Generalized Star $\alpha b$-Separation Axioms in Bigeneralized Topological Spaces

Lady Lee Lim Lusanta and Helen Moso Rara

Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
Tibanga, Iligan City, Philippines

Abstract

This study introduces the notion of generalized star $\alpha b$-open (briefly $g^*\alpha b$-open) sets using the concepts of $g\alpha b$-open sets due to Vinayagamoorthi [12]. With the idea of $g^*\alpha b$-open sets, we define and study new separation axioms, namely, $g^*\alpha b\mu_{(m,n)}$-$T_1^2$, $g^*\alpha b\mu_{(m,n)}$-$T_1$, $g^*\alpha b\mu_{(m,n)}$-$T_2$, $g^*\alpha b\mu_{(m,n)}$-regular, $g^*\alpha b\mu_{(m,n)}$-normal, and $g^*\alpha b\mu_{(m,n)}$-$T_4$ spaces in bigeneralized topological spaces.

Keywords: bigeneralized topological spaces, $g^*\alpha b\mu_{(m,n)}$-$T_1^2$ spaces, $g^*\alpha b\mu_{(m,n)}$-$T_1$ spaces, $g^*\alpha b\mu_{(m,n)}$-$T_2$ spaces, $g^*\alpha b\mu_{(m,n)}$-regular spaces, $g^*\alpha b\mu_{(m,n)}$-normal spaces, $g^*\alpha b\mu_{(m,n)}$-$T_4$ spaces

1 Introduction


\textsuperscript{1}Research is funded by the Department of Science and Technology-Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRD).
introduced the concepts of bigeneralized topological spaces and studied \((m,n)\)-closed sets and \((m,n)\)-open sets in bigeneralized topological spaces. In 2012, Torton et al. [11] studied some separation axioms in bigeneralized topological spaces.

In this paper, we introduce the notions of some \(g^{*}ab\)-separation axioms in bigeneralized topological spaces using the \(g^{*}ab\)-open sets and investigate some of their properties.

2 Preliminaries

Let us recall some terms used in this study. A topology in a set \(X\) is a family \(\tau\) of subsets of \(X\) that satisfies: (i) \(\varnothing \in \tau\), (ii) \(X \in \tau\), (iii) each union of members of \(\tau\) is also a member of \(\tau\), and (iv) each finite intersection of members of \(\tau\) is also a member of \(\tau\). The couple \((X,\tau)\) is called a topological space [6]. If \(A \subseteq X\), then the interior of \(A\), denoted by \(\text{int}(A)\), is the union of all open sets contained in \(A\) and the closure of \(A\), denoted by \(\text{cl}(A)\), is the intersection of all closed sets containing \(A\).

If the properties (ii) and (iv) are omitted, then \(\tau\) is called a generalized topology and is usually denoted by \(\mu\). If \(\mu\) is a generalized topology on \(X\), then \((X,\mu)\) is called a generalized topological space (or briefly GTS) and if \(X \in \mu\), then \(X\) is a strong GTS. The elements of \(\mu\) are called \(\mu\)-open sets [5]. A subset \(A\) of \(X\) is said to be \(\mu\)-closed if the complement \(X \setminus A\) of \(A\) is \(\mu\)-open. The \(\mu\)-interior and the \(\mu\)-closure of \(A\) are denoted by \(i_{\mu}(A)\) and \(c_{\mu}(A)\), respectively.

**Definition 2.1** [10, 2] Let \((X,\tau)\) be a topological space and \(A \subseteq X\). Then \(A\) is said to be

(i) \(\alpha\)-open if \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\). A set is \(\alpha\)-closed if its complement is \(\alpha\)-open.

(ii) \(b\)-open if \(A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))\). A set is \(b\)-closed if its complement is \(b\)-open.

If \(\tau\) is a generalized topology \(\mu\), then the \(\alpha\)-open and \(b\)-open sets are denoted by \(\mu\alpha\)-open and \(\mu b\)-open, respectively.

**Definition 2.2** [4] Let \((X,\tau)\) be a topological space and \(A \subseteq X\).

(i) The \(b\)-closure of \(A\), denoted by \(\text{bcl}(A)\), is the intersection of all \(b\)-closed sets containing \(A\). Thus, \(A \subseteq \text{bcl}(A)\).

(ii) The \(b\)-interior of \(A\), denoted by \(\text{bint}(A)\), is the union of all \(b\)-open sets contained in \(A\). Hence, \(\text{bint}(A) \subseteq A\).
If $\tau$ is a generalized topology $\mu$, then the $b$-closure and $b$-interior of $A$ are denoted by $bc_\mu(A)$ and $bi_\mu(A)$, respectively.

**Definition 2.3** [12] Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $A$ is said to be a generalized $\alpha_b$-closed (briefly $g\alpha_b$-closed) set if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$. The complement of a $g\alpha_b$-closed set is called a $g\alpha_b$-open set.

If $\tau$ is a generalized topology $\mu$, then the $g\alpha_b$-closed (resp. $g\alpha_b$-open) sets are denoted by $g\mu\alpha_b$-closed (resp. $g\mu\alpha_b$-open) set.

**Definition 2.4** Let $(X, \mu)$ be a GTS and $A \subseteq X$. Then

(i) The $g\mu\alpha_b$-closure of $A$, denoted by $g\alpha bc_\mu(A)$, is the intersection of all $g\mu\alpha_b$-closed sets containing $A$. Thus, $A \subseteq g\alpha bc_\mu(A)$.

(ii) The $g\mu\alpha_b$-interior of $A$, denoted by $g\alpha bi_\mu(A)$, is the union of all $g\mu\alpha_b$-open sets contained in $A$. Thus, $g\alpha bi_\mu(A) \subseteq A$.

**Theorem 2.5** Let $(X, \mu)$ be a GTS and $A \subseteq X$.

(i) If $A$ is $\mu$-open (resp. $\mu$-closed), then $A$ is $\mu\alpha$-open (resp. $\mu\alpha$-closed).

