Determination of Stress-Strain State of Geometrically Nonlinear Sandwich Plate

I. B. Badriev
Kazan Federal University
18 Kremlyovskaya Street
420008, Kazan, Russian Federation

V. V. Banderov
Kazan Federal University
18 Kremlyovskaya Street
420008, Kazan, Russian Federation

M. V. Makarov
Kazan Federal University
18 Kremlyovskaya Street
420008, Kazan, Russian Federation
Kazan National Research Technical University
10 K.Marks Street
420111, Kazan, Russian Federation

V. N. Paimushin
Kazan Federal University
18 Kremlyovskaya Street
420008, Kazan, Russian Federation
Kazan National Research Technical University
10 K.Marks Street
420111, Kazan, Russian Federation

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Abstract

By using the two-layer iterative method we obtain the basic characteristics of the equilibrium position of sandwich plate with a transversely soft filler in geometrical nonlinear one-dimensional statement. To solving problem we previously construct its finite-difference approximation. Analysis of the results of numerical experiments is performed. Obtained data testify to the effectiveness of the proposed method.

Keywords: sandwich plate, geometric nonlinearity, transversely soft filler, iterative method, numerical experiment

Introduction

In this paper we consider the problem of determining the stress-strain states of sandwich plate with transversal-soft filler. The kinematic relations for the filler are obtained by successive integration over the transverse coordinates of the initial three-dimensional equations of elasticity theory, preliminary simplified by the introduction by introducing the assumption of vanishing of the tangential components of the stress [1-3]. The numerical solution is carried out by using a two-layer iterative method with preconditioner being a linear part of the operator of constructed difference scheme. The results of numerical experiments are given. Analysis of the numerical results is carried out. Note that in [4-15], the generalized formulations of problems of the theory of soft shells mesh with constraints, as well as methods for their numerical solution.

1 Problem statement

The problem of equilibrium sandwich plate with transversal-soft filler in one-dimensional geometrically nonlinear formulation is described by the following system of differential equations:

\[
\begin{align*}
\frac{dT_{(k)}^{11}}{dx} + X_{(k)}^{1} + (3 - 2k) q_{1}^{1} = 0, & \quad 0 < x < a, \quad k = 1, 2, \\
\frac{dS_{(k)}^{1}}{dx} + c_{3} (3 - 2k) (w^{(2)} - w^{(1)}) + X_{(k)}^{3} = 0, & \quad 0 < x < a, \quad k = 1, 2, \\
u^{(1)} - u^{(2)} - H_{(1)} \frac{dw^{(1)}}{dx} - H_{(2)} \frac{dw^{(2)}}{dx} + \frac{q_{1}^{1}}{G_{13}} \frac{2h}{3E_{3}} \frac{d^{2}q_{1}^{1}}{dx^{2}} = 0, & \quad 0 < x < a,
\end{align*}
\]

where (below we put \(k = 1, 2\)) \(H_{(k)} = h_{(k)} + h\), \(a\) – the length of the plate, \(2h\) – thickness of filler, \(2h_{(k)}\) – thickness of the \(k\)-th layer, \(T_{(k)}^{11}\) – membrane
forces in \( k \)-th layer, \( X_{1}^{(k)}, X_{3}^{(k)} \) – component of the surface load normalized to the median surface of \( k \)-th layer, \( w^{(k)} \) and \( u^{(k)} \) – bending and axial displacements of the points of the median surface of \( k \)-th layer, \( S_{1}^{(k)} \) – generalized shear forces in \( k \)-th layer, \( G_{13}, E_{3} \) – modules of transverse shear and compression of filler, \( q^{1} \) – tangential stresses in the filler,

\[
S_{1}^{(k)} = \frac{dM_{11}^{(k)}}{dx} + T_{11}^{(k)} \omega^{(k)} + M_{1}^{(k)} + H_{(k)} q^{1}, \quad T_{11}^{(k)} = B_{(k)} \left( \frac{du^{(k)}}{dx} + \frac{1}{2} (\omega^{(k)})^2 \right), \quad M_{11}^{(k)} = -D_{(k)} \frac{d^{2}w^{(k)}}{dx^2}, \quad B_{(k)} = 2h_{(k)} E_{(k)} (1 - \nu_{12}^{(k)} \nu_{21}^{(k)}) \quad \text{stiffness of} \; k \text{-th layer}
\]

