Restrained Convex Dominating Sets in the Corona and the Products of Graphs

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Abstract

In this paper, we characterize the restrained convex dominating sets in the corona, lexicographic and Cartesian products of two connected graphs and then determine the corresponding restrained convex domination numbers of these graphs.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $S \subseteq V(G)$, then the open neighborhood of $S$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. 
The closed neighborhood of $S$ is $N_G[S] = N[S] = S \cup N(S)$. A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in (V(G) \setminus S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The domination number $\gamma(G)$ of $G$ is the smallest cardinality of a dominating set of $G$. A vertex $u \in V(G) \setminus S$, where $S$ is a dominating set of $G$, is an external private neighbor of $v \in S$ if $N_G(u) \cap S = \{v\}$. The set of external private neighbors of $v \in S$ is denoted by $epn(v; S)$.

Let $x, y \in V(G)$. Any $x$-$y$ path of length equal to $d_G(x, y)$ (the distance between the vertices $x$ and $y$ in $G$) is called an $x$-$y$ geodesic. The interval $I[x, y] = I_G[x, y]$ consists of $x$, $y$ and all vertices lying on any $x$-$y$ geodesic. If $S \subseteq V(G)$, then the geodetic closure of $S$ is the set $I[S] = I_G[S] = \bigcup_{x,y \in S} I[x, y]$.

$S$ is convex if $I[x, y] \subseteq S$ for any $x, y \in S$, i.e., $I_G[S] = S$. A dominating set $S$ which is also convex is called a convex dominating set of $G$. The convex domination number $\gamma_{con}(G)$ of $G$ is the smallest cardinality of a convex dominating set of $G$.

A subset $S$ of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $(V(G) \setminus S)$ is a subgraph without isolated vertices.

A convex dominating set $S$ of $V(G)$ is a restrained convex dominating set of $G$ if for each $u \in V(G) \setminus S$, there exists $z \in V(G) \setminus S$ such that $uz \in E(G)$. The minimum cardinality of a restrained convex dominating set of $G$, denoted by $\gamma_{rcon}(G)$, is called the restrained convex domination number of $G$.

Convexity in graphs has been discussed and studied in [1,2,3,5]. On the other hand, domination and convex domination in graphs has been studied in [4,6,7,8].

## 2 Results

Let $G$ and $H$ be graphs of order $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. The join of vertex $v$ of $G$ and a copy $H^v$ of $H$ in the corona of $G$ and $H$ is denoted by $v + H^v$.

**Theorem 2.1** Let $G$ be a connected graph. Then $V(G)$ is a $\gamma_{rcon}$-set in $G \circ H$ if and only if $H$ is a graph without isolated vertices.

**Proof:** Suppose $V(G)$ is a $\gamma_{rcon}$-set in $G \circ H$. Then $\bigcup_{v \in V(G)} H^v = (V(G \circ H) \setminus V(G))$ has no isolated vertices. If follows that $H^v$ has no isolated vertices for each $v \in V(G)$. Thus, $H$ has no isolated vertices.
For the converse, suppose that \( H \) is a graph without isolated vertices. Then each \( H' \) has no isolated vertices. Therefore \( \bigcup_{v \in V(G)} H' = (V(G \circ H) \setminus V(G)) \) has no isolated vertices. Thus, \( V(G) \) is a restrained convex dominating set of \( G \circ H \) and \( \gamma_{rcon}(G \circ H) \leq |V(G)| \). Hence, let \( C \) be a \( \gamma_{rcon} \)-set of \( G \circ H \). Then \( |C \cap V(v + H')| \geq 1 \) for each \( v \in V(G) \). Therefore \( \gamma_{rcon}(G \circ H) = |C| = \sum_{v \in V(G)} |C \cap V(v + H')| \geq |V(G)| \). Therefore \( \gamma_{rcon}(G \circ H) = |V(G)| \), that is, \( V(G) \) is a \( \gamma_{rcon} \)-set in \( G \circ H \). □

The next result follows from Theorem 2.1.

