Efficient Spectral Approach to SISO Problems of 

H₂-Optimal Synthesis

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Abstract

The paper is devoted to H₂-optimization problem for linear time invariant (LTI) systems with scalar control, external disturbance and measurement noise. This problem can be numerically solved with the help of the well-known universal approaches based on Riccati equations or LMI technique. Nevertheless, for particular cases there exists a possibility to increase a computational efficiency of the synthesis using a special spectral approach to the problem in frequency domain. Some theoretical details are discussed and numerical algorithms are proposed for practical implementation of this approach. One of its virtues is a possibility to present optimal solutions in a specific form that is convenient for investigations of the optimal system features.

Keywords: control system, optimization, H₂-control, feedback, functional

1. Introduction

One of the most important problems in a practice of controlled systems analytical design is LTI synthesis problem of the external disturbances and measurement noises optimal rejection for closed-loop system. This problem has determined a vast area of investigations in control theory and signal processing, and nowadays its multiple descendants are joined in the framework of the modern H-optimization theory.

Computational basis of H-optimal control is substantially connected with two approaches: first of them is based on a solution of the algebraic matrix Riccati equations ("2-Riccati" approach) (Doyle et al. [5], Bhattacharyya et al. [2]), and second – on a solution of linear matrix inequalities ("LMI" technique) (Boyd et al.
Nevertheless, in our opinion these approaches are not fully efficient for a partial situation of $H_2$ SISO LTI synthesis, where controlled plants of no large order have scalar controlling and disturbing inputs. In our opinion, it is more suitable to use spectral frequency methods for this case based on a polynomial factorization. This direction was intensively developed by F.A. Aliev, V.B. Larin, K. I. Naumenko, and V. N. Suntsev (see, for example [1] and references therein). Some other viewing of the problem was presented by H. Kwakernaak [7], V. Kucera [6], M. Vidyasagar [12] and their successors [8]. Here we are going to discuss particular variant of polynomial approach, which seems to be quite convenient and effective for analysis and synthesis of $H$-optimal controllers.

Note that the computational efficiency of the algorithms for optimal synthesis is highly crucial issue for control systems with an adoptive changeover in real-time regime of operating. Actually, the time gap of calculations is not so significant for laboratory conditions, but we cannot say the same with respect to embedded systems or for onboard control systems of autonomous moving robots. This work is aimed to present the special spectral form of the $H_2$-optimal controller based on polynomial approach. Relative simplicity of the situation allows obtaining of this form by the method which is more effective in computational sense in comparison with the "2-Riccati" or "LMI" techniques.

The paper is organized as follows. In the next section, equations of a controlled plant are presented and the problem of $H_2$-optimal synthesis is posed. Section 3 is devoted to the special spectral representation of the $H_2$-optimal controller. In Section 4, we consider description of a singular situation and discuss the computational effectiveness of proposed approach. In section 5, we give illustrative examples of synthesis. Finally, Section 6 concludes this paper by discussing the overall results.

2. Problem Statement

Let us consider a problem of feedback control laws synthesis for LTI plants with mathematical models of the form

$$
\begin{align*}
\dot{x} &= Ax + bu + pd(t), \\
y &= cx + \psi(t), \quad \xi = cx, \\
\xi_1 &= \xi, \\
\xi_2 &= ku,
\end{align*}
$$

where $x \in E^n$ is the state space vector, $y$, $u$, $\xi$, $d$ and $\psi$ are the scalar values: $y$ and $\xi$ are the measured and controlled variables respectively, $u$ is the control, $d(t)$ represents an external disturbance, $\psi(t)$ is the measurement noise. All components of the matrices and parameter $k$ are given constants, the pairs $\{A,b\}$ and $\{A,c\}$ are controllable and observable respectively.

