajić et al. [36] obtained some very interesting fixed point theorems for mappings satisfying different contractive conditions. Recently, T. Abdeljawad et al. [8], proved one fixed point result for generalized contraction principle with control functions on partial metric spaces. For various new results on partial metric spaces see [3]-[8], [10]-[14], [15], [22]-[24], [26]-[33], [39]-[42].

The aim of this paper is to continue the study of common fixed points of mappings in $\sigma$-complete metric-like spaces via new functions. Consistent with Matthews [26], [27] and O’Neill [30], [31] the following definitions and results are well known:

**Definition 1.1.** A partial metric on a nonempty set $X$ is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

- (p1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
- (p2) $p(x, x) \leq p(x, y)$,
- (p3) $p(x, y) = p(y, x)$,
- (p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

For a partial metric $p$ on $X$, the function $p^* : X \times X \to R^+$ given by,

$$ p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1} $$

is a (usual) metric on $X$. Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base of the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

**Definition 1.2.** [27], [28]) Let $(X, p)$ be a partial metric space. Then:

(i) a sequence $\{x_n\}$ in a partial metric space $(X, p)$ converges to $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x)$;

(ii) a sequence $\{x_n\}$ in a partial metric space $(X, p)$ is called Cauchy if and only if $\lim_{n, m \to \infty} p(x_n, x_m)$ exists (and finite); a sequence $\{x_n\}$ in a partial metric space $(X, p)$ is called 0-Cauchy if $\lim_{n, m \to \infty} p(x_n, x_m) = 0$;

(iii) a partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)$.

(iv) a partial metric space $(X, p)$ is said to be 0-complete if every 0-Cauchy sequence in $X$ converges (with respect to $\tau_p$) to point $x \in X$ such that $p(x, x) = 0$;
(v) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f (B_{\delta} (x_0)) \subset B_{\varepsilon} (f x_0)$. 

Lemma 1.3. \([27], [28]\) Let $(X, p)$ be a partial metric space. Then:

(1) The sequence $\{x_n\}$ is a Cauchy in a partial metric space $(X, p)$ if and only if $\{x_n\}$ is a Cauchy in a metric space $(X, p^*)$;

(2) A partial metric space $(X, p)$ is complete if and only if a metric space $(X, p^*)$ is complete; Moreover, $\lim_{n \to \infty} p^* (x_n, x) = 0$ if and only if

$$p (x, x) = \lim_{n \to \infty} p (x_n, x) = \lim_{n,m \to \infty} p (x_n, x_m) \quad (1.2)$$

Remark 1.4. (1) \([28]\) Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p (x_n, y_n) \to p (x, y)$. For example, if $X = [0, +\infty)$ and $p (x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}, p (x_n, x) = x = p (x, x)$ for each $x \geq 1$ and so, e.g., $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

(2) \([8]\) However, if $p (x_n, x) \to p (x, x) = 0$ then $p (x_n, y) \to p (x, y)$ for all $y \in X$.

Consistent with P. Hitzler \([19], [20]\) (see also and \([1], [17]\) and \([37]\)) the following definitions and results will be needed in the sequel:

Definition 1.5. A mapping $\sigma : X \times X \to [0, +\infty)$, where $X$ is a nonempty set, is said to be metric-like on $X$ if for any $x, y, z \in X$, the following three conditions hold true:

$(\sigma 1)$ $\sigma (x, y) = 0 \Rightarrow x = y$;

$(\sigma 2)$ $\sigma (x, y) = \sigma (y, x)$;

$(\sigma 3)$ $\sigma (x, z) \leq \sigma (x, y) + \sigma (y, z)$.

A metric-like space is a pair $(X, \sigma)$ such that $X$ is a nonempty set and $\sigma$ is a metric-like function on $X$. Note that a metric-like function satisfies all the conditions of metric except that $\sigma (x, x)$ may be positive for $x \in X$. Each metric-like function $\sigma$ on $X$ generates a topology $\tau_\sigma$ on $X$ whose base is the family of open $\sigma$-balls

$$B_\sigma (x, \varepsilon) = \{y \in X : |\sigma (x, y) - \sigma (x, x)| < \varepsilon\} \quad \text{for all } x \in X \text{ and } \varepsilon > 0.$$ 

A sequence $\{x_n\}$ in $X$ converges to a point $x \in X$ if and only if $\lim_{n \to \infty} \sigma (x_n, x) = \sigma (x, x)$. A sequence $\{x_n\}$ is said to be $\sigma$-Cauchy if $\lim_{n,m \to \infty} \sigma (x_n, x_m)$ exists and is finite. A metric-like space $(X, \sigma)$ is called complete if for each $\sigma$-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n \to \infty} \sigma (x_n, x) = \sigma (x, x) = \lim_{n,m \to \infty} \sigma (x_n, x_m).$$

Every partial metric space is a metric-like space but the converse may not be true.
Example 1.6. [37] Let $X = \{0, 1\}$ and $\sigma : X \times X \to [0, +\infty)$ be defined by

$$\sigma(x, y) = \begin{cases} 
2 & \text{if } x = y = 0, \\
1 & \text{otherwise.}
\end{cases}$$

Then $(X, \sigma)$ is a metric-like space, but it is not a partial metric space, as $\sigma(0, 0) \not\leq \sigma(0, 1)$.

