On the Solvability of Geometrically Nonlinear Problem of Sandwich Plate Theory

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Abstract

The problem of determining the stress-strain state rigidly fixed sandwich plate with transversely soft core in the presence of constraints (i.e., material nonlinearity) corresponding to the ideal elastic-plastic model for the core material is considered. Solvability of the generalized statement of the problem as a problem of finding the saddle point of some functionals is investigated.

Keywords: Sandwich plate, transversely soft filler, saddle point, theorem of solvability, uniqueness theorem

Introduction

The stress-strain state of sandwich plates is considered with transversely soft core in the presence of constraints on transverse tangential stresses formed in the core layer, which corresponds to the use of the ideal elastic-plastic model for the core material and so adding nonlinearity into the statement of the problem. Sandwich constructions possess good sound and thermal insulation properties, as well as high adaptability and vibration resistance [9]. This explains their widespread use not only in aerospace engineering, shipbuilding and transport machinery, but also in construction. Generalized statement of the problem is formulated as a problem of finding a saddle point of some functionals. We prove a theorem of solvability by using the general results of the existence of saddle points [20]. The generalized formulations of the theory of soft network shells with constraints are considered in [4-8, 13-15, 17-19, 29, 30] and the methods for their numerical solutions are given.

1 Problem statement

We consider, in one-dimensional spatial coordinates, the stress-strain state of a sandwich plate consisting of two outer load carrying layers and transversely soft core adhesively bonded to them. Load carrying layers are described with refined Kirchhoff-Love hypothesis as in [26, 27] and for the core layer, equations of elasticity theory simplified in accordance with the accepted model of the transversely soft material and integrated over the thickness satisfying conjugate conditions of the layers for the displacements in the transverse direction. Let in accordance with introduced in [26, 27] notation, \( a \) be the length the plate, \( 2h \), \( 2h_{(k)} \) be the thickness of the filler and the \( k \)-th layer, respectively (here and in what follows we assume that \( k = 1, 2 \)), \( H_{(k)} = h + h_{(k)} \), \( X_{(k)}^1 \), \( X_{(k)}^3 \) be the components of the surface load given to the middle surface of \( k \)-th layer, \( w_{(k)} \).
and \( u^{(k)} \) be the bending and axial displacement surface of the middle points of \( k \)-th layer, \( X_{11}^{(k)}, M_{11}^{(k)} \) – the membrane forces and inner bending moments in the \( k \)-th layer, respectively. The edges of the plate carrier layers we assume rigidly fixed so that the conditions hold \( u^{(k)}(x) = 0, \ w^{(k)}(x) = 0, \ d w^{(k)} / dx = 0, \) at \( x = 0, \ x = a \). The problem is considered in the geometrically linear statement, that is, we assume that \( T_{11}^{(k)} = B_{(k)} d u^{(k)} / dx, \ M_{11}^{(k)} = D_{(k)} d^2 u^{(k)} / dx^2, \) where

\[
B_{(k)} = 2h E^{(k)} / (1 - \nu_{12}^{(k)})(1 - \nu_{21}^{(k)})
\]

is the stiffness of the \( k \)-th layer on the tension-compression, \( E^{(k)} \) and \( \nu_{12}^{(k)}, \nu_{21}^{(k)} \) are the first kind modulus and Poisson’s coefficients of the material of the \( k \)-th carrying layer, \( D_{(k)} = B_{(k)} h^2 / 3 \) is the flexural rigidity of the \( k \)-th layer. Let \( U = (w^{(1)}, w^{(2)}, u^{(1)} w^{(2)}) \) be the displacement vector points of the middle surface of the \( k \)-th layer, and \( q^1 \) be tangential stresses in the core. For \( q^1 \) we assume that the boundary conditions \( q^1(0) = q^1(a) = 0 \) are satisfied. Let in accordance with [25, 26] take into consideration the functional \( L = P - A - A_q \), where

\[
P(U, q^1) = \frac{1}{2} \left\{ \sum_{k=1}^{a} \left[ B_{(k)} (d u^{(k)} / dx)^2 + D_{(k)} (d^2 w^{(k)} / dx^2)^2 \right] + c_1 (q^1)^2 + 
+ c_2 (d q^1 / dx)^2 + c_3 (w^{(2)} - w^{(1)})^2 \right\} dx
\]

is the strain potential, \( c_1 = 2h / G_{13}, \ c_2 = h^3 / 3 E_3, \ c_3 = E_3 / (2h), \ G_{13}, \ E_3 \) are the transverse shear and compression modules of a core,

\[
A(U, q^1) = \int_0^a \sum_{k=1}^{21} \left[ X_{11}^{(k)} u^{(k)} + M_{11}^{(k)} d w^{(k)} / dx + X^{3} w^{(k)} \right] dx
\]

is the work of given external forces and moments, \( M_{11}^{(k)} \) is the surface moment of external forces, reduced to the middle surface of the \( k \)-th layer, and

\[
A_q(U, q^1) = \int_0^a \left[ (u^{(1)} - u^{(2)}) - \sum_{k=1}^{21} H_{(k)} d w^{(k)} / dx + c_1 q^1 - c_2 d^2 q^1 / dx^2 \right] q^1 dx
\]

is the work of unknown tangential stresses on corresponding displacements. As shown in [25, 26] a solution of the problem is a saddle point of the functional \( L \).