External inputs $d$ and $\psi$ of the system (1) will be treated bellow as the outputs of systems.
\[ d = S_{d1}(s)i_1, \quad \psi = S_{\psi 1}(s)i_2 \] (2)
correspondently, where \( i_1(t) \) and \( i_2(t) \) are mutually independent Gaussian white noises,

\[ S_{d1}(s) = N_d(s)/T_d(s), \quad S_{\psi 1}(s) = N_{\psi}(s)/T_{\psi}(s), \]
polynomials \( N_d, T_d, N_{\psi}, T_{\psi} \) are Hurwitz. Really, external disturbances \( d(t) \) and \( \psi(t) \) are treated as random stationary processes with given rational spectral power densities \( S_d(\omega) = S_{d1}(s)S_{d1}(-s)|_{s=j\omega}, \quad S_{\psi}(\omega) = S_{\psi 1}(s)S_{\psi 1}(-s)|_{s=j\omega}. \)

Let us accept that controller to be designed has tf-form

\[ u = W(s)y, \] (3)
where \( W(s) = W_1(s)/W_2(s), \ W_1, W_2 \) are polynomials. If the transfer function \( W \) of the controller (3) is found as a solution of the analytical synthesis problem, we obtain a closed-loop connection (1) – (3) presented in Fig. 1 by its block-scheme.

![Fig. 1. Closed-loop connection scheme.](image)

It is easy to see that the closed-loop system (1) – (3) has the input \( i = (i_1, i_2)^T \) and the output \( e = (e_1, e_2)^T \) and its mathematical model can be presented as \( e = H(s,W)i \), where \( H(s,W) \) is the transfer matrix of the system. Let also introduce auxiliary transfer functions of the system (1) – (3) with respect to inner inputs \( d, \psi \) and outputs \( \xi, u \) : \( F_{\xi d}(s) = P/(A - BW) \), \( F_{\psi d}(s) = BW/(A - BW) \), \( F_{\xi u}(s) = PW/(A - BW) \), \( F_{\psi u}(s) = AW/(A - BW) \), \( A = A(s) = \det(ES - A) \), \( B = B(s) = A(s)c(ES - A)^{-1}b \), \( P = P(s) = A(s)c(ES - A)^{-1}p \).

Using the introduced notations, we can represent the transfer matrix \( H(s,W) \) as follows:

\[
H(s,W) \equiv \begin{pmatrix}
F_{\xi d}(s) & F_{\psi d}(s) & S_{d1}(s) & 0 \\
0 & 0 & S_{\psi 1}(s)
\end{pmatrix}.
\] (4)

Finally, let us introduce the generalized weighted transfer function \( H_w(s,W) \) such that the following identity holds:
The matter of $H_2$-optimal synthesis is to find a solution of the control problem

$$J_2(W) = \|H_w(s,W)\|^2_2 \rightarrow \min_{W \in \Omega_2},$$

where $\Omega_2 = \{ W : H_w(s,W) \in RH_2 \}$, $RH_2$ is the set of strictly proper fractions with Hurwitz denominators, having the norm $\|H_w\|^2_2 = \left( \frac{1}{\pi} \int_0^\infty |H_w(j\omega)|^2 \ d\omega \right)$.

The essence of the problem (6) is to suppress input weighted noises as much as possible with respect to controlled variable $\xi$ and control $u$. Note that parameter $k$ can be treated as weight multiplier governing the relationship between the intensity of control action and the achieved measure of suppression for the closed loop connection.

Remark that the problem (6) direct solution is appreciably obstructed by the nonlinear dependency of the functional $J_2$ from the adjustable function $W$ [13]. To avoid this difficulty, we can use the parameterization technique [1], [11].

In accordance with this method, let introduce the adjustable function-parameter $\Phi$ as

$$\Phi = L^{-1}_\Phi(W) = \frac{\alpha + \beta W}{A - B W} \Rightarrow W = L_\Phi(\Phi) = \frac{A\Phi - \alpha}{B\Phi + \beta}, \quad (7)$$

where $\alpha$ and $\beta$ are any polynomials such that the polynomial $Q(s) = A(s)\beta(s) + B(s)\alpha(s)$

is Hurwitz. Formulae (7) allow us to presents transfer functions of the closed-loop system as

$$F_{d\xi} = P(B\Phi + \beta)/Q, \ F_{d\xi} = B(A\Phi - \alpha)/Q,$$
$$F_{du} = P(A\Phi - \alpha)/Q, \ F_{yu} = A(A\Phi - \alpha)/Q. \quad (9)$$

It is easy to see that optimization problem (6) is equivalent to the problem

$$I_2(\Phi) = \|H(s,\Phi)\|^2_2 \rightarrow \min_{\Phi \in \Omega_2^\Phi},$$

where the admissible set $\Omega_2^\Phi = L^{-1}_\Phi(\Omega_2)$ includes rational fractions $\Phi$ with Hurwitz denominators. In accordance with (5), we have for the function $H = H(s,\Phi) = H_w(s, L_\Phi(\Phi))$:

$$|H|^2 = (F_{d\xi} F_{d\xi}^* + k^2 F_{du} F_{du}^*) S_d + (F_{d\xi} F_{d\xi}^* + k^2 F_{yu} F_{yu}^*) S_y, \quad (11)$$
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where the notation $\overline{\rho} = \overline{\rho}(s) = \rho(-s)$ is used.