**Corollary 2.2** Let \( G \) be a connected graph and \( H \) any graph without isolated vertices. Then \( \gamma_{rcon}(G \circ H) = |V(G)| \).

The lexicographic product \( G[H] \) of two graphs \( G \) and \( H \) is the graph with \( V(G[H]) = V(G) \times V(H) \) and \( (u, u')(v, v') \in E(G[H]) \) if and only if either \( uv \in E(G) \) or \( u = v \) and \( u'v' \in E(H) \).

The next result is obtained in [4] and characterizes the dominating sets in the lexicographic product \( G[K_n] \).

**Theorem 2.3** [4] Let \( G \) be a connected graph and \( n \geq 2 \). Then a subset \( C = \bigcup_{x \in S} \{x\} \times T_x \) of \( V(G[K_n]) \) is a convex dominating set in \( G[K_n] \) if and only if \( S \) is a convex dominating set in \( G \) and \( T_x = V(K_n) \) for each \( x \in S \cap I(S) \).

Note that a corollary of Theorem 2.3 given in [4] and its proof contain some errors. We now make a rectification of such a result.

**Corollary 2.4** Let \( G \) be a connected graph and \( n \geq 2 \). Then \( \gamma_{con}(G[K_n]) = \min\{|S| + (n - 1)|S^o| : S \text{ is a convex dominating set in } G\} \), where \( S^o = S \cap I(S) \).

**Proof:** Let \( k = \min\{|S| + (n - 1)|S^o| : S \text{ is a convex dominating set in } G\} \). Let \( C = \bigcup_{x \in S} \{x\} \times T_x \) be a \( \gamma_{con} \)-set of \( G[K_n] \). Then \( S \) is a convex dominating set in \( G \) and \( T_x = V(K_n) \) \( \forall x \in S^o \) by Theorem 2.3. It follows that \( \gamma_{con}(G[K_n]) = |C| = \sum_{x \in (S \setminus S^o)} |T_x| + n|S^o| \geq k \). Now, let \( S_1 \) be a convex dominating set in \( G \) such that \( k = |S_1| + (n - 1)|S_1^o| \). Choose any \( a \in V(K_n) \) and set \( T_x = \{a\} \) if \( x \in S_1 \setminus S_1^o \) and \( T_x = V(K_n) \) if \( x \in S_1^o \). Then \( C_1 = \bigcup_{x \in S_1} \{x\} \times T_x \) is a convex dominating set in \( G[K_n] \) by Theorem 2.3. Hence, \( \gamma_{con}(G[K_n]) \leq |C_1| = |S_1| + m|S_1^o| = k \). Therefore, \( \gamma_{con}(G[K_n]) = k \). □

**Theorem 2.5** Let \( G \) be a connected graph and \( n \geq 2 \). A subset \( C = \bigcup_{x \in S} \{x\} \times T_x \) of \( V(G[K_n]) \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(K_n) \) for each \( x \in S \), is a restrained convex dominating set in \( G[K_n] \) if and only if
(i) $S$ is a convex dominating set in $G$;

(ii) $T_x = V(K_n)$ for each $x \in S \cap I(S)$; and

(iii) $|V(K_n) \setminus T_y| \geq 2$ for each $y \in S_1 \setminus [N_G(V(G) \setminus S) \cup N_G(S_1)]$, where $S_1 = \{z \in S : T_z \neq V(K_n)\}$.

Proof: Suppose that $C$ is a restrained convex dominating set in $G[K_n]$. Then, by Theorem 2.3, (i) and (ii) hold. Let $y \in S \setminus [N_G(V(G) \setminus S) \cup I(S) \cup N_G(S_1)]$. Since $y \notin S_1$, $T_y \neq V(K_n)$. Let $a \in V(K_n) \setminus T_y$. Since $C$ is a restrained dominating set and $(y, a) \notin C$, there exists $(w, b) \in [V(G[K_n]) \setminus C] \cap N_G(K_n)((y, a))$. Since $y \notin N_G(V(G) \setminus S) \cup N_G(S_1)$, it follows that $y = w$. Thus, $b \in (V(K_n) \setminus T_y)$. Therefore $|V(K_n) \setminus T_y| \geq 2$.