Example 1.7. [37] Let $X = \mathbb{R}$, $k \geq 0$ and $\sigma : X \times X \to [0, +\infty)$ be defined by

$$\sigma(x, y) = \begin{cases} 
2k & \text{if } x = y = 0, \\
k & \text{otherwise.}
\end{cases}$$

Then $(X, \sigma)$ is a metric-like space, but for $k > 0$, it is not a partial metric space, as $\sigma(0, 0) \not\leq \sigma(0, 1)$.

Definition 1.8. [37] Let $(X, \sigma)$ be a metric-like space. A sequence $\{x_n\}$ in $X$ is called a $0 -$ $\sigma -$ Cauchy sequence if $\lim_{n,m \to \infty} \sigma(x_n, x_m) = 0$. The space $(X, \sigma)$ is said to be $0 -$ $\sigma -$ complete if every $0 -$ $\sigma -$ Cauchy sequence in $X$ converges with respect to $\tau_\sigma$ to point $x \in X$ such that $\sigma(x, x) = 0$.

It is obvious that every $0 -$ $\sigma -$ Cauchy sequence is a $\sigma -$ Cauchy sequence in $(X, \sigma)$ and every $\sigma -$ complete metric-like space is $0 -$ $\sigma -$ complete. Also, every $0 -$ complete partial metric space is a $0 -$ $\sigma -$ complete metric-like space. The following example shows that the converse assertions of these facts do not hold.

Example 1.9. [37] Let $X = [0, +\infty) \cap \mathbb{Q}$ and $\sigma : X \times X \to [0, +\infty)$ be defined by

$$\sigma(x, y) = \begin{cases} 
2x & \text{if } x = y, \\
\max\{x, y\} & \text{otherwise}
\end{cases}$$

for all $x, y \in X$. Then $(X, \sigma)$ is a metric-like space. Note that $(X, \sigma)$ is not a partial metric space, as $\sigma(1, 1) = 2 \not\leq \sigma(1, 0) = 1$. Now, it is easy to see that $(X, \sigma)$ is a $0 -$ $\sigma -$ complete metric-like space, while it is not a $\sigma -$ complete metric-like space.

Remark 1.10.

(a) [37] It is not hard to see that, if $\sigma(x_n, x) \to \sigma(x, x) = 0$, then $\sigma(x_n, y) \to \sigma(x, y)$ for all $y \in X$;

(b) [24] if $\sigma(x, y) = 0$, then $\sigma(x, y) = \sigma(y, y) = 0$;

(c) if $\{x_n\}$ is a sequence such that $\lim_{n \to \infty} \sigma(x_n, x_{n+1}) = 0$, then we have $\lim_{n \to \infty} \sigma(x_n, x_n) = \lim_{n \to \infty} \sigma(x_{n+1}, x_{n+1}) = 0$;

(d) if $x \neq y$, then $\sigma(x, y) > 0$;

(e) $\sigma(x, x) \leq \frac{2}{n} \sum_{i=1}^{n} \sigma(x, x_i)$ holds for all $x_i, x \in X$, where $1 \leq i \leq n$.

Assertions similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers [28], [34].
Lemma 1.11. Let \((X,d)\) be a metric space and let \(\{x_n\}\) be a sequence in \(X\) such that
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]
If \(\{x_n\}\) is not a Cauchy sequence, then there exist \(\varepsilon > 0\) and two sequences \(\{m_k\}\) and \(\{n_k\}\) of positive integers such that \(m_k > n_k > k\) and the following four sequences tend to \(\varepsilon^+\) when \(k \to \infty\):
\[
d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k+1}), d(x_{m_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k+1}).
\]
In the frame of metric-like spaces we also have the similar:

Lemma 1.12. [37] Let \((X,\sigma)\) be a metric-like space and let \(\{x_n\}\) be a sequence in \(X\) such that
\[
\lim_{n \to \infty} \sigma(x_n, x_{n+1}) = 0.
\]
If \(\{x_n\}\) is not a \(0-\sigma\)-Cauchy sequence in \((X,\sigma)\), then there exist \(\varepsilon > 0\) and two sequences \(\{m_k\}\) and \(\{n_k\}\) of positive integers such that \(m_k > n_k > k\) and the following four sequences tend to \(\varepsilon^+\) when \(k \to \infty\):
\[
\sigma(x_{m_k}, x_{n_k}), \sigma(x_{m_k}, x_{n_k+1}), \sigma(x_{m_k-1}, x_{n_k}), \sigma(x_{m_k-1}, x_{n_k+1}).
\]
For the following definition and proposition we refer to [2]

Definition 1.13. Let \(f\) and \(g\) be self maps of a set \(X\). If \(w = fx = gx\) for some \(x \in X\), then \(x\) is called a coincidence point of \(f\) and \(g\), and \(w\) is called a point of coincidence of \(f\) and \(g\). The pair \(f, g\) of self maps is weakly compatible if they commute at their coincidence points.

Proposition 1.14. Let \(f\) and \(g\) be weakly compatible self maps of a set \(X\). If \(f\) and \(g\) have a unique point of coincidence \(w = fx = gx\), then \(w\) is the unique common fixed point of \(f\) and \(g\).