on the tension-compression, \( E_{(k)} \) – modulus of elasticity of the first kind of \( k \)-th layer, \( D_{(k)} = B_{(k)} h_{(k)}^2 / 3 \) – flexural stiffness of \( k \)-th layer, \( c_{3} = E_{3} / (2h) \), \( \omega^{ sol } = dw^{(k)} / dx \) – rotation angles of the normal to the median surface of \( k \)-th layer, \( \Omega_{1}^{(k)} \) – the components of the surface moments of \( k \)-th layer, \( \nu_{12}^{(k)} \), \( \nu_{21}^{(k)} \) – Poisson’s coefficients of the material of \( k \)-th layer. We assume that the edges of the plate carrier layers are rigidly fixed, so the equations (1) are supplemented by the boundary conditions (at \( x = 0, \; x = a \))

\[
u^{(k)} = 0, \; w^{(k)} = 0, \; \omega^{(k)} = 0, \; q^{1} = 0, \quad k = 1, 2. \tag{2}
\]

Let us formulate a generalized statement of the problem. We introduce the following notation for Sobolev spaces (see., e.g. [16]) \( V_{m} = W_{m}^{(m)}(0,a), \; W_{m} = V_{m} \times V_{m}, \; m = 1, 2 \). By a generalized solution considered problem (1), (2) we mean a functions \( u = (u^{(1)}, u^{(2)}) \in W_{1}, \; w = (w^{(1)}, w^{(2)}) \in W_{2}, \; q^{1} \in V_{1} \), satisfying for all \( \eta = (\eta^{(1)}, \eta^{(2)}) \in W_{1}, \; z = (z^{(1)}, z^{(2)}) \in W_{2}, \; y \in V_{1} \) the integral identities

\[
\begin{align*}
&\int_{a}^{0} \left[ -D_{(k)} \frac{d^{2}w^{(k)}}{dx^2} \frac{d^{2}z^{(k)}}{dx^2} - B_{(k)} \left( \frac{du^{(k)}}{dx} + \frac{1}{2} (\omega^{(k)})^2 \right) \frac{dw^{(k)}}{dx} \frac{dz^{(k)}}{dx} + 
\right. \\
&\left. - H_{(k)} q^{1} \frac{dz^{(k)}}{dx} + ((3 - 2k)c_{3}(w^{(2)} - w^{(1)}) + \frac{dM_{1}^{(k)}}{dx} + X_{k}^{(3)} z^{(k)}) \right] dx = 0, \\
&\int_{a}^{0} \left[ -B_{(k)} \left( \frac{du^{(k)}}{dx} + \frac{1}{2} (\omega^{(k)})^2 \right) \frac{d\eta^{(k)}}{dx} + (3 - 2k)q^{1} + X_{k}^{(1)} \eta^{(k)} \right] dx = 0, \\
&\int_{0}^{a} \left( w^{(2)} - w^{(1)} \right) \frac{dw^{(1)}}{dx} - H_{(1)} \frac{dw^{(2)}}{dx} + \frac{2h}{G_{13}} q^{1} \bigg( y + \frac{2h^{3}}{3E_{3}} \frac{dq^{1}}{dy} \bigg) dx = 0.
\end{align*}
\]
2 Finite difference scheme