(ii) If $A$ is $\mu$-open (resp. $\mu$-closed), then $A$ is $\mu b$-open (resp. $\mu b$-closed).

(iii) If $A$ is $\mu\alpha$-open (resp. $\mu\alpha$-closed), then $A$ is $g\mu\alpha b$-open (resp. $g\mu\alpha b$-closed).

(iv) If $A$ is $\mu b$-open (resp. $\mu b$-closed), then $A$ is $g\mu\alpha b$-open (resp. $g\mu\alpha b$-closed).

**Definition 2.6** [3] Let $X$ be a nonempty set and let $\mu_1, \mu_2$ be generalized topologies on $X$. The triple $(X, \mu_1, \mu_2)$ is said to be a bigeneralized topological space (briefly BGTS). If $\mu_1$ and $\mu_2$ are strong generalized topologies, then $(X, \mu_1, \mu_2)$ is a strong BGTS.

Throughout this paper, let $m$ and $n$ be elements of the set $\{1, 2\}$ where $m \neq n$.

**Definition 2.7** [7] A subset of a BGTS $(X, \mu_1, \mu_2)$ is said to be $(m, n)$ generalized closed (briefly $\mu_{(m,n)}$-closed) set if $c_{\mu_n}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\mu_m$-open in $X$. The complement of $\mu_{(m,n)}$-closed set is called $(m, n)$-generalized open (briefly $\mu_{(m,n)}$-open) set.

In [7], Duangthaisong et al. and in [11], Torton et al. defined some separation axioms in BGTS. These definitions are stated below.
Definition 2.8 [7, 11] A bigeneralized topological space \((X, \mu_1, \mu_2)\) is said to be

(i) \(\mu_{(m,n)} T_{\frac{1}{2}}\) space if for every \(\mu_{(m,n)}\)-closed set is \(\mu_n\)-closed.

(ii) \(\mu_{(m,n)} T_1\) space if for any \(x, y \in X\) with \(x \neq y\), there exist a \(\mu_m\)-open set \(U\) and a \(\mu_n\)-open set \(V\) such that \(x \in U\) but \(y \notin U\) and \(y \in V\) but \(x \notin V\).

(iii) \(\mu_{(m,n)}\)-regular space if for any \(x \in X\) and for any \(\mu_m\)-closed subset \(F\) of \(X\) with \(x \notin F\), there exist a \(\mu_m\)-open set \(U\) and a \(\mu_n\)-open set \(V\) such that \(x \in U\), \(F \subseteq V\) and \(U \cap V = \emptyset\).

(iv) \(\mu_{(m,n)}\)-normal space if for any \(\mu_m\)-closed set \(F\) and for any \(\mu_n\)-closed set \(K\) with \(F \cap K = \emptyset\), there exist a \(\mu_m\)-open set \(U\) and a \(\mu_n\)-open set \(V\) such that \(F \subseteq V\), \(K \subseteq U\) and \(U \cap V = \emptyset\).

(v) \(\mu_{(m,n)} T_4\) if it is both \(\mu_{(m,n)} T_1\) and \(\mu_{(m,n)}\)-normal.

Remark 2.9 The concept of \(\mu_{(1,2)}\)-regular space was introduced by Min in [9].

3 \(g^*\mu_{ab}\)-closed sets and \(g^*\mu_{ab}\)-open sets

In this section, the concepts of \(g^*\mu_{ab}\)-closed and \(g^*\mu_{ab}\)-open sets are introduced.

Definition 3.1 Let \((X, \mu)\) be a GTS and \(A \subseteq X\). Then

(i) \(A\) is said to be a generalized star \(\mu_{ab}\)-closed (briefly \(g^*\mu_{ab}\)-closed) set if \(gabc_\mu(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\mu\)-open in \(X\). The complement of a \(g^*\mu_{ab}\)-closed set is called a \(g^*\mu_{ab}\)-open set. Equivalently, \(A\) is \(g^*\mu_{ab}\)-open if \(U \subseteq gab_\mu(A)\) whenever \(U \subseteq A\) and \(U\) is \(\mu\)-closed. We denote the collection of all \(g^*\mu_{ab}\)-open sets in \(X\) by \(g^*\mu_{ab}O(X)\).

(ii) The \(g^*\mu_{ab}\)-closure of \(A\), denoted by \(g^*\alpha bc_\mu(A)\), is the intersection of all \(g^*\mu_{ab}\)-closed sets containing \(A\). Thus, \(A \subseteq g^*\alpha bc_\mu(A)\).

(iii) The \(g^*\mu_{ab}\)-interior of \(A\), denoted by \(g^*\alpha bi_\mu(A)\), is the union of all \(g^*\mu_{ab}\)-open sets contained in \(A\). Thus, \(g^*\alpha bi_\mu(A) \subseteq A\).

Example 3.2 Let \(X = \{a, b, c\}\) and consider the generalized topology \(\mu = \{\emptyset, \{a\}, \{a, b\}, X\}\). The \(g^*\mu_{ab}\)-open subsets of \(X\) are \(\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\), and \(X\). Moreover, \(g^*\alpha bc_\mu(\{a\}) = \{a, c\}\) and \(g^*\alpha bi_\mu(\{b, c\}) = \{b\}\).

Remark 3.3 The union of two \(g^*\mu_{ab}\)-open sets need not be \(g^*\mu_{ab}\)-open as we can see in the next example.
Example 3.4 Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, \{d\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. The $g^*\mu ab$-open sets of $X$ are $X$, $\{b, c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, d\}$, $\{b, c\}$, $\{a, d\}$, $\{c, d\}$, $\{a, b\}$, $\{d\}$, $\{c\}$, $\{b\}$, $\{a\}$, and $\varnothing$. Then $\{a\}$ and $\{b, c\}$ are $g^*\mu ab$-open but $\{a\} \cup \{b, c\} = \{a, b, c\}$ is not $g^*\mu ab$-open.

Remark 3.5 The union of two $g^*\mu ab$-closed sets need not be $g^*\mu ab$-closed as we can see in the next example.