For the converse, suppose that properties (i), (ii), and (iii) hold. Then $C$ is a convex dominating set in $G[K_n]$ by Theorem 2.3. Let $(z, p) \in V(G[K_n]) \setminus C$. Consider the following cases:

Case 1. $z \notin S$ Since $n \geq 2$, there exists $q \in V(K_n) \setminus \{p\}$. This implies that $(z, q) \notin C$ and $(z, p)(z, q) \in E(G[K_n])$.

Case 2. $z \in S$

If $z \in N_G(V(G) \setminus S) \cup N_G(S_1)$, then $\exists v \in [(V(G) \setminus S) \cup S_1] \cap N_G(z)$. Choose any $t \in V(K_n)$ such that $t \notin T_v$. Then $(v, t) \in (V(G[K_n]) \setminus C) \cap N_G(K_n)((z, p))$. If $z \notin N_G(V(G) \setminus S) \cup N_G(S_1)$, then $|V(G[K_n]) \setminus T_z| \geq 2$ by condition (iii). Hence, $\exists (z, p^*) \in (V(G[K_n]) \setminus C) \cap N_G(K_n)((z, p))$.

Accordingly, $C$ is a restrained convex dominating set in $G[K_n]$. □

Corollary 2.6 Let $G$ be a connected graph and $n \geq 2$. Then

$$\gamma_{rcon}(G[K_n]) \geq \min\{|S| + (n - 1)|S^0| : S \text{ is a convex dominating set in } G\}.$$  

Proof: This follows from the fact that $\gamma_{rcon}(G[K_n]) \leq \gamma_{rcon}(G[K_n])$. □

The lower bound in Corollary 2.6 is tight as the following result shows.

Corollary 2.7 $\gamma_{rcon}(P_m[K_n]) = mn - 4n + 2 \forall m \geq 3$ and $\forall n \geq 2$.

Labendaia and Canoy in [7] characterized the convex dominating sets in the lexicographic products $G[H]$ of connected non-complete graphs $G$ and $H$.

Theorem 2.8 [7] Let $G$ and $H$ be non-complete connected graphs with $\gamma_d(G) \geq 2$. A subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$ of $V(G[H])$ is a convex dominating set in $G[H]$ if and only if $S$ is a clique dominating set in $G$ and $T_x$ is a clique in $H$ for each $x \in S$. 
Theorem 2.9 Let $G$ and $H$ be non-complete connected graphs with $\gamma_{cl}(G) \geq 2$. A subset $R = \bigcup_{x \in S} \{x\} \times T_x$ of $V(G[H])$ is a restrained convex dominating set in $G[H]$ if and only if $S$ is a clique dominating set in $G$ and $T_x$ is a clique in $H$ for each $x \in S$.

Proof: Suppose $R = \bigcup_{x \in S} \{x\} \times T_x$ is a restrained convex dominating set of $G[H]$. By Theorem 2.8, $S$ is a clique dominating set in $G$ and $T_x$ is a clique in $H$ for each $x \in S$.

For the converse, suppose that $S$ is a clique dominating set in $G$ and $T_x$ is a clique in $H$ for each $x \in S$. Then by Theorem 2.8, $R = \bigcup_{x \in S} \{x\} \times T_x \subseteq V(G[H])$ is a convex dominating set of $G[H]$. Let $(x, a) \in V(G[H]) \setminus R$. Consider the following cases:

Case 1. Suppose that $x \in S$. Since $S$ is a clique and $|S| \geq 2$, there exists $y \in S$ such that $xy \in E(G)$. Since $T_y$ is a clique and $H$ is non-complete, $V(H) \setminus T_y \neq \emptyset$. Let $b \in V(H) \setminus T_y$. Then $(y, b) \notin R$ and $(x, a)(y, b) \in E(G[H])$.