To approximate solving problem (1), (2) we previously construct its finite-difference approximation. To this aim on the interval $[0,a]$ we introduce with the step $h_c = a/N$ the uniform grids $\omega = \{x_i, i = 1,2,\ldots,N-1\}$, $\overline{\omega} = \{x_i, i = 0,1,2,\ldots,N\}$. Let us denote as usual $v_{x,i} = (v_{i+1} - v_i)/h_c$; $v_{x,j} = (v_j - v_{j-1})/h_c$; $v_{o,i} = (v_{i+1} - v_{i-1})/(2h_c)$. Then the difference scheme is written as (for the difference functions we leave the same notation as in the differential case, wherein the functions $w^{(k)}(x)$ are defined at $\overline{\omega}$, functions $u^{(k)}(x)$, $q_1^{(k)}(x) - \omega$)

\[
-D_{(k)}w_{\omega x}(x) + B_{(k)}((u_{\omega x}^{(k)})^2) w_{\omega x}(x) + (3 - 2k)c_3(w^{(2)}_i - w^{(1)}_i) + H_{(k)}q_1^{(k)} + X_{(k)}^{(i)} = 0, \quad i = 1,2,\ldots,N-1, \quad k = 1,2,
\]

\[
B_{(k)}((u_{\omega x}^{(k)})^2 + 1/2(w_{\omega x}^{(k)})^2 + 2x_{(k)}^{(i)} + 3 - 2k)q_1^{(k)} + X_{(k)}^{(i)} = 0, \quad i = 1,2,\ldots,N-1, \quad k = 1,2,
\]

\[
(w^{(2)}_i - w^{(1)}_i) - H_{(k)}w^{(1)}_{\omega x,i} - H_{(2)}w^{(2)}_{\omega x,i} + \frac{2h}{G_{13}}q_i - \frac{2h^3}{3E_3}q_1^{(k)} = 0, \quad i = 1,2,\ldots,N-1.
\]

The boundary conditions (2) are approximated as follows $u^{(k)}_0 = 0$, $w^{(k)}_0 = 0$, $q_0^{(k)} = 0$, $u^{(k)}_N = 0$, $w^{(k)}_N = 0$, $q_1^{(k)} = 0$, $w^{(k)}_{x,n} = 0$. We denote $U = (w^{(1)}, w^{(2)}, u^{(1)}, u^{(2)}, q_1)$. Let $V_{2h}$ be a set of difference functions $z$, defined on $\omega$, such that $z_0 = 0$, $z_{\omega,0} = 0$, $z_\omega = 0$, $z_{\omega,n} = 0$, $V_{2h}$ be a set of difference functions $\eta$, defined on $\omega$, such that $\eta_{\omega} = 0$, $\eta_{\omega,n} = 0$. We denote $V_h = V_{1h} \times V_{1h} \times V_{2h} \times V_{2h} \times V_{1h}$. We also introduce into consideration the difference operators $A_{ju}^{(k)}: V_h \rightarrow V_{1h}$, $A_{ju}^{(k)}: V_h \rightarrow V_{2h}$, $k = 1,2$, $A_{ju}^{(k)}: V_h \rightarrow V_{1h}$, $j = 1,2$ according to the formulas

\[
A_{ju}^{(k)}U(x) = B_{(k)}u^{(k)}_{x,x} + (3 - 2k)q_1^{(k)}, \quad k = 1,2, \quad x \in \overline{\omega},
\]

\[
A_{ju}^{(k)}U(x) = -D_{(k)}w_{\omega x}^{(k)} + H_{(k)}q_1^{(k)} + (3 - 2k)c_3(w^{(2)}_i - w^{(1)}_i), \quad k = 1,2, \quad x \in \overline{\omega},
\]

\[
A_{ju}^{(k)}U(x) = -((u^{(2)}_i - u^{(1)}_i) - H_{(1)}w^{(1)}_{x,i} - H_{(2)}w^{(2)}_{x,i} + \frac{2h}{G_{13}}q_i - \frac{2h^3}{3E_3}q_1^{(k)}), \quad x \in \overline{\omega},
\]

\[
A_{ju}^{(k)}U(x) = -\frac{1}{2}B_{(k)}((w_{\omega x}^{(k)})^2), \quad k = 1,2, \quad x \in \overline{\omega}, \quad A_{ju}^{(k)}U(x) = 0, \quad x \in \omega,
\]