Example 3.6 Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. The $g^*\mu ab$-closed sets of $X$ are $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{b, c, d\}$, and $X$. Then $\{a\}$ and $\{c\}$ are $g^*\mu ab$-closed but $\{a\} \cup \{c\} = \{a, c\}$ is not $g^*\mu ab$-closed.

The proof of the next theorem is straightforward and thus, it is omitted.

Theorem 3.7 Let $(X, \mu)$ be a GTS and $A \subseteq X$.

(i) If $A$ is $\mu$-open (resp. $\mu$-closed), then $A$ is $g^*\mu ab$-open (resp. $g^*\mu ab$-closed).

(ii) If $A$ is $\mu ab$-open (resp. $\mu ab$-closed), then $A$ is $g^*\mu ab$-open (resp. $g^*\mu ab$-closed).

(iii) $x \in g^*\alpha bc_\mu(A)$ if and only if for every $g^*\mu ab$-open set $O$ with $x \in O$, $O \cap A \neq \varnothing$.

Proposition 3.8 Let $(X, \mu)$ be a GTS. For each $x \in X$, $\{x\}$ is $\mu$-closed or $X \setminus \{x\}$ is $g^*\mu ab$-closed.

Proposition 3.9 Let $(X, \mu)$ be a strong GTS and $A \subseteq X$. If $A$ is $g^*\mu ab$-closed, then $g\alpha bc_\mu(A) \setminus A$ contains no nonempty $\mu$-closed set.

Corollary 3.10 Let $(X, \mu)$ be a strong GTS and $A \subseteq X$. If $A$ is $g^*\mu ab$-closed, then $c_\mu(A) \setminus A$ contains no nonempty $\mu$-closed set.

4 $g^*\alpha b$-Separation Axioms

In this section, we introduce some $g^*\alpha b\mu_{(m,n)}$-separation axioms in bigeneralized topological spaces parallel to the separation axioms defined by Dungthaisong et al. in [7] and Torton et al. in [11].

Definition 4.1 A bigeneralized topological space $(X, \mu_1, \mu_2)$ is said to be

(i) $g^*\alpha b\mu_{(m,n)}-T_{\frac{1}{2}}$ space if every $g^*\mu_m\alpha b$-closed set is $g\mu_n\alpha b$-closed.
(ii) $g^*\alpha\mu_{(m,n)} T_1$ space if for any $x, y \in X$ with $x \neq y$, there exist a $g^*\mu_m \alpha$-open set $U$ and a $g^*\mu_n \alpha$-open set $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

(iii) $g^*\alpha\mu_{(m,n)} T_2$ space if for any $x, y \in X$ with $x \neq y$, there exist a $g^*\mu_m \alpha$-open set $U$ and a $g^*\mu_n \alpha$-open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

(iv) $g^*\alpha\mu_{(m,n)}$-regular space if for any $x \in X$ and for any $\mu_m$-closed subset $F$ of $X$ with $x \notin F$, there exist a $g^*\mu_m \alpha$-open set $U$ and a $g^*\mu_n \alpha$-open set $V$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.

(v) $g^*\alpha\mu_{(m,n)}$-normal space if for any $\mu_m$-closed set $F$ and for any $\mu_n$-closed set $K$ with $F \cap K = \emptyset$, there exist a $g^*\mu_m \alpha$-open set $U$ and a $g^*\mu_n \alpha$-open set $V$ such that $F \subseteq V$, $K \subseteq U$ and $U \cap V = \emptyset$.

(vi) $g^*\alpha\mu_{(m,n)}$-$T_4$-space if it is both $g^*\alpha\mu_{(m,n)} T_1$ and $g^*\alpha\mu_{(m,n)}$-normal.

**Remark 4.2** Every $\mu_{(m,n)} T_1$ (resp. $\mu_{(m,n)}$-regular) space is $g^*\alpha\mu_{(m,n)}$-$T_1$ (resp. $g^*\alpha\mu_{(m,n)}$-regular) space.

**Remark 4.3** The converses of Remark 4.2 need not be true as we can see in the next example.

**Example 4.4** Let $X = \{a, b, c\}$ and consider the two generalized topologies $\mu_1 = \{\emptyset, \{a\}, \{a,b\}, \{b\}\}$ and $\mu_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then all subsets of $X$ are $g^*\mu_1 \alpha$-open and $g^*\mu_2 \alpha$-open. Thus, $(X, \mu_1, \mu_2)$ is $g^*\alpha\mu_{(1,2)} T_1$ but not $\mu_{(1,2)} T_1$. Moreover, $(X, \mu_1, \mu_2)$ is $g^*\alpha\mu_{(1,2)}$-regular but not $\mu_{(1,2)}$-regular.

**Remark 4.5** Every $\mu_{(m,n)}$-normal space is a $g^*\alpha\mu_{(m,n)}$-normal space.

**Remark 4.6** Every $g^*\alpha\mu_{(m,n)} T_2$ space is a $g^*\alpha\mu_{(m,n)} T_1$ space.

**Theorem 4.7** A strong BGTS $(X, \mu_1, \mu_2)$ is a $g^*\alpha\mu_{(m,n)} T_{\frac{3}{2}}$ space if and only if $\{x\}$ is $g\mu_n \alpha$-open or $\mu_m$-closed for each $x \in X$.

**Proof:** Suppose that $\{x\}$ is not $\mu_m$-closed. By Proposition 3.8, $X \setminus \{x\}$ is $g^*\mu_m \alpha$-closed. Since $X$ is a $g^*\alpha\mu_{(m,n)} T_{\frac{3}{2}}$ space, $X \setminus \{x\}$ is $g\mu_n \alpha$-open. Hence, $\{x\}$ is $g\mu_n \alpha$-open.

Conversely, let $F$ be a $g^*\mu_m \alpha$-closed set. By assumption, $\{x\}$ is $g\mu_n \alpha$-open or $\mu_m$-closed for any $x \in c_{\mu_n}(F)$.