Lemma 1: The identity (11) can be transformed to the form

$$H(\Phi)\overline{H}(\Phi) \equiv (T_1 - T_2 \Phi)(\overline{T}_1 - \overline{T}_2 \Phi) + T_3,$$

where the rational fraction $T_2(s)$ with Hurwitz denominator and functions $T_1(s) \in RL, T_3(s) \in RL$ are determined as

$$T_1 = (k^2 \alpha A - \beta B^2 D/(GQ) + S_y \overline{A}B / (G\overline{D}), T_2 = \overline{G}D/Q,$$

$$T_3 = k^2 D\overline{D} / (G\overline{G}) + S_y (B\overline{B} - k^2 A\overline{A}) / (G\overline{G}) - S_y^2 A\overline{A}BB / (G\overline{G}D\overline{D}).$$

Here a Hurwitz polynomial $G(s)$ and a fraction $D(s) \equiv N(s)/T(s)$ with Hurwitz polynomials $N, T$ are the results of the following factorizations:

$$k^2 A\overline{A} + B\overline{B} \equiv G\overline{G}, S_y A\overline{A} + S_y^2 \overline{P} \equiv D\overline{D}.\quad (15)$$

Proof: A possibility of the mentioned representation directly follows from the substitution of the formulae (13) – (15) to the right part of (12). ■

3. $H_2$-Optimal Transfer Function

Remark that Lemma 1 allows us to attract well-known idea of the model matching for the treatment of the presented questions (Doyle J.C. et al. [5]).

Theorem 1: Optimization problem (10) is equivalent to the model-matching problem

$$E_2(\Phi) = \|T_1 - T_2 \Phi\|^2 \rightarrow \min_{\Phi \in \Omega^d_2}$$

for the given model with the transfer matrix $T_1$ (13).

Proof: In accordance with Lemma 1 we have

$$I_2(\Phi) = \frac{1}{\pi} \int_0^\infty |H|^2 d\omega = \frac{1}{\pi} \int_0^\infty |T_1 - T_2 \Phi|^2 d\omega + \frac{1}{\pi} \int_0^\infty T_3 d\omega,$$

and it follows directly from (15) that

$$\|H(\Phi)\|^2 = \|T_1 - T_2 \Phi\|^2 + \frac{1}{\pi} \int_0^\infty T_3 d\omega.$$

Observe that the second term in the right part of (17) does not depend on $\Phi$, so a minimum value of $I_2$ can be achieved if and only if the first term reaches its minimum. ■
Corollary 1: The real value \( I_a = \frac{1}{\pi} \int_0^\infty T_3 \, d\omega \) is a lower bound for the minimum values of the functionals \( J_2(W) \) and \( I_2(\Phi) \) for the problems (6), (12) correspondently:

\[
J_{20} = \min_{W \in \Omega_2} J_2(W) = \min_{\Phi \in \Phi_2^\phi} I_2(\Phi) \leq I_a. \tag{18}
\]

Next, let us consider the problem (16).

Theorem 2: There exists the unique function-parameter \( \Phi = \Phi_{02} \in \Omega_{2\Phi} \) providing a minimum of the functional \( E_2(\Phi) \) in (16) such that

\[
\Phi_{02} = \left[ (k^2 \alpha \overline{A} - \beta \overline{B})DT_\psi - RQ_T \psi + P_1Q \overline{G} \right] / (G \overline{G} DT_\psi), \tag{19}
\]

\[
R(s) = \sum_{i=1}^n \frac{G(-s)}{g_i} - \frac{B(-g_i)N(g_i)}{g_i - s} A(g_i)T(g_i)G'(-g_i), \tag{20}
\]

\[
P_i(s) = \sum_{i=1}^{q_i} \frac{T_\psi(s)}{s - \theta_i} \frac{A(-\theta_i)B(-\theta_i)S_\psi(-g_i)}{G(-\theta_i)D(-\theta_i)T_\psi(\theta_i)}.	ag{21}
\]

where \( g_i, i = 1, n \) and \( \theta_i, i = 1, q \) are the roots of the polynomials \( G(-s) \) and \( T_\psi(s) \) correspondently (for simplicity here we assume that all the roots are distinct).