Definition 1.15. [3], [9], [16], [25], [34], [36] The two classes of following mappings are defined as
- \(\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)\} \) is continuous, non-decreasing and \(\psi^{-1}([0]) = \{0\}\) and
- \(\Phi = \{\varphi \mid \varphi : [0, \infty) \to [0, \infty)\} \) is lower semi-continuous and \(\varphi^{-1}([0]) = \{0\}\).
- \(\Phi_1 = \{\varphi \mid \varphi : [0, \infty) \to [0, \infty)\} \) is lower semi-continuous and \(\varphi(0) \geq 0, \varphi(t) > 0, t \neq 0\).
- A function \(\varphi : [0, \infty) \to [0, \infty)\) is called an ultra-altering distance function if \(\varphi\) is continuous, and \(\varphi(0) \geq 0, \varphi(t) > 0, t \neq 0\).
- We say that \(F : [0, \infty)^2 \to \mathbb{R}\) is called \(C\)-class function if it is continuous and satisfies following conditions:
  - (a) \(F(s, t) \leq s\);
  - (b) \(F(s, t) = s\) implies that either \(s = 0\) or \(t = 0\) for all \(s, t \in [0, \infty)\).
Note that \(F(0, 0) = 0\).
We denote $C$-class functions as $C$.

The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $C$.

Example 1.16.

(1) $F(s, t) = s - t, F(s, t) = s \Rightarrow t = 0$;

(2) $F(s, t) = ks, 0 < k < 1, F(s, t) = s \Rightarrow s = 0$;

(3) $F(s, t) = \frac{s}{(1 + t)^r}; r \in (0, \infty), F(s, t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $F(s, t) = \log(t + a^s)/(1 + t), a > 1, F(s, t) = s \Rightarrow s = 0$ or $t = 0$;

(5) $F(s, 1) = \ln(1 + a^s)/2, a > e, F(s, 1) = s \Rightarrow s = 0$;

(6) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l, l > 1, r \in (0, \infty), F(s, t) = s \Rightarrow t = 0$;

(7) $F(s, t) = s \log_{t+a} a, a > 1, F(s, t) = s \Rightarrow s = 0$ or $t = 0$.

(8) $F(s, t) = s - \frac{l}{t+1}, F(s, t) = s \Rightarrow t = 0$.

(9) $F(s, t) = s - \varphi(s), F(s, t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$.

(10) $F(s, t) = sh(s, t), F(s, t) = s \Rightarrow s = 0$, here $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$.

(11) $F(s, t) = s - \frac{(2+t)t}{t+1}, F(s, t) = s \Rightarrow t = 0$.

2 Main results

In this section, we prove some new common fixed point results defined on $0-\sigma$-complete metric-like spaces. Obtained results generalize and complement many existing results in the literature.

Theorem 2.1. Let $(X, \sigma)$ be a $0-\sigma$-complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(\sigma(fx, fy)) \leq F(\psi(\sigma(gx, gy)), \varphi(\sigma(gx, gy))) \quad (2.1)$$

for all $x, y \in X$ where $\psi \in \Psi, \varphi \in \Phi \ (\varphi \in \Phi_1)$ and $F \in C$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point (say $z$) and $\sigma(z, z) = 0 = \sigma(fz, fz) = \sigma(gz, gz)$.

Proof. Let us prove first that the point of coincidence of $f$ and $g$ is unique (if it exists). Suppose that $u_1$ and $u_2$ are two distinct point of coincidence of $f$ and $g$. From this follows that there exist two point $u_1$ and $u_2$ such that $fu_1 = gu_1 = w_1 \neq w_2 = fu_2 = gu_2$. If $\sigma(w_1, w_1) > 0$, then (2.1) implies that

$$\psi(\sigma(w_1, w_1)) = \psi(\sigma(fu_1, fu_1)) \leq F(\psi(\sigma(gu_1, gu_1)), \varphi(\sigma(gu_1, gu_1))) = F(\psi(\sigma(w_1, w_1)), \varphi(\sigma(w_1, w_1))) < \psi(\sigma(w_1, w_1)),$$
a contradiction. Therefore $\sigma(w_1, w_1) = 0$. Similarly, $\sigma(w_2, w_2) = 0$. Further, if $\sigma(w_1, w_2) > 0$, we have

$$\psi(\sigma(w_1, w_2)) = \psi(\sigma(fu_1, fu_2)) \leq F(\psi(\sigma(gu_1, gu_2)), \varphi(\sigma(gu_1, gu_2)))$$

$$= F(\psi(\sigma(w_1, w_2)), \varphi(\sigma(w_1, w_2))) < \psi(\sigma(w_1, w_2)),$$

which is a contradiction. Hence, $\sigma(w_1, w_2) = 0$, that is, $w_1 = w_2$. Thus, the point of coincidence of $f$ and $g$ is unique (if it exists).

Now, let $x_0$ be an arbitrary point in $X$. Let a choose a point $x_1 \in X$ such that $fx_0 = gx_1$. This can be done, since the range of $g$ contains the range of $f$. Continuing this process, having chosen $x_n$ in $X$, we obtain $x_{n+1}$ in $X$ such that $fx_n = gx_{n+1}$. Consider the two possible cases.