Case 1: Suppose that $\{x\}$ is $g\mu_m \alpha$-open. Since $\{x\} \cap F \neq \emptyset$, we have $x \in F$. Hence, $F = c_{\mu_n}(F)$. This implies that $F$ is $\mu_n$-closed. By Theorem 2.5, $F$ is $g\mu_n \alpha$-closed.

Case 2: Suppose that $\{x\}$ is $\mu_m$-closed. If $x \notin F$, then $\{x\} \subseteq c_{\mu_n}(F) \setminus F$. This
is a contradiction to Corollary 3.10. Thus, \( x \in F \) and \( F = c_{\mu_n}(F) \). Thus, \( F \) is \( g\mu_n\alpha b \)-closed. 

In both cases, \( F \) is \( g\mu_n\alpha b \)-closed. Therefore, \( X \) is a \( g^*\alpha b\mu_{(m,n)}-T_{\frac{1}{2}} \) space. \( \square \)

Remark 3.3 states that the union of two \( g^*\mu\alpha b \)-open sets need not be \( g^*\mu\alpha b \)-open. With this, we now define a property of \( g^*\mu\alpha b O(X) \) so that arbitrary union of elements in the set is also in the set.

**Definition 4.8** We say that the family \( g^*\mu\alpha b O(X) \) in a GTS \((X, \mu)\) has property \( \vartheta \) if \( g^*\mu\alpha b O(X) \) is closed under arbitrary union of \( g^*\mu\alpha b \)-open sets.

**Remark 4.9** If \( g^*\mu\alpha b O(X) \) has property \( \vartheta \), then an arbitrary intersection of \( g^*\mu\alpha b \)-closed sets in \( X \) is \( g^*\mu\alpha b \)-closed.

Let \((X, \mu_1, \mu_2)\) be a BGTS and \( A \subseteq X \). The closure of \( A \) and the interior of \( A \) with respect to \( \mu_m \) are denoted by \( c_{\mu_m}(A) \) and \( i_{\mu_m}(A) \), respectively. The \( g^*\alpha b \)-closure of \( A \) and the \( g^*\alpha b \)-interior of \( A \) with respect to \( \mu_m \) are denoted by \( g^*\alpha b c_{\mu_m}(A) \) and \( g^*\alpha b i_{\mu_m}(A) \), respectively. Moreover, using the property \( \vartheta \), we now characterize \( g^*\alpha b\mu_{(m,n)}-T_1 \), \( g^*\alpha b\mu_{(m,n)}-T_2 \), \( g^*\alpha b\mu_{(m,n)} \)-regular, \( g^*\alpha b\mu_{(m,n)} \)-normal, and \( g^*\alpha b\mu_{(m,n)} \)-spaces.

**Theorem 4.10** Let \((X, \mu_1, \mu_2)\) be a BGTS such that \( g^*\mu_m\alpha b O(X) \) and \( g^*\mu_n\alpha b O(X) \) have property \( \vartheta \). Then \((X, \mu_1, \mu_2)\) is a \( g^*\alpha b\mu_{(m,n)}-T_1 \) space if and only if for all \( x, y \in X \) with \( x \neq y \), \( \{x\} \) is a \( g^*\mu_n\alpha b \)-closed set and \( \{y\} \) is a \( g^*\mu_m\alpha b \)-closed set.

**Proof:** Let \( x \in X \) and suppose that \( X \) is a \( g^*\alpha b\mu_{(m,n)}-T_1 \) space. Then for every \( y \in X \) such that \( x \neq y \), there exist a \( g^*\mu_m\alpha b \)-open set \( U_y \) and a \( g^*\mu_n\alpha b \)-open set \( V_y \) with \( x \in U_y \) but \( x \notin V_y \) and \( y \in V_y \) but \( y \notin U_y \). Then \( X \setminus \{x\} = \bigcup_{y \neq x} V_y \) and \( X \setminus \{y\} = g^*\mu_m\alpha b \)-open and \( X \setminus \{x\} \) is \( g^*\mu_n\alpha b \)-open.

Hence, \( \{y\} \) is \( g^*\mu_m\alpha b \)-closed and \( \{x\} \) is \( g^*\mu_n\alpha b \)-closed.

Conversely, let \( x, y \in X \) with \( x \neq y \). By assumption, \( \{x\} \) is a \( g^*\mu_n\alpha b \)-closed set and \( \{y\} \) is a \( g^*\mu_m\alpha b \)-closed set. Set \( U = X \setminus \{y\} \) and \( V = X \setminus \{x\} \). Thus, \( U \) is a \( g^*\mu_m\alpha b \)-open set and \( V \) is a \( g^*\mu_n\alpha b \)-open set. Moreover, \( x \in U \) but \( y \notin U \) and \( y \in V \) but \( x \notin V \). Hence, \((X, \mu_1, \mu_2)\) is \( g^*\alpha b\mu_{(m,n)}-T_1 \) space. \( \square \)

**Theorem 4.11** Let \((X, \mu_1, \mu_2)\) be a BGTS such that \( g^*\mu_n\alpha b O(X) \) has property \( \vartheta \). Then the following are equivalent:

(i) \( X \) is a \( g^*\alpha b\mu_{(m,n)}-T_2 \) space.

(ii) Let \( p \in X \). For each \( q \in X \) where \( q \neq p \), there exists a \( g^*\mu_m\alpha b \)-open set \( U \) such that \( p \in U \) and \( q \notin g^*\alpha b c_{\mu_n}(U) \).
(iii) For each \( p \in X \), \( \cap \{ g^*\alpha bc_{\mu_n}(U) \mid U \text{ is } g^*\mu_m\alpha b \text{-open with } p \in U \} = \{ p \} \).

Proof:

(i) \( \Rightarrow \) (ii): Let \( p \in X \). Suppose that \( X \) is a \( g^*\alpha b\mu_{(m,n)}-T_2 \) space. Then for each \( q \in X \) such that \( q \neq p \), there exist a \( g^*\mu_m\alpha b \)-open set \( U \) and a \( g^*\mu_n\alpha b \)-open set \( V \) such that \( U \cap V = \emptyset \) where \( p \in U \) and \( q \in V \). Suppose that \( q \in g^*\alpha bc_{\mu_n}(U) \).