Proof: Let consider an expression for the model matching error in (16). After substitution the formulae (13) obtain

\[
E_2(\Phi) = \left\| (k^2 \alpha \overline{A} - \beta \overline{B})D/(QG) + S_\psi \overline{A} \overline{B} / (G \overline{G}) - \overline{G} D \Phi / Q \right\|_2^2
\]

that can be transformed to the equality \( E_2(\Phi) = \| M - L \Phi \|_2^2 \) by the dividing to the fraction \( \overline{G} / G \), taking into account that \( \| \overline{G} / G \|_2^2 = 1 \).

Here \( M = (k^2 \alpha \overline{A} - \beta \overline{B})D / (Q \overline{G}) + S_\psi \overline{A} \overline{B} / (G \overline{G}), \ L = GD / Q \). Now let expand the fraction \( M \in RL_2 \) to the sum of orthogonal elements \( M_2 \in RH_2 \) and \( M_2^\perp \in RH_2^\perp \):

\[
E_2(\Phi) = \| M_2 + M_2^\perp - L \Phi \|_2^2 = \| M_2 - L \Phi \|_2^2 + \| M_2^\perp \|_2^2. \tag{23}
\]

This equality holds because \( (M_2 - L \Phi) \in RH_2^\perp \), \( L \Phi \in RH_2 \) i.e. \( (M_2 - L \Phi) \perp M_2^\perp \).

To find the function \( M_2 \) explicitly, let make a separation
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\[
\frac{k^2 \alpha A - \beta B}{QG} \equiv \frac{R_1(s)}{Q(s)T(s)} + \frac{R(s)}{G(-s)},
\]

from where it directly follows that

\[
R(g_i) = -B(-g_i)D(g_i)/A(g_i), \quad i = 1, n
\]

on the base of (8) and (15). Formula (25) allows us to construct Lagrange interpolation polynomial \( R(s) \) (20). By the same way, it is easy to obtain the polynomial \( P_i(s) \) (21), separating the fraction

\[
\frac{S_{\psi}(s)A(-s)B(-s)}{G(s)D(-s)} \equiv \frac{P_i(s)}{T_{\psi}(s)} + K(s).
\]

Next, from (24) and (26) we have

\[
M_2 = \left( k^2 \alpha A - \beta B \right) D/\bar{QG} - R/\bar{G} + P_1/T_{\psi},
\]

assuming that the polynomials \( R(s) \) and \( P_i(s) \) are known.

It is easy to see that the second term in the right part of (23) does not depend on the function \( \Phi \), so a minimum of \( E_2 \) can be achieved only under the condition

\[
M_2 - \Phi \Phi = 0
\]

that is possible for the only variant \( \Phi = \Phi_{02} = M_2/L \), from where the expression (19) follows taking into account (27).

Observe that the polynomial \( R(s) \) satisfies the equalities

\[
[k^2 \alpha A - \beta B]D - RQ \bigg|_{s=g_i} = 0, \quad (i = 1, n)
\]

in accordance with the expression (25). It means that a numerator of the fraction \( \Phi_{02}(s) \) is totally divided to the polynomial \( G(-s) \). However, because of the polynomials \( G(s) \) and \( N_{\psi}(s) \) are Hurwitz, then a denominator of the function \( \Phi_{02}(s) \) is also Hurwitz polynomial. Besides that, it is easy to verify that the functional \( E_2 \) accepts a finite value, so the value \( \|H(\Phi)\|_2^2 \) is also finite, i.e. \( \Phi_{02} \in \Omega_2^\Phi \).

**Theorem 3:** Controller (3) with the transfer function

\[
W_0 = \frac{T_{\psi}(ATR + \bar{B}N)/\bar{G} - AP_T}{T_{\psi}(BTR - k^2 \bar{A}N)/\bar{G} - BP_T}
\]

is the unique solution of the problem (6). A division to the polynomial \( G(-s) \) in (25) is realized totally (without a remainder). A characteristic polynomial of the closed loop system (1) – (3) can be presented as

\[
\Delta_0(s) = -N(s)G(s)T_{\psi}(s).
\]
Proof: The expression (28) with the mentioned features follows from the substitution (19) to the formula \( W_0 = L\Phi(0,2) \), taking into account (7), (8), (15).