Suppose that $gx_n = gx_{n+1}$ for some $n \in \mathbb{N}$. Hence, $gx_n = fx_n$ is a point of coincidence and then the proof is finished. Thus, suppose that $gx_n \neq gx_{n+1}$ for any $n \geq 0$. In this case, we have

$$\psi(\sigma(gx_{n+1}, gx_n)) = \psi(\sigma(fx_n, fx_{n-1}))$$

$$\leq F(\psi(\sigma(gx_n, gx_{n-1})), \varphi(\sigma(gx_n, gx_{n-1})))$$

$$< \psi(\sigma(gx_n, gx_{n-1})).$$

(2.2)

Now, according to the properties of function $\psi$ it follows that the sequence $\sigma(gx_{n+1}, gx_n)$ is non-increasing. Therefore, $\sigma(gx_{n+1}, gx_n) \to \sigma^* \geq 0$ when $n \to \infty$.

We prove now that $\sigma^* = 0$. Indeed, if $\sigma^* > 0$, then passing to the limit in (2.2) when $n \to \infty$, we obtain that $\psi(\sigma^*) \leq F(\psi(\sigma^*), \varphi(\sigma^*)) \leq \psi(\sigma^*)$. In both cases $F(\psi(\sigma^*), \varphi(\sigma^*)) = \psi(\sigma^*)$ or $F(\psi(\sigma^*), \varphi(\sigma^*)) < \psi(\sigma^*)$ we obtain a contradiction. Hence, $\lim_{n \to \infty} \sigma(gx_{n+1}, gx_n) = 0$.

We next prove that $\{y_n\} = \{fx_n\} = \{gx_{n+1}\}$ is a $0-\sigma$-Cauchy sequence in the $0-\sigma$-complete metric-like space $(X, \sigma)$. Suppose that is not the case. Then using Lemma 1.12, we get that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers and sequences

$$\sigma(y_{m_k}, y_{n_k}), \sigma(y_{m_k}, y_{n_k+1}), \sigma(y_{m_k-1}, y_{n_k}), \sigma(y_{m_k-1}, y_{n_k+1}),$$

all tend to $\varepsilon^+$ when $k \to \infty$. Applying condition (2.1) to elements $x = x_{m_k}$ and $y = x_{n_k+1}$ and since $y_n = fx_n = gx_{n+1}$ for each $n \geq 0$, we get that

$$\psi(\sigma(y_{m_k}, y_{n_k+1})) \leq F(\psi(\sigma(y_{m_k-1}, y_{n_k})), \varphi(\sigma(y_{m_k-1}, y_{n_k}))).$$

(2.3)

Letting $k \to \infty$ in (2.3), we obtain

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)),$$
which is a contradiction if $\varepsilon > 0$.

This shows that $\{g x_n\}$ is a $0 - \sigma -$Cauchy sequence in $0 - \sigma -$complete metric-like space $(X, \sigma)$.

Since $g (X)$ is closed in $0 - \sigma -$complete metric-like space $(X, \sigma)$, then it is $0 - \sigma -$complete metric-like space. Therefore, there exists $u, v \in X$ such that $v = gu$ and
\[
\lim_{n \to \infty} \sigma (y_n, v) = \lim_{n, n \to \infty} \sigma (y_n, y_m) = \sigma (v, v) = 0.
\] (2.4)

Now, putting $x = x_n, y = u, gu = v$ and $y_n = f x_n = g x_{n+1}$ in (2.1) we obtain
\[
\psi (\sigma (y_n, fu)) \leq F (\psi (\sigma (y_{n-1}, v)), \varphi (\sigma (y_{n-1}, v))).
\] (2.5)

Letting $n \to \infty$ in (2.5) and applying Remark 1.10. (a), we get
\[
\psi (\sigma (v, fu)) \leq F (\psi (\sigma (v, v)), \varphi (\sigma (v, v))) = F (0, 0) = 0.
\]

Hence, we obtain that $\sigma (v, fu) = 0$, that is $v = fu = gu$. Thus $f$ and $g$ have a unique point of coincidence.

By the Proposition 1.14. $f$ and $g$ have the unique common fixed point.

In the case when $f (X)$ is closed subset in $(X, \sigma)$ the proof is similar. □

In the following result we deduce another variation of Theorem 5, from [8], that is of Theorem 2.2, from [36] on weak contraction for a family of self-mappings. The difference is that we do not use the maximum of some set $\{\sigma (x, y), \sigma (x, f x), \sigma (y, g y), \frac{1}{4} (\sigma (x, g y) + \sigma (y, f x))\}$, but its arbitrary element.

**Theorem 2.2.** Let $(X, \sigma)$ be a complete partial metric space and $f, g : X \to X$ be two mappings such that for some $\psi \in \Psi, \varphi \in \Phi (\varphi \in \Phi_1)$ and $F \in \mathcal{C}$ and for all $x, y \in X$ there exists
\[
u (x, y) \in \left\{\sigma (x, y), \sigma (x, f x), \sigma (y, g y), \frac{1}{4} (\sigma (x, g y) + \sigma (y, f x))\right\}
\]

such that
\[
\psi (\sigma (f x, g y)) \leq F (\psi (\sigma (u (x, y))), \varphi (\sigma (u (x, y)))).
\] (2.6)

Then $f$ and $g$ have a unique common fixed point.