Assuming that the connection between tangential stress \( q^1 \) and shear deformations corresponds to the ideal elastic-plastic model, we consider the problem under the constraint \( |q^1(x)| \leq q^* \), \( 0 < x < a \), where \( q^* \) is the given value of limiting stress in filler.
2 Solvability of the generalized problem

Let \( V_k = W_2^{(k)}(0,a) \) be Sobolev spaces with inner products
\[
(u, \eta)_k = \int_0^a d^k u/dx^k \ d^k \eta/dx^k \ dx, \quad V_+ \ be \ Sobolev \ space \ with \ the \ inner \ product
\]
\[
(y, z)_+ = \int_0^a [c_2 y(x) z(x) + c_3 d y/dx \ d z/dx] \ dx.
\]
We denote \( V = V_2 \times V_2 \times V_1 \times V_1 \), \( K = \{ y \in V_+ : | y(x)| \leq q^*, \ 0 < x < a \} \). For the sake of easiness let rewrite the functional
\( L : V \times V_+ \to R^1 \) in the form
\[
L(U, q^1) = \Phi(U) + \Phi_3(U, q^1) - \Phi_4(q^1),
\]
where
\[
\Phi(U) = \frac{1}{2} \int_0^a \left[ B_{(k)} (d u^{(k)} / dx)^2 + D_{(k)} (d^2 w^{(k)} / dx^2)^2 \right] dx -
\]
\[
- \int_0^a (X_{(k)} u^{(k)} + X_{(k)} w^{(k)}) + M_{(k)} d w^{(k)} / dx, \]
\[
\Phi_3(U, q^1) = \int_0^a \left[ \sum_{k=1}^2 H_{(k)} \frac{d w^{(k)}}{dx} + (u^{(2)} - u^{(1)}) \right] q^1 \ dx,
\]
\[
\Phi_4(q^1) = \frac{1}{2} \int_0^a \left[ \frac{2h}{G_{13}} (q^1)^2 + \frac{h^3}{3E_3} \left( \frac{d q^1}{dx} \right)^2 \right] dx, \quad \Phi_0(U) = \frac{1}{2} \int_0^a c_3 (w^{(2)} - w^{(1)})^2 \ dx,
\]

It is easy to see that the functional \( \Phi \) is well defined on \( V \), the functionals \( L \), \( \Phi_3 \) are well defined on \( V \times V_+ \), and functional \( \Phi_4 \) is well defined on \( V_+ \). The inner product in \( V \) is denoted by (\( ., . \))_\( V \). By a generalized solution of the problem of determining the stress-strain state of the sandwich plate with transversal-soft filler we mean a function \( (\hat{U}, \hat{q}^1) \in V \times K \) such that
\[
L(\hat{U}, \hat{q}^1) = \inf_{U \in V} \sup_{q^1 \in K} L(U, q^1).
\]

**Lemma 1.** The functionals \( \Phi_j, \ j=0, 1, 2, 4 \), are convex and weakly lower semicontinuous, functionals \( \Phi_j, j=1, 2, 4 \), are strictly convex.

**Proof.** Convexity (and strict convexity) of functionals \( \Phi_j, j=1, 2, 4 \) follows from the strict convexity of the quadratic function. Convexity of the function \( \phi(x, y) = (x - y)^2 \) and hence of the functional \( \Phi_{(0)} \) follows from the obvious
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inequality \( \varphi(x_1, y_1) \geq \varphi(x_2, y_2) + 2(x_2 - x_1)(y_2 - y_1) \) which holds for any vectors \((x_1, y_1), (x_2, y_2)\) from \( \mathbb{R}^2 \). Also considered functional, as is easily verified, are continuous, therefore, by virtue of their convexity they are weakly lower semicontinuous [20].

**Lemma 2.** The functional \( \Phi_{(3)} \) is linear and continuous in both arguments.

**Proof.** The lemma follows immediately from the definition of the \( \Phi_{(3)} \).

**Lemma 3.** The set \( K \) is weakly closed.

**Proof.** The lemma follows from the convexity of \( K \), and compactness of the embedding \( L_2(0,a) \) in \( V_1 \) [2].

**Lemma 4.** Functionals \( \Phi_{(0)} + \Phi_{(1)} + \Phi_{(2)} \) and \( \Phi_{(4)} \) are coercive [24] in \( V \) and \( V_+ \) respectively.

**Proof.** The lemma follows immediately from the definition of the functional \( \Phi_{(0)} + \Phi_{(1)} + \Phi_{(2)} \) and \( \Phi_{(4)} \).

**Theorem 1.** The problem (1) has a unique saddle point.

**Proof.** Note in the first that \( 0 \in K \). From Lemmas 1, 2, 4 it implies that the functional \( L \) satisfies the following conditions: functional \( q^1 \rightarrow L(U, q^1) \) is concave and upper semicontinuous for all \( U \in V \); functional \( U \rightarrow L(U, q^1) \) is convex and lower semicontinuous for all \( q^1 \in K \); \( L(U, 0) \rightarrow +\infty \) as \( \| U \|_v \rightarrow +\infty \); \( L(0, q^1) \rightarrow -\infty \) as \( \| q^1 \|_+ \rightarrow +\infty \). Therefore, from Proposition VI.2.2 [20] follows that there exists of at least one saddle point \((\hat{U}, \hat{q}^1) \in V \times K \) of the functional \( L \). The uniqueness of the saddle point follows from the fact that for any \( U \in V \) the functional \( U \rightarrow L(U, q^1) \) is strictly concave, and for any \( q^1 \in K \) functional \( U \rightarrow L(U, q^1) \) is strictly convex.

The solving of problem (1) can be carried out by methods proposed in the articles [1, 3, 10-12, 21-23, 28, 31].

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