Let us consider a particular situation with respect to the \( H_2 \)-problem presented above under assumption that the spectral power densities of the disturbances \( d(t) \) and \( \psi(t) \) are constants \( S_d(\omega) = 1 \), \( S_\psi(\omega) = \gamma^2 \).

In this case the optimization problem (6) is equivalent to the well known LQG-problem with the scalar control and disturbances. To find the optimal controller (3) by the «2-Riccati» approach, we can apply the following algorithm.

Algorithm 1:
1. Find solutions of two Riccati equations with respect to the matrices \( S \) and \( P \) correspondently:
   \[
   -Sb'S/k^2 + A'S + SA + c'c = 0, \quad -Pc'cP/\gamma^2 + AP + PA' + pp' = 0.
   \]
2. Compute two auxiliary matrices \( m = -b'S/k^2, \quad l = Pc'/\gamma^2 \).
3. Represent an optimal controller by the equations
   \[
   z = (A + bm - lc)z + ly, \quad u = mz,
   \]
   that can be uniquely transformed to the form (3) with the transfer function
   \[
   W(s) = W_0(s) = m(Ed - A - bm + lc)^{-1}.\]

Together with this classical scheme of the LQG-problem solution, let us also consider a spectral algorithm based on the theorems from the preceding section.

Algorithm 2:
1. Execute the factorizations of two polynomials
   \[
   k^2A(s)A(-s) + B(s)B(-s) \equiv G(s)G(-s), \quad \gamma^2A(s)A(-s) + P(s)P(-s) \equiv N(s)N(-s).\]  

2. Construct the auxiliary polynomial
   \[
   R(s) = \sum_{i=1}^{n} \frac{G(-s)B(-g_i)N(g_i)}{g_i - s} A(g_i)G'(-g_i),
   \]
   where \( g_i, i = 1, n \) are the distinct roots of \( G(-s) \).
3. Represent a transfer function of the optimal controller
   \[
   W = W_0(s) = \frac{[A(s)R(s) + B(-s)N(s)]/G(-s)}{[B(s)R(s) - k^2A(-s)N(s)]/G(-s)}, \]
   where a division to the polynomial \( G(-s) \) is done totally.

4. Effectiveness of Synthesis

The matter of this section is to discuss some computational advantages of the
proposed approach. Nevertheless, let us preliminary refer to one more particular situation with respect to the problem (6) when the measurement noise is absent in the plant model (1), i.e. \( \psi(t) \equiv 0 \) (non-standard \( H_2 \)-problem [9]). From the practical point of view, this irregular situation is not senseless. However, the problem (6) is directly unsolvable by «2-Riccati» approach for this case due to the degeneration essence of the statement. Of course, there exist many ways to overcome this difficulty (one of them is realized in MATLAB), but these ways only give us the minimizing sequences of regular controllers with no accurate low bound.

As for the spectral approach, one can easy see that the mentioned situation is not irregular here; it is a usual particular case with the initial data \( N_{\psi} \equiv 0 \), \( T_{\psi} \equiv 1 \).

Really, here we have the following optimal solution:

\[
N(s) \equiv N_{\psi}(s), \quad T(s) = T_d(s), \quad P_1(s) \equiv 0; \quad k^2 A(s) A(-s) + B(s) B(-s) \equiv G(s) G(-s);
\]

\[
R(s) = \sum_{i=1}^{n} \frac{G(-s)}{g_i} B(-g_i) N(g_i) / A(g_i) T(g_i) G'(-g_i);
\]

\[
W = W_0(s) = \frac{(A(s) T(s) R(s) + B(-s) N(s)) / G(-s)}{(B(s) T(s) R(s) - k^2 A(s) N(s)) / G(-s))}.
\]

We omit the issue of the controller (32) practical applicability, because this problem is not simple in comparison with (28) for the regular case. However, this particular result can be successfully used for the lower estimation, more realistic then \( I_d \) (18), for irregular situation. Now let us return to the initial problem (6) and consider computations leading to the optimal transfer function (28).