**Proof.** Firstly, let $a$ be a common fixed point of $f$ and $g$. Then $\sigma (a, a) = 0$. Indeed, (2.6) for $x = y = a$ implies that
\[
\psi (\sigma (a, a)) = \psi (\sigma (f a, g a)) \leq F (\psi (u (a, a)), \varphi (u (a, a))),
\]

where
\[
u (a, a) \in \left\{\sigma (a, a), \sigma (a, f a), \sigma (a, g a), \sigma (a, f a)\right\}
\]

\[
= \left\{\sigma (a, a), \frac{1}{2} \sigma (a, a)\right\}.
\]
Hence, from both cases:

\[ \psi(\sigma(a,a)) \leq F(\psi(\sigma(a,a)), \varphi(\sigma(a,a))) \quad \text{and} \quad \psi(\sigma(a,a)) \leq F\left(\psi\left(\frac{1}{2}\sigma(a,a)\right), \frac{1}{2}\varphi(\sigma(a,a))\right), \]

follows that \( \sigma(a,a) = 0 \). Secondly, let us prove that the common fixed point of \( f \) and \( g \) is unique (if it exists). If \( a \) and \( b \) are two distinct common fixed points of \( f \) and \( g \), (2.6) for \( x = a, y = b \) implies

\[ \psi(\sigma(a,b)) = \psi(\sigma(fa,gb)) \leq F(\psi(u(a,b)), \varphi(u(a,b))), \]

where

\[ u(a,b) = \left\{ \sigma(a,b), 0, 0, \frac{1}{4}(\sigma(a,b) + \sigma(a,b)) \right\} \]

In all three possible cases we get that \( \sigma(a,b) = 0 \), i.e., \( a = b \).

Now, in order to prove the existence of a common fixed point, let \( x_0 \) be an arbitrary point in \( X \). Further, let \( x_{2n+1} = fx_{2n} \) and \( x_{2n+2} = gx_{n+1} \) for all \( n \geq 0 \) be a Jungck sequence. Consider the two possible cases.

Suppose that \( \sigma(x_n, x_{n+1}) = 0 \) for some \( n \in \mathbb{N} \). Then \( x_{n+1} = x_{n+2} \) and the sequence is eventually constant, and so convergent. Indeed, let, e.g., \( n = 2k \) (in the case \( n = 2k+1 \) the proof is similar). Then, putting \( x = x_{2k}, y = x_{2k+1} \) in (2.6), we get that there exists

\[ u(x_{2k}, x_{2k+1}) = \left\{ \sigma(x_{2k}, x_{2k+1}), \sigma(x_{2k}, x_{2k+1}), \sigma(x_{2k+1}, x_{2k+2}), \right. \]

\[ \frac{1}{2}(\sigma(x_{2k}, x_{2k+2}) + \sigma(x_{2k+1}, x_{2k+1})) \]

such that

\[ \psi(\sigma(x_{2k+1}, x_{2k+2})) \leq F(\psi(u(x_{2k}, x_{2k+1})), \varphi(u(x_{2k}, x_{2k+1}))). \]

Consider the three possible cases:

1) \( u(x_{2k}, x_{2k+1}) = 0 \); it trivially follows that \( \sigma(x_{2k+1}, x_{2k+2}) = 0 \), i.e., \( x_{2k+1} = x_{2k+2} \).

2) \( u(x_{2k}, x_{2k+1}) = \sigma(x_{2k+1}, x_{2k+2}) \); it follows that

\[ \psi(\sigma(x_{2k+1}, x_{2k+2})) \leq F(\psi(\sigma(x_{2k+1}, x_{2k+2})), \varphi(\sigma(x_{2k+1}, x_{2k+2}))), \]

and by the properties of functions \( \psi, \varphi \) and \( F \) we get that \( x_{2k+1} = x_{2k+2} \).
3^0 \ u (x_{2k}, x_{2k+1}) = \frac{1}{4} (\sigma (x_{2k}, x_{2k+2}) + \sigma (x_{2k+1}, x_{2k+1}))
\text{since}
\begin{align*}
u (x_{2k}, x_{2k+1}) &\leq \frac{1}{4} (\sigma (x_{2k}, x_{2k+1}) + \sigma (x_{2k+1}, x_{2k+2}) + \sigma (x_{2k+1}, x_{2k}) + \sigma (x_{2k+1}, x_{2k+2})) \\
&= \frac{1}{2} (\sigma (x_{2k}, x_{2k+1}) + \sigma (x_{2k+1}, x_{2k+2})) = \frac{1}{2} \sigma (x_{2k+1}, x_{2k+2}),
\end{align*}
\text{it follows that}
\begin{align*}
\psi (\sigma (x_{2k+1}, x_{2k+2})) &\leq F (\psi \left( \frac{1}{2} \sigma (x_{2k+1}, x_{2k+2}) \right), \varphi \left( \frac{1}{2} \sigma (x_{2k+1}, x_{2k+2}) \right)),
\end{align*}
wherefrom \( \psi (\sigma (x_{2k+1}, x_{2k+2})) \leq \frac{1}{2} \sigma (x_{2k+1}, x_{2k+2}) \) which is only possible if \( x_{2k+1} = x_{2k+2} \).