Then for all \( g^*\mu_m\alpha b \)-open set \( O \) with \( q \in O \), \( O \cap U \neq \emptyset \). This is a contradiction since \( V \) is a \( g^*\mu_n\alpha b \)-open set with \( q \in V \) and \( V \cap U = \emptyset \). Thus, \( q \notin g^*\alpha bc_{\mu_n}(U) \).

(ii) \( \Rightarrow \) (iii): Let \( q \in X \) such that \( q \neq p \). Then by (ii), there exists a \( g^*\mu_m\alpha b \)-open set \( U \) such that \( p \in U \) and \( q \notin g^*\alpha bc_{\mu_n}(U) \). Thus,

\[
q \notin \cap \{ g^*\alpha bc_{\mu_n}(U) \mid U \text{ is } g^*\mu_m\alpha b \text{-open with } p \in U \}.
\]

Since \( p \neq q \), then \( \cap \{ g^*\alpha bc_{\mu_n}(U) \mid U \text{ is } g^*\mu_m\alpha b \text{-open with } p \in U \} = \{ p \} \).

(iii) \( \Rightarrow \) (i): Let \( p, q \in X \) with \( p \neq q \). By (iii),

\[
L = \cap \{ g^*\alpha bc_{\mu_n}(U) \mid U \text{ is } g^*\mu_m\alpha b \text{-open with } p \in U \} = \{ p \}.
\]

Since \( p \neq q \), then \( q \notin L \). Thus, there exists a \( g^*\mu_m\alpha b \)-open set \( U \) containing \( p \) such that \( q \notin g^*\alpha bc_{\mu_n}(U) \). Let \( V = X \setminus g^*\alpha bc_{\mu_n}(U) \). By assumption, \( V \) is a \( g^*\mu_n\alpha b \)-open set with \( q \in V \). Since \( U \subseteq g^*\alpha bc_{\mu_n}(U) \), \( X \setminus g^*\alpha bc_{\mu_n}(U) \subseteq X \setminus U \).

Hence, \( U \cap V = \emptyset \). Therefore, \( X \) is a \( g^*\alpha b\mu_{(m,n)}-T_2 \) space.

\]

Theorem 4.12 Let \( (X, \mu_1, \mu_2) \) be a BGTS such that \( g^*\mu_m\alpha bO(X) \) and \( g^*\mu_n\alpha bO(X) \) have property \( \vartheta \). Then the following are equivalent:

(i) \( X \) is \( g^*\alpha b\mu_{(m,n)} \)-regular.

(ii) For any \( p \in X \) and for any \( \mu_m \)-closed subset \( F \) of \( X \) with \( p \notin F \), there exist a \( g^*\mu_m\alpha b \)-open set \( U \) and a \( g^*\mu_n\alpha b \)-open set \( V \) such that \( p \in U \), \( F \subseteq V \) and \( g^*\alpha bc_{\mu_n}(U) \cap V = \emptyset \).

(iii) If \( p \in X \) and \( F \) is a \( \mu_m \)-closed subset of \( X \) with \( p \notin F \), then there exists a \( g^*\mu_m\alpha b \)-open set \( U \) with \( p \in U \) and \( g^*\alpha bc_{\mu_n}(U) \cap F = \emptyset \).

(iv) If \( p \in X \) and \( G \in \mu_m \) with \( p \in G \), then there exists a \( g^*\mu_m\alpha b \)-open set \( V \) with \( p \in V \) and \( p \in V \subseteq g^*\alpha bc_{\mu_n}(V) \subseteq G \).

(v) \( F = \cap \{ g^*\alpha bc_{\mu_n}(V) \mid V \text{ is } g^*\mu_m\alpha b \text{-open set with } F \subseteq V \} \) for each \( \mu_m \)-closed subset \( F \) of \( X \).

Proof:

(i) \( \Rightarrow \) (ii): Suppose that \( (X, \mu_1, \mu_2) \) is a \( g^*\alpha b\mu_{(m,n)} \)-regular space. Let \( p \in X \) and let \( F \) be a \( \mu_m \)-closed subset of \( X \) with \( p \notin F \). Then there exist a \( g^*\mu_m\alpha b \)-open set \( U \) and a \( g^*\mu_n\alpha b \)-open set \( V \) such that \( p \in U \), \( F \subseteq V \) and \( U \cap V = \emptyset \).
Suppose that $g^*abc_{\mu_n}(U) \cap V \neq \emptyset$. Then there exists $q \in g^*abc_{\mu_n}(U)$ and $q \in V$. Since $V$ is a $g^*\mu_n\alpha\beta$-open set, $U \cap V \neq \emptyset$. This is a contradiction since $U \cap V = \emptyset$. Therefore, $g^*abc_{\mu_n}(U) \cap V = \emptyset$.

(ii) $\Rightarrow$ (iii): Let $p \in X$ and $F$ be a $\mu_m$-closed subset of $X$ with $p \notin F$. By (ii), there exist a $g^*\mu_m\alpha\beta$-open set $U$ and a $g^*\mu_m\alpha\beta$-open set $V$ such that $p \in U$, $F \subseteq V$ and $g^*abc_{\mu_n}(U) \cap V = \emptyset$. Thus, $g^*abc_{\mu_n}(U) \cap F \subseteq g^*abc_{\mu_n}(U) \cap V = \emptyset$. Therefore, $g^*abc_{\mu_n}(U) \cap F = \emptyset$.

(iii) $\Rightarrow$ (iv): Let $p \in X$ and $G \subseteq \mu_m$ with $p \in G$. Then $X \setminus G$ is $\mu_m$-closed with $p \notin X \setminus G$. By (iii), there exists a $g^*\mu_m\alpha\beta$-open set $V$ with $p \in V$ and $g^*abc_{\mu_n}(V) \cap (X \setminus G) = \emptyset$. Hence, $p \in V \subseteq g^*abc_{\mu_n}(V) \subseteq G$.