Case 2. Suppose that $x \notin S$. Since $H$ is a connected non-trivial graph, there exists $c \in V(H) \cap N_H(a)$. Hence, $(x, c) \notin R$ and $(x, a)(x, c) \in E(G[H])$. Therefore, $R$ is a restrained convex dominating set in $G[H]$.

Since $G$ is a non-complete connected graph, there exists $x \in V(G) \setminus S$ such that $(x, x')(x, y') \in E(G[H])$ for all $x'y' \in E(H)$ with $(x', x'), (x, y') \in V(G[H]) \setminus R$. Similarly, since $H$ is a non-complete connected graph, there exists $x' \in V(H) \setminus T_x$ such that $(x, x')(y, x') \in E(G[H])$ for all $xy \in E(G)$ with $(x, x'), (y, x') \in V(G[H]) \setminus R$. Thus, $\langle V(G[H]) \setminus R \rangle$ is a subgraph without isolated vertex. Hence, $R$ is a restrained convex dominating set of $G[H]$. □

Corollary 2.10 Let $G$ and $H$ be non-complete connected graphs with $\gamma_{cl}(G) \geq 2$. Then $\gamma_{rcon}(G[H]) = \gamma_{cl}(G)$.

The Cartesian product of two graphs $G$ and $H$ is the graph $G \Box H$ with vertex-set $V(G \Box H) = V(G) \times V(H)$ and edge-set $E(G \Box H)$ satisfying the following conditions: $(x, a)(y, b) \in E(G \Box H)$ if and only if either $xy \in E(G)$ and $a = b$ or $x = y$ and $ab \in E(H)$.

Labendia and Canoy in [7] also characterized the convex dominating sets in the Cartesian product $G \Box H$ of connected non-complete graphs $G$ and $H$, and then determined its convex domination number.

Theorem 2.11 [7] Let $G$ and $H$ be connected graphs. A subset $C$ of $V(G \times H)$ is a convex dominating set in $G \times H$ if and only if $C = C_1 \times C_2$ and

(i) $C_1$ is a convex dominating set in $G$ and $C_2 = V(H)$, or

(ii) $C_2$ is a convex dominating set in $H$ and $C_1 = V(G)$.
Corollary 2.12 [7] Let $G$ and $H$ be connected graphs of orders $m$ and $n$ respectively. Then $\gamma_{\text{con}}(G \square H) = \min\{m\gamma_{\text{con}}(H), n\gamma_{\text{con}}(G)\}$.

Theorem 2.13 Let $G$ and $H$ be connected non-trivial graphs. A subset $C$ of $V(G \square H)$ is a restrained convex dominating set in $G \square H$ if and only if

(i) $C_1$ is a convex dominating set in $G$ and $C_2 = V(H)$ or

(ii) $C_1 = V(G)$ and $C_2$ is a convex dominating set in $H$.

Proof: Suppose that $C$ is a restrained convex dominating set in $G \square H$. By Theorem 2.11, $C = C_1 \times C_2$ and either $C_1$ is a convex dominating set in $G$ and $C_2 = V(H)$ or $C_1 = V(G)$ and $C_2$ is a convex dominating set in $H$.

For the converse, suppose $C = C_1 \times C_2$ and (i) holds. Then $C$ is a convex dominating set in $G \square H$ by Theorem 2.11. Next, let $(x, a) \notin C$. Then $x \notin C_1$. Since $H$ is a connected non-trivial graph, there exists $b \in V(H)$ such that $ab \in E(H)$. Hence, $(x, b) \notin C$ and $(x, a)(x, b) \in E(G \square H)$. Therefore $C$ is a restrained convex dominating set in $G \square H$. The same conclusion holds if (ii) holds. \[ \square \]

The next result is a consequence of Theorem 2.13 and Corollary 2.12.

Corollary 2.14 Let $G$ and $H$ be connected non-trivial graphs of orders $m$ and $n$ respectively. Then $\gamma_{\text{rcon}}(G \square H) = \gamma_{\text{con}}(G \square H) = \min\{m\gamma_{\text{con}}(H), n\gamma_{\text{con}}(G)\}$.

References


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