\[
A_{ju}^{(k)}U(x) = -B_{(k)}((u_{\omega x}^{(k)})^2), \quad k = 1,2, \quad x \in \overline{\omega}.
\]
and function \( F = (f_w^{(1)}, f_w^{(2)}, f_u^{(1)}, f_u^{(2)}, f_q) \in V_h \), where \( f_{av}^{(k)} = -X_k^{(1)} \), \( f_w^{(k)} = M_k^{(1)} + X_k^{(3)} \), \( k = 1, 2 \), \( f_q = 0 \) on \( \partial \omega \). Note that \( A_{lw}^{(k)} \), \( A_{iu}^{(k)} \), \( A_{iq} \) are the linear operators and \( A_{2w}^{(k)} \), \( A_{2u}^{(k)} \) are the nonlinear one. Then the difference scheme can be written as

\[
(A_1 + A_2) U = F, \tag{3}
\]

where \( A_j = (A_{ju}^{(1)}, A_{ju}^{(2)}, A_{ju}^{(3)}, A_{ju}^{(4)}, A_{ju}^{(5)}) \), \( j = 1, 2 \).

3 Iterative method and numerical experiments

To solve the difference scheme (3), we use the following two-layer iterative process with the lowering of the nonlinearity on the lower layer [17-23]:

\[
A_1 (U^{(n+1)} - U^{(n)}) / \tau + (A_1 + A_2) U^{(n)} = F, \tag{4}
\]

where \( U^{(0)} \) be given an initial approximation, \( \tau > 0 \) be iteration parameter. In the software environment Matlab was developed software package for numerical implementation of the iterative method (4). For the model problem, numerical experiments were carried out. Iteration parameter was chosen empirically. The calculations were performed for the following characteristics: \( a = 1 \), \( h_1 = h_2 = 0.005 \), \( h = 0.05 \), \( G_{13} = 15 \text{ MPa} \), \( E_3 = 25 \text{ MPa} \), \( X_{(1)}^3 = 0.0319 \text{ MPa} \), \( X_{(2)}^3 = 0 \), \( E^{(k)} = 7 \cdot 10^4 \text{ MPa} \), \( \nu_{12}^{(k)} = \nu_{21}^{(k)} = 0.3 \), \( X_{(k)}^1 = 0 \), \( M_{(k)}^1 = 0 \), \( k = 1, 2 \).

The number of grid points be \( N = 100 \). The initial approximation \( U^{(0)} \) be zero. Calculations according to (4) were carried out as long as the residual norm \( \| F - (A_1 + A_2) U^{(n)} \| \) remained greater than given accuracy \( \varepsilon = 5 \cdot 10^{-6} \). As the norm of the vector \( g = (g_1, g_2, \ldots, g_m) \) chosen \( \| g \| = \max \{|g_1|, |g_2|, \ldots, |g_m|\} \), optimal (by the number of iterations) value of iteration parameter was \( \tau = 1 \), the number of iterations in this case was equal to 17. The results of numerical experiments are shown in Fig. 1–3.

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**Fig. 1.** Axial movement of the carrier layers  
**Fig. 2.** Shear stresses in the filler
It should be noted that formulated for $q^1$ boundary conditions (2) correspond to the presence of, at the edges $x = 0$, $x = a$ the diaphragms, this leads to the formation of the maximum transverse tangential stresses in sections the filler a distance of about its thickness $2h$, as observed in Fig 2. The limiting the free end sections $x = 0$, $x = a$ displacements in axial direction leads to the formation to the formation of a significant in magnitude membrane forces $T_{(1)}^{11}$, $T_{(2)}^{11}$ in the carrying layers $T_{(1)}^{11}$, $T_{(2)}^{11}$, the force $T_{(3)}^{11}$ of which is a contraction in cross-section $x = a/2$. For this reason, in the neighborhood of this section should be expected buckling carrier layers in a mixed form. It is easy to verify that the first two equations of (1) implies that $T_{(1)}^{11} + T_{(2)}^{11} = \text{const}$, in the implementation of which can be verified on the basis of the results shown in Fig. 3.

![Fig. 3. Membrane forces in the bearing layers](image)

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Determination of stress-strain state


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