(iv) $\Rightarrow$ (v): Let $F$ be a $\mu_m$-closed subset of $X$,

$$L = \cap \{g^*abc_{\mu_n}(V) \mid V \text{ is } g^*\mu_n\alpha\beta \text{-open set with } F \subseteq V \}$$

and $q \in X$ with $q \notin F$. Then $X \setminus F$ is $\mu_m$-open and $q \in X \setminus F$. By (iv), there exists a $g^*\mu_m\alpha\beta$-open set $V$ with $q \in V \subseteq g^*abc_{\mu_n}(V) \subseteq X \setminus F$. Thus, $F \subseteq X \setminus g^*abc_{\mu_n}(V) \subseteq X \setminus V$ and $q \notin X \setminus V$. Take $M = X \setminus g^*abc_{\mu_n}(V)$. Then $M$ is $g^*\mu_m\alpha\beta$-open and $F \subseteq M$. Since $X \setminus V$ is $g^*\mu_m\alpha\beta$-closed and $q \notin X \setminus V \supseteq M$, $q \notin g^*abc_{\mu_n}(M)$. Hence, $q \notin L$. Thus, $L \subseteq F$. Therefore, $F = L$.

(v) $\Rightarrow$ (i): Let $p \in X$ and $F$ be a $\mu_m$-closed set such that $p \notin F$. By (v), there exists a $g^*\mu_m\alpha\beta$-open set $V$ such that $F \subseteq V$ and $p \notin g^*abc_{\mu_n}(V)$. Take $U = X \setminus g^*abc_{\mu_n}(V)$. Then $U$ is $g^*\mu_m\alpha\beta$-open and $p \in U$. Moreover, $U \cap V = \emptyset$. Therefore, $(X, \mu_1, \mu_2)$ is $g^*\alpha\beta\mu_{(m,n)}$-regular. \square

**Theorem 4.13** Let $(X, \mu_1, \mu_2)$ be a BGTS such that $g^*\mu_m\alpha\beta O(X)$ and $g^*\mu_n\alpha\beta O(X)$ have property $\vartheta$. Then the following are equivalent:

(i) $X$ is a $g^*\alpha\beta\mu_{(m,n)}$-normal space.

(ii) If $F$ is $\mu_m$-closed and $K$ is $\mu_n$-closed such that $F \cap K = \emptyset$, then there exist a $g^*\mu_m\alpha\beta$-open set $U$ and a $g^*\mu_n\alpha\beta$-open set $V$ such that $F \subseteq V$, $K \subseteq U$ and $g^*abc_{\mu_n}(U) \cap V = \emptyset$.

(iii) If $F$ is $\mu_m$-closed and $K$ is $\mu_n$-closed such that $F \cap K = \emptyset$, then there exist a $g^*\mu_m\alpha\beta$-open set $U$ such that $K \subseteq U$ and $g^*abc_{\mu_n}(U) \cap F = \emptyset$.

(iv) If $F$ is $\mu_m$-closed and $G$ is $\mu_n$-open such that $F \subseteq G$, then there exists a $g^*\mu_n\alpha\beta$-open set $V$ such that $F \subseteq V \subseteq g^*abc_{\mu_n}(V) \subseteq G$.

**Proof:**

(i) $\Rightarrow$ (ii): Let $F$ be a $\mu_m$-closed set and $K$ a $\mu_n$-closed set in $X$ such that $F \cap K = \emptyset$. By (i), there exist a $g^*\mu_m\alpha\beta$-open set $U$ and a $g^*\mu_n\alpha\beta$-open set $V$ such that $F \subseteq V$, $K \subseteq U$ and $U \cap V = \emptyset$. Suppose that $g^*abc_{\mu_n}(U) \cap V \neq \emptyset$. 


Then there exists \( y \in g^*\alpha bc_{\mu_n}(U) \) and \( y \in V \). Since \( V \) is a \( g^*\mu_n\alpha b\)-open set, \( U \cap V \neq \emptyset \), which is a contradiction. Therefore, \( g^*\alpha bc_{\mu_n}(U) \cap V = \emptyset \).

(ii) \( \Rightarrow \) (iii): It is straightforward since \( F \subseteq V \).

(iii) \( \Rightarrow \) (iv): Assume that \( F \) is \( \mu_m \)-closed and \( G \) is \( \mu_n \)-open such that \( F \subseteq G \). Then \( X \setminus G \) is a \( \mu_n \)-closed and \( F \cap (X \setminus G) = \emptyset \). By (iii), there exists a \( g^*\mu_m\alpha b\)-open set \( U \) such that \( X \setminus G \subseteq U \) and \( g^*\alpha bc_{\mu_n}(U) \cap F = \emptyset \). Hence, \( F \subseteq X \setminus g^*\alpha bc_{\mu_n}(U) \subseteq X \setminus U \subseteq G \). Let \( V = X \setminus g^*\alpha bc_{\mu_n}(U) \). Thus, \( V \) is \( g^*\mu_n\alpha b\)-open and \( F \subseteq V \subseteq g^*\alpha bc_{\mu_m}(V) \subseteq X \setminus U \subseteq G \).

(iv) \( \Rightarrow \) (i): Let \( F \) be a \( \mu_m \)-closed set and \( K \) a \( \mu_n \)-closed set such that \( F \cap K = \emptyset \). Then \( X \setminus K \) is a \( \mu_n \)-open set and \( F \subseteq X \setminus K \). By (iv), there exists a \( g^*\mu_n\alpha b\)-open set \( V \) such that \( F \subseteq V \subseteq g^*\alpha bc_{\mu_m}(V) \subseteq X \setminus K \). Take \( U = X \setminus g^*\alpha bc_{\mu_n}(V) \). Then \( U \) is a \( g^*\mu_m\alpha b\)-open set and \( K \subseteq U \). Moreover, \( U \cap V = (X \setminus g^*\alpha bc_{\mu_n}(V)) \cap V \subseteq (X \setminus V) \cap V = \emptyset \). Hence, \( (X, \mu_1, \mu_2) \) is \( g^*\alpha bc_{\mu_{(m,n)}} \)-normal.