Suppose now that \( \sigma (x_n, x_{n+1}) > 0 \) for all \( n \in \mathbb{N} \). Putting \( x = x_{2n}, y = x_{2n-1} \) in (2.6), we get that there exists
\begin{align*}
u (x_{2n}, x_{2n-1}) &\in \{ \sigma (x_{2n}, x_{2n-1}), \sigma (x_{2n}, x_{2n+1}), \sigma (x_{2n-1}, x_{2n}) \} \\
&= \left\{ \sigma (x_{2n}, x_{2n-1}), \sigma (x_{2n}, x_{2n+1}), \frac{1}{4} \left( \sigma (x_{2n}, x_{2n}) + \sigma (x_{2n-1}, x_{2n+1}) \right) \right\},
\end{align*}
such that
\begin{align*}
\psi (\sigma (x_{2n+1}, x_{2n})) &\leq F (\psi (u (x_{2n}, x_{2n-1})), \varphi (u (x_{2n}, x_{2n-1}))).
\end{align*}

Consider three possible cases:
1^0 u (x_{2n}, x_{2n-1}) = \sigma (x_{2n}, x_{2n-1}); it follows that
\begin{align*}
\psi (\sigma (x_{2n+1}, x_{2n})) &\leq F (\psi (\sigma (x_{2n}, x_{2n-1})), \varphi (\sigma (x_{2n}, x_{2n-1}))) < \psi (\sigma (x_{2n}, x_{2n-1}))
\end{align*}
and \( \sigma (x_{2n+1}, x_{2n}) < \sigma (x_{2n}, x_{2n-1}) \).

2^0 u (x_{2n}, x_{2n-1}) = \sigma (x_{2n}, x_{2n+1}); it follows that
\begin{align*}
\psi (\sigma (x_{2n+1}, x_{2n})) &\leq F (\psi (\sigma (x_{2n+1}, x_{2n})), \varphi (\sigma (x_{2n+1}, x_{2n}))) < \psi (\sigma (x_{2n}, x_{2n+1})),
\end{align*}
which is impossible.

3^0 u (x_{2n}, x_{2n-1}) = \frac{1}{4} (\sigma (x_{2n}, x_{2n}) + \sigma (x_{2n-1}, x_{2n+1})); it follows that
\begin{align*}
\psi (\sigma (x_{2n+1}, x_{2n})) &\leq F (\psi \left( \frac{1}{4} (\sigma (x_{2n}, x_{2n}) + \sigma (x_{2n-1}, x_{2n+1})) \right), \varphi \left( \frac{1}{4} (\sigma (x_{2n}, x_{2n}) + \sigma (x_{2n-1}, x_{2n+1})) \right)).
\end{align*}

By the properties of functions \( \psi \) and \( \varphi \) we obtain that
\begin{align*}
\sigma (x_{2n+1}, x_{2n}) &\leq \frac{1}{4} (\sigma (x_{2n}, x_{2n}) + \sigma (x_{2n-1}, x_{2n+1})) \\
&\leq \frac{1}{2} (\sigma (x_{2n-1}, x_{2n}) + \sigma (x_{2n}, x_{2n+1})).
\end{align*}
and $\sigma(x_{2n+1}, x_{2n}) \leq \sigma(x_{2n-1}, x_{2n})$.

Hence, in any possible cases, $\sigma(x_{2n+1}, x_{2n}) \leq \sigma(x_{2n-1}, x_{2n})$, and, similarly, $\sigma(x_{2n+2}, x_{2n+1}) \leq \sigma(x_{2n+1}, x_{2n})$. Thus, the sequence $\{\sigma(x_n, x_{n+1})\}$ is non-increasing; moreover,

$$\sigma(x_{2n+2}, x_{2n+1}) \leq u(x_{2n+1}, x_{2n}) \leq \sigma(x_{2n+1}, x_{2n}), \quad (2.7)$$

$$\sigma(x_{2n+1}, x_{2n}) \leq u(x_{2n}, x_{2n-1}) \leq \sigma(x_{2n}, x_{2n-1}). \quad (2.8)$$

Passing to the limit in (2.7) and (2.8) when $n \to \infty$, we obtain that $\sigma(x_n, x_{n+1}) \to \Sigma$ and $u(x_n, x_{n+1}) \to \Sigma \ (n \to \infty)$ for some $\Sigma \geq 0$. If $\Sigma > 0$, then passing to the limit in

$$\psi(\sigma(x_{2n+1}, x_{2n+2})) \leq F(\psi(u(x_{2n}, x_{2n+1})), \varphi(u(x_{2n}, x_{2n+1}))),$$

we obtain that $\psi(\Sigma) \leq F(\psi(\Sigma), \varphi(\Sigma))$ and $\Sigma = 0$ by the properties of functions $\psi \in \Psi$ and $F$. Hence $\sigma(x_n, x_{n+1}) \to 0$.

We next prove that $\{x_n\}$ is a $0-\sigma$-Cauchy sequence. Suppose that this is not a case. Applying Lemma 1.6, we obtain that there exist $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that the sequences

$$\sigma(x_{m_k}, x_{n_k}), \sigma(x_{m_k}, x_{n_k+1}), \sigma(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, x_{n_k+1})$$

all tend to $\varepsilon^+$ when $k \to \infty$.