First, we need to make factorizations (15). It is easy to see that the expressions \( N_{\psi} \bar{N}_{\psi} \bar{T}_d \bar{T}_d \bar{A} \bar{A} + N_{d} \bar{N}_{d} \bar{T}_d \bar{T}_d \bar{P} \bar{P} \) and \( k^2 \bar{A} \bar{A} + \bar{B} \bar{B} \) are polynomials with respect to \( s^2 \) of the degrees \( n_p = \max\{n + p_w + q_d, \deg \bar{P} + p_d + q_y\} \) and \( n \) respectively. Natural way to obtain theirs roots is to find the eigenvalues of \( n \times n \) and \( n_p \times n_p \) correspondent companion matrices using theirs Schur transformations. After that we find \( G \) and \( N \) polynomials.

Second, we have to construct the Lagrange polynomials \( R(s) \) (20) and \( P_1(s) \) (21). Observe that theirs computations can be execute by parallel way for every root of \( \bar{G} \) and \( N \).

Finally, it is a matter of simple calculations to find the fraction \( W_0 \), realizing division to polynomial \( \bar{G} \) in both numerator and denominator of (28).

As for solution of the same problem, using «2-Riccati» approach, first, we need to expand a state-space vector from dimension \( n \) to \( n_s = n + q_d + q_y \), including formative filters (2) to a plant model. Then we have to decide correspondent LQG problem with the help of Algorithm 1 that requires solving of two algebraic Riccati equations (ARE). In turn, here also can be used Schur transformation, but for matrices with dimensions \( n_s \times n_s \) that in general case more then twice large then for
preceding case. Besides, the reordering of the Schur form eigenvalues obstructs the above transformation.

Remark that the detailed estimation of computational complexity for the both approaches seems to be no simple problem and is not a matter of this work.

Nevertheless, we can point the following reasons to be taken into attention:
1. To solve ARE or to make factorizations (15) usually the transformation of matrices to Schur form are used. However, the dimensions of the matrices to be transformed for factorization are more then twice less then for the ARE solution.
2. Using ARE, we have to make additional computations such as eigenvalues reordering and transition from ss- to tf-form of the controllers' model.
3. If the optimal synthesis is realized in real-time regime with adaptation to initially unknown disturbance, we need no repeat only the first factorization in (15) for every step of adaptation to the spectrums $S_d$ and $S_\psi$. As for «2-Riccati» method, it is necessary to solve the both ARE or to take special actions to optimize calculations.
4. Proposed spectral method is particularly well suited to parallelization of computational process, especially for construction of the auxiliary polynomials $R(s)$ (20) and $P_1(s)$ (21), where parallel calculations seem to be natural.
5. Spectral approach allows solving degenerated problems that are directly unsolvable using ARE.

All the mentioned reasons produce an impressive argument to prefer spectral method in the sense of computational effectiveness for the solution of the posed problem (6).

5. Examples of Synthesis

Example 1: Let us use Algorithm 2 for the solution of LQG-problem with the model (1) of a control plant, having the following matrices:

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 2 & -3 & 0 \\ -1 & -2 & -4 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = (0 \ 0 \ 1);$$

accepted also $k = 1$, $\gamma = 1$.

As a result of factorizations (15), obtain the polynomials

$$G(s) = s^3 + 8.04s^2 + 18.3s + 5.39, \quad N(s) = s^3 + 8.59s^2 + 22.4s + 15.8.$$  

Then construct $R(s) = 0.146s^2 - 3.20s + 3.73$ for the roots $g_{1,2} = 3.85 \pm 0.923j, g_3 = 0.344$ of the polynomial $G(-s)$. Next, represent optimal transfer function

$$W_0(s) = \frac{0.1455s^2 + 1.135s + 2.410}{s^3 + 8.627s^2 + 22.71s + 16.07}.$$  

Naturally, the same transfer function we also obtain with the help of Algorithm 1. Nevertheless, the spectral variant 2 operates with essentially higher speed.
Experiments show that the time of computations can be decreased by more than a factor ten, using spectral approach, in comparison with standard MATLAB procedures of LQG-synthesis.