\textbf{Theorem 4.14} Let \((X, \mu_1, \mu_2)\) be a BGTS such that \( g^*\mu_m\alpha bO(X) \) and \( g^*\mu_n\alpha bO(X) \) have property \( \vartheta \). Then the following are equivalent:

(i) \( X \) is a \( g^*\alpha bc_{\mu_{(m,n)}} \)-\( T_4 \) space.

(ii) If \( F_1 \) is \( \mu_m \)-closed and \( F_2 \) is \( \mu_n \)-closed such that \( F_1 \cap F_2 = \emptyset \), then there exist a \( g^*\mu_m\alpha b\)-open set \( U_1 \) and a \( g^*\mu_n\alpha b\)-open set \( U_2 \) such that \( F_1 \subseteq U_1 \), \( F_2 \subseteq U_2 \) and \( g^*\alpha bc_{\mu_n}(U_1) \cap U_2 = \emptyset \).

(iii) If \( F_1 \) is \( \mu_m \)-closed and \( F_2 \) is \( \mu_n \)-closed such that \( F_1 \cap F_2 = \emptyset \), then there exists a \( g^*\mu_m\alpha b\)-open set \( U \) such that \( F_2 \subseteq U \) and \( g^*\alpha bc_{\mu_n}(U) \cap F_1 = \emptyset \);

(iv) If \( F \) is a \( \mu_m \)-closed and \( U \) is \( \mu_n \)-open such that \( F \subseteq U \), then there exists a \( g^*\mu_n\alpha b\)-open set \( V \) such that \( F \subseteq V \subseteq g^*\alpha bc_{\mu_m}(V) \subseteq U \).

\textit{Proof:} Similar to the proof of Theorem 4.13.

Now, we define some classes of functions which will be used in the invariance properties of the separation axioms stated in Definition 4.1.

\textbf{Definition 4.15} Let \((X, \mu_X^1, \mu_X^2)\) and \((Y, \mu_Y^1, \mu_Y^2)\) be BGTS and let \( f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2) \) be a function. Then \( f \) is said to be

(i) \( \text{regular strongly } g^*\mu^{(m,n)}\alpha b\)-continuous \( \left( rs-g^*\mu^{(m,n)}\alpha b\text{-continuous} \right) \) if for each \( g^*\mu_{\mu_n}^b \)-open subset \( U \) of \( Y \), \( f^{-1}(U) \) is \( \mu_{\mu_1}^b \)-open in \( X \).

(ii) \( \text{pairwise } rs-g^*\mu_{\alpha b} \)-continuous if \( f \) is both \( rs-g^*\mu^{(1,2)}_{\alpha b} \)-continuous and \( rs-g^*\mu^{(2,1)}_{\alpha b} \)-continuous.

(iii) \( \text{quasi } g^*\mu^{(m,n)}\alpha b \)-open if for each \( g^*\mu_{\mu_n}^m \alpha b \)-open set \( U \) in \( X \), \( f(U) \) is \( \mu_{\mu_2}^n \)-open in \( Y \).
Lemma 4.16 Let \((X, \mu_X^1, \mu_X^2)\) and \((Y, \mu_Y^1, \mu_Y^2)\) be BGTS and \(f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2)\) be a bijective function. Then the following are equivalent:

(i) \(f\) is rs-\(g^*\mu^{(m,n)}\alpha\)-continuous.

(ii) For each \(g^*\mu_m\alpha\)-closed subset \(F\) of \(Y\), \(f^{-1}(F)\) is a \(\mu_n\)-closed subset of \(X\).

Lemma 4.17 Let \((X, \mu_X^1, \mu_X^2)\) and \((Y, \mu_Y^1, \mu_Y^2)\) be BGTS and \(f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2)\) be a bijective function. Then the following are equivalent:

(i) \(f\) is quasi \(g^*\mu^{(m,n)}\alpha\)-open.

(ii) For each \(g^*\mu_X^m\alpha\)-closed subset \(G\) in \(X\), \(f(G)\) is a \(\mu_Y^n\)-closed set in \(Y\).

We will now determine under which type of functions previously defined do some spaces are invariant.

Theorem 4.18 Every \(g^*\alpha \mu_{m,n}^\alpha\)-\(T_1\) space is invariant under pairwise quasi \(g^*\mu\alpha\)-open bijections.

Proof: Let \(f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2)\) be a pairwise quasi \(g^*\mu\alpha\)-open bijections. Suppose that \(X\) is a \(g^*\alpha \mu_{m,n}^\alpha\)-\(T_1\) space. Let \(y_1, y_2 \in Y\) such that \(y_1 \neq y_2\). Since \(f\) is a bijection, there exist \(x_1, x_2 \in X\) such that \(f(x_1) = y_1\), \(f(x_2) = y_2\), and \(x_1 \neq x_2\). Since \(X\) is a \(g^*\alpha \mu_{m,n}^\alpha\)-\(T_1\) space, there exist a \(g^*\mu_b\alpha\)-open set \(U\) and a \(g^*\mu_Y^b\alpha\)-open set \(V\) such that \(x_1 \in U\) but \(x_1 \notin V\) and \(x_2 \in V\) but \(x_2 \notin U\). Hence, \(f(x_1) = y_1 \in f(U)\) but \(f(x_1) = y_1 \notin f(V)\) and \(f(x_2) = y_2 \in f(V)\) but \(f(x_2) = y_2 \notin f(U)\). Since \(f\) is pairwise quasi \(g^*\mu\alpha\)-open, \(f(U)\) is a \(\mu_Y^b\alpha\)-open set in \(Y\) and \(f(V)\) is a \(\mu_Y^b\alpha\)-open set in \(Y\). By Theorem 3.7(i), \(f(U)\) is \(g^*\mu_Y^b\alpha\)-open and \(f(V)\) is \(g^*\mu_Y^b\alpha\)-open. Therefore, \(Y\) is a \(g^*\alpha \mu_{m,n}^\alpha\)-\(T_1\) space.