Now, from (2.7), (2.8) and the obtained limits, we have

$$\lim_{k \to \infty} u(x_{n_k}, x_{m_k-1}) = \varepsilon, \quad (2.9)$$

for any

$$u(x_{n_k}, x_{m_k-1}) \in \{\sigma(x_{n_k}, x_{m_k-1}), \sigma(x_{n_k}, x_{n_k+1}), \sigma(x_{m_k-1}, x_{m_k}), \frac{1}{4}(\sigma(x_{n_k}, x_{m_k}) + \sigma(x_{m_k-1}, x_{n_k+1}))\}.$$
This shows that \( \{ x_n \} \) is a 0-\( \sigma \)-Cauchy sequence in 0-\( \sigma \)-complete metric-like space \( (X, \sigma) \). Therefore, there exists \( z \in X \) such that

\[
\sigma (z, z) = \lim_{n \to \infty} \sigma (x_n, z) = \lim_{n,m \to \infty} p(x_n, x_m) = 0. \tag{2.10}
\]

Now, putting \( x = x_{2n}, y = z \) in (2.6) we get

\[
\psi (\sigma (fx_{2n}, gz)) \leq F (\psi (u(x_{2n}, z)), \varphi (u(x_{2n}, z))), \tag{2.11}
\]

where

\[
u(x_{2n}, z) \in \left\{ \sigma (x_{2n}, z), \sigma (x_{2n}, x_{2n+1}), \sigma (z, gz), \frac{1}{4} (\sigma (x_{2n}, gz) + \sigma (z, x_{2n+1})) \right\}.
\]

So, in this case we obtain four possibilities:

1. \( \psi (\sigma (fx_{2n}, gz)) \leq F (\psi (\sigma (x_{2n}, z)), \varphi (\sigma (x_{2n}, z))) \);

2. \( \psi (\sigma (fx_{2n}, gz)) \leq F (\psi (\sigma (x_{2n}, x_{2n+1})), \varphi (\sigma (x_{2n}, x_{2n+1}))) \);

3. \( \psi (\sigma (fx_{2n}, gz)) \leq F (\psi (\sigma (z, gz)), \varphi (\sigma (z, gz))) \);

4. \( \psi (\sigma (fx_{2n}, gz)) \leq F \left( \psi \left( \frac{1}{4} (\sigma (x_{2n}, gz) + \sigma (z, x_{2n+1})) \right) \right), \varphi \left( \frac{1}{4} (\sigma (x_{2n}, gz) + \sigma (z, x_{2n+1})) \right). \)

Passing to the limit when \( n \to \infty \) in these four relations and applying Remark 1.10. (a) we get respectively the next four inequalities:

- \( \psi (\sigma (z, gz)) \leq F (\psi (\sigma (z, z)), \varphi (\sigma (z, z))) = F (\psi (0), \varphi (0)) \leq \psi (0) = 0; \)

- \( \psi (\sigma (z, gz)) \leq F (\psi (0), \varphi (0)) \leq \psi (0) = 0; \)

- \( \psi (\sigma (z, gz)) \leq F (\psi (\sigma (z, gz)), \varphi (\sigma (z, gz))) ; \)

- \( \psi (\sigma (z, gz)) \leq F \left( \psi \left( \frac{1}{4} (\sigma (z, gz) + \sigma (z, z)) \right) \right), \varphi \left( \frac{1}{4} (\sigma (z, gz) + \sigma (z, z)) \right). \)

In each of these cases it easily follows that \( gz = z. \)

Now, putting \( x = y = z \) in (2.6), one gets \( \psi (\sigma (fz, gz)) \leq F (\psi (u(z, z)), \varphi (u(z, z))) \), where

\[
u(z, z) \in \left\{ \sigma (z, z), p(z, fz), \sigma (z, gz), \frac{1}{4} (\sigma (z, fz) + \sigma (z, gz)) \right\} = \left\{ 0, \sigma (z, fz), \frac{1}{4} \sigma (z, fz) \right\}
\]

and in each of the possible three cases it easily follows that \( fz = z. \) Hence, \( z \)

is the unique common fixed point of \( f \) and \( g. \) \( \square \)

Putting \( g = f \) in Theorem 2.2, one obtains

**Corollary 2.3.** Let \( (X, \sigma) \) be a 0-\( \sigma \)-complete metric-like space and \( f : X \to X \) be such that for some \( \psi \in \Psi, \varphi \in \Phi (\varphi \in \Phi_1) \) and \( F \in \mathcal{C} \) and for all \( x, y \in X \) there exists

\[
u(x, y) \in \left\{ \sigma (x, y), \sigma (x, fx), \sigma (y, fy), \frac{1}{4} (\sigma (x, fy) + \sigma (y, fx)) \right\}
\]
such that $\psi(\sigma(fx, fy)) \leq F(\psi(u(x, y)), \varphi(u(x, y)))$.

Then $f$ has a unique fixed point.

**Example 2.4.** Let $X = \{0, 1, 2\}$. Define $\sigma : X \times X \to [0, +\infty)$ as follows:

$$
\begin{align*}
\sigma(0, 0) &= 0, \sigma(1, 1) = 3, \sigma(2, 2) = 1, \sigma(0, 1) = \sigma(1, 0) = 7, \\
\sigma(0, 2) &= \sigma(2, 0) = 3, \sigma(1, 2) = \sigma(2, 1) = 4.
\end{align*}
$$

Then $(X, \sigma)$ is a $0-\sigma$-complete metric-like space. Note that $\sigma$ is not partial metric on $X$ because

$$
\sigma(0, 1) \nless \sigma(0, 2) + \sigma(2, 1) - \sigma(2, 2).
$$

Define the maps $f, g : X \to X$ by

$$
f0 = 0, f1 = 2, \text{ and } f2 = 0,
$$

and $gx = x$ for all $x \in X$. Then all the required hypothesis of Theorem 2.1 are satisfied and $f$ has a unique fixed point.