Example 2: Let synthesize the optimal transfer function (32) for the irregular case, using the same matrices (33) of the plant model, and accepting \( k = 0.1 \),

\[
S_{d1}(s) = \frac{s + 4}{s^2 + 2s + 3}, \quad \text{i.e. } N_g(s) = s + 4, \quad T_g(s) = s^2 + 2s + 3.
\]

Here we obtain \( G = 0.1s^3 + 0.962s^2 + 3.23s + 2.06, \quad R = 0.0349s^2 - 0.267s + 0.430. \)

As a result, an optimal transfer function (32) assumes the form

\[
W_0(s) = \frac{0.349s^4 + 4.18s^3 + 19.5s^2 + 46.7s + 55.1}{0.1s^3 + 0.264s^2 + 1.06s + 2.71}, \quad (34)
\]

providing the minimum value \( J_{20} = J_2(W_0) = 0.328 \) of the functional (6). Remark, that obtained controller cannot be directly realized due to degeneration of the problem. Yet, this result has a practical application as a limit point for regular synthesis that we illustrate by the next example.

Example 3: Let us consider the problem (6) for the same initial data as for the preceding case, but additionally introduce a measurement white noise with the intensity \( \gamma^2 : N_w(s) = \gamma, \; T_w(s) = 1. \) For the value \( \gamma = \gamma_1 = 1 \) we consequently obtain

\[
G(s) = 0.1s^3 + 0.962s^2 + 3.23s + 2.06,
\]

\[
N(s) = s^5 + 10.7s^4 + 44.7s^3 + 94.9s^2 + 117s + 61.9,
\]

\[
R(s) = 0.075s^2 - 2.72s + 2.84, \quad P_1(s) = 0.
\]

that allows us to form the optimal transfer function (28)

\[
W_{01}(s) = \frac{0.755s^4 + 7.62s^3 + 28.2s^2 + 47.6s + 39.4}{0.1s^5 + 1.23s^4 + 6.35s^3 + 14.2s^2 + 18.6s + 9.76}.
\]

Correspondently we have \( J_{21} = J_2(W_{01}) = 0.884 > J_{20} \) for the closed-loop system with no noise. Consequently executing calculations in the same manner for the multipliers \( \gamma = \gamma_2 = 0.5, \; \gamma = \gamma_3 = 0.1, \; \gamma = \gamma_4 = 0.01 \) we obtain the following values of the functional: \( J_{22} = 0.637, \quad J_{23} = 0.408, \quad J_{24} = 0.344. \) Later this sequence is obviously converges to the limit \( J_{20} \). We can accept its any member as a regular approximate solution for the irregular problem. In particular, for the value \( \gamma = \gamma_4 = 0.01 \) we have \( J_{24} = 0.344 \) that is only slightly more then the accurate low bound \( J_{20} = 0.328. \) The correspondent transfer function is

\[
W_{04} = \frac{0.448s^5 + 5.13s^3 + 22.9s^2 + 50.9s + 54.2}{0.001s^5 + 0.0246s^4 + 0.289s^3 + 0.836s^2 + 2.06s + 3.06}.
\]
6. Conclusion

The main goal of the paper is to propose and to discuss a special spectral approach in frequency domain to partial cases of $H_2$-optimization problems for LTI controlled plants. Despite a particularity, the mentioned problem is of considerable importance for a variety of practical applications such as marine and flight autopilots [10], tokamak plasma control [9], robots control and many others.

Proposed approach is based on a polynomial representation of initial and temporary data and on the special parameterization method for stabilizing controllers' set. Algorithm of optimization does not require using any initial stabilizing controller. Instead of the Riccati equation (or linear matrix inequalities) solutions, polynomial factorizations are used that can substantially simplify an optimal synthesis.

Remark that reliable and efficient methods for the numerical solution of matrix algebraic Riccati equations (ARE) now are exclusively well developed [3]. This technique is commonly used as the universal computational tool in control theory and various applications. Importance of ARE is difficult to overprize, especially in MIMO large order $H$-optimization problems. Nevertheless, theirs using for the low order ($n \leq 5$) problem of the mentioned type seems to be too much powerful and unjustified in the sense of computational effectiveness.

From this point of view, proposed approach can be useful for practical situations such that the computational consumptions play crucial role. In the first place, this relates to adoptive turning systems, particularly for embedded systems or for onboard control systems of autonomous moving robots with essentially limited computational resources. Naturally, a special analysis must be preliminarily done to avoid various numerical pitfalls connected with polynomial calculations.

It seems suitable to note that spectral approach allows presenting a solution of $H$-problem in special polynomial form of optimal controllers' transfer function. This representation is convenient for various investigations of the optimal solution such as a structure of the transfer function (28), its limit behavior with respect to $k \to 0$ and $k \to \infty$, robust features of the controller, situations with non-fractional representation of the spectrum for external disturbances, transport delays, etc.

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References


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