Now, if we endow $X$ with the partial metric $p$ given by $p(x, y) = \max\{x, y\}$ for each $x, y \in X$ then $(X, p)$ is a complete partial metric space. Since

$$
\psi(p(f0, f1)) \nless \psi(p(0, 1)) - \varphi(p(0, 1)) \text{ for each } \psi \in \Psi, \varphi \in \Phi \ (\Phi_1)
$$

then we cannot invoke the main result of [36] (Theorem 2.1) to show the existence of a fixed point for $f$. Hence our Theorem 2.1. is genuine generalization of result from [36].

Further, putting $g = f$ in this example we get that all the conditions of Theorem 2.2 are satisfied and $f$ has a unique fixed point. If we consider $X$ with the same partial metric $p$, it is easy to see that

$$
\psi(p(f0, f1)) \nless \psi(u(0, 1)) - \varphi(u(0, 1)) \text{ for each } \psi \in \Psi, \varphi \in \Phi,
$$

where obviously

$$
u(0, 1) \in \left\{p(0, 1), p(0, 0), p(1, 2), \frac{p(0, 2) + p(1, 0)}{2}\right\} = \left\{1, 0, 2, \frac{3}{2}\right\}.
$$

This example also implies that our Theorem 2.2. can be applied to show the existence of a fixed point for $f$, while the main result (Theorem 2.2) in [36] cannot also for each $F \in C$ above discussion is true.

With choose $F(s, t)$ we have the following:
3 Applications

In the sequel with choose of the function $F(s,t)$ we obtained the following results in the frame of $0-\sigma-$complete metric-like spaces:

**Corollary 2.4.** [25] Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(\sigma(fx, fy)) \leq cF(\psi(\sigma(gx, gy)), \varphi(\sigma(gx, gy))),$$

for all $x, y \in X$ where $0 < c < 1$ and $\psi \in \Psi, \varphi \in \Phi (\varphi \in \Phi_1)$ and $F \in \mathcal{C}$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

**Corollary 2.5.** Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Assume that $f : X \to X$ is a mapping such that

$$\psi(\sigma(fx, fy)) \leq cF(\psi(\sigma(x, y)), \varphi(\sigma(x, y))),$$

for all $x, y \in X$ where $0 < c < 1$ and $\psi \in \Psi, \varphi \in \Phi (\varphi \in \Phi_1)$ and $F \in \mathcal{C}$. Then $f$ has a unique fixed point (say $u$) and $\sigma(fu, fu) = 0 = \sigma(u, u)$.

**Corollary 2.6.** [36] Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(\sigma(fx, fy)) \leq F(\psi(\sigma(gx, gy)), \varphi(\sigma(gx, gy))),$$

for all $x, y \in X$ where $\psi \in \Psi, \varphi \in \Phi (\varphi \in \Phi_1)$ and $F \in \mathcal{C}$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point (say $z$) and $\sigma(z, z) = 0 = \sigma(fz, fz) = \sigma(gz, gz)$.

**Corollary 2.7.** Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(pfx, fy) \leq \frac{\psi(p(gx, gy))}{(1 + \varphi(p(gx, gy)))r}, \quad r \in (0, \infty),$$

for all $x, y \in X$ where $\psi \in \Psi, \varphi \in \Phi (\varphi \in \Phi_1)$ and $F \in \mathcal{C}$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point (say $z$) and $\sigma(z, z) = 0 = \sigma(fz, fz) = \sigma(gz, gz)$.

**Corollary 2.8.** Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(\sigma(fx, fy)) \leq \psi(\sigma(gx, gy))\log_{a + \varphi(p(gx, gy))} a, \quad a > 1,$$
for all $x, y \in X$ where $\psi \in \Psi$, $\varphi \in \Phi$ ($\varphi \in \Phi_1$) and $F \in C$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point (say $z$) and $\sigma(z, z) = 0 = \sigma(fz, fz) = \sigma(gz, gz)$.

**Corollary 2.9.** Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(\sigma(fx, fy)) \leq (\psi(\sigma(gx, gy)) + l) \left(\frac{1}{1+\psi(\sigma(gx, gy))}\right) - l, \ l > 1,$$

for all $x, y \in X$ where $\psi \in \Psi$, $\varphi \in \Phi$ ($\varphi \in \Phi_1$) and $F \in C$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point (say $z$) and $\sigma(z, z) = 0 = \sigma(fz, fz) = \sigma(gz, gz)$.

**Corollary 2.10.** Let $(X, \sigma)$ be a $0-\sigma-$ complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy

$$\psi(\sigma(fx, fy)) \leq \psi(\sigma(gx, gy)) - \left(\frac{1+\psi(\sigma(gx, gy))}{2+\psi(\sigma(gx, gy))}\right) \left(\frac{\varphi(\sigma(gx, gy))}{1+\varphi(\sigma(gx, gy))}\right),$$

for all $x, y \in X$ where $\psi \in \Psi$, $\varphi \in \Phi$ ($\varphi \in \Phi_1$) and $F \in C$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point (say $z$) and $\sigma(z, z) = 0 = \sigma(fz, fz) = \sigma(gz, gz)$.

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