Theorem 4.19 Every \(g^*\alpha \mu_{m,n}^\alpha\)-\(T_2\) space is invariant under pairwise quasi \(g^*\alpha \mu_{m,n}^\alpha\)-open bijections.

Proof: Similar to the proof of Theorem 4.18

Theorem 4.20 Let \((X, \mu_X^1, \mu_X^2)\) and \((Y, \mu_Y^1, \mu_Y^2)\) be BGTS and let \(f : (X, \mu_X^1, \mu_X^2) \rightarrow (Y, \mu_Y^1, \mu_Y^2)\) be a bijective pairwise rs-\(g^*\mu\alpha\)-continuous and pairwise quasi \(g^*\mu\alpha\)-open function. Then \(X\) is \(g^*\alpha \mu_{m,n}^\alpha\)-regular if and only if \(Y\) is \(g^*\alpha \mu_{m,n}^\alpha\)-regular.
**Proof:** Let $f$ be a bijective pairwise $rs$-$g^*\mu ab$-continuous and pairwise quasi $g^*\mu ab$-open function. Suppose that $X$ is a $g^*\alpha y\mu_{(m,n)}$-regular space. Let $y \in Y$ and $F$ be a $\mu^m_{\alpha b}$-closed subset of $Y$ such that $y \notin F$. By Theorem 3.7(i), $F$ is $g^*\mu^m_{\alpha b}$-closed. Since $f$ is an $rs$-$g^*\mu^{(m,n)}_{\alpha b}$-continuous bijective function, there exists $x \in X$ such that $f(x) = y$ and $f^{-1}(F)$ is a $\mu^m_X$-closed subset of $X$ with $x \notin f^{-1}(F)$. Thus, there exist a $g^*\mu^m_{\alpha b}$-open set $U$ and a $g^*\mu^m_{\alpha b}$-open set $V$ such that $x \in U$, $f^{-1}(F) \subseteq V$, and $U \cap V = \emptyset$. Hence, $y = f(x) \in f(U)$, $F \subseteq f(V)$, and $f(U) \cap f(V) = \emptyset$. Since $f$ is pairwise quasi $g^*\mu ab$-open, $f(U)$ is $\mu^m_Y$-open in $Y$ and $f(V)$ is $\mu^m_Y$-open in $Y$. By Theorem 3.7(i), $(f(U)$ is $g^*\mu^m_{\alpha b}$-open and $f(V)$ is $g^*\mu^m_{\alpha b}$-open. Therefore $Y$ is $g^*\alpha y\mu_{(m,n)}$-regular.

Conversely, suppose $Y$ is $g^*\alpha y\mu_{(m,n)}$-regular. Let $x \in X$ and let $G$ be a $\mu^m_X$-closed set in $X$ with $x \notin G$. Then $G$ is $g^*\mu^m_X$-open-closed in $X$. Since $f$ is quasi $g^*\mu^{(m,n)}_{\alpha b}$-open and by Lemma 4.17, $f(G)$ is $\mu^m_Y$-closed in $Y$ and $f(x) \notin f(G)$. Since $Y$ is $g^*\alpha y\mu_{(m,n)}$-regular, there exist a $g^*\mu^m_{\alpha b}$-open set $U$ and a $g^*\mu^m_{\alpha b}$-open set $V$ such that $f(x) \in U$, $f(G) \subseteq V$, and $U \cap V = \emptyset$. Since $f$ is pairwise $rs$-$g^*\mu ab$-continuous, $f^{-1}(U)$ is a $\mu^m_X$-open set in $X$ and $f^{-1}(V)$ is a $\mu^m_X$-open set in $X$. Thus, there exist a $g^*\mu^m_X$-open set $f^{-1}(U)$ and a $g^*\mu^m_X$-open set $f^{-1}(V)$ such that $x \in f^{-1}(U)$, $G \subseteq f^{-1}(V)$, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore, $X$ is a $g^*\alpha y\mu_{(m,n)}$-regular space.

**Theorem 4.21** Every $g^*\alpha y\mu_{(m,n)}$-normal space is invariant under pairwise $rs$-$g^*\mu ab$-continuous and pairwise quasi $g^*\mu ab$-open bijective function.

**Proof:** Let $f : (X, \mu^1_X, \mu^3_X) \rightarrow (Y, \mu^1_Y, \mu^3_Y)$ be a pairwise $rs$-$g^*\mu ab$-continuous and pairwise quasi $g^*\mu ab$-open bijective function. Suppose that $X$ is a $g^*\alpha y\mu_{(m,n)}$-normal space. Let $F_1$ be a $\mu^m_Y$-closed set and $F_2$ be a $\mu^m_Y$-closed set where $F_1 \cap F_2 = \emptyset$. Then by Theorem 3.7(i), $F_1$ is a $g^*\mu^m_Y$-open-closed and $F_2$ is $g^*\mu^m_{\alpha b}$-open-closed. Since $f$ is pairwise $rs$-$g^*\mu ab$-continuous bijective, $f^{-1}(F_1)$ is $\mu^m_X$-closed, $f^{-1}(F_2)$ is $\mu^m_X$-closed, and $f^{-1}(F_1) \cap f^{-1}(F_2) = \emptyset$. Hence, there exist a $g^*\mu^m_X$-open set $V$ and $g^*\mu^m_{\alpha b}$-open set $K$ such that $f^{-1}(F_1) \subseteq V$, $f^{-1}(F_2) \subseteq K$, and $V \cap K = \emptyset$. Since $f$ is pairwise quasi $g^*\mu ab$-open, $f(V)$ is $\mu^m_Y$-open and $f(K)$ is $\mu^m_Y$-open. By Theorem 3.7(i), $(f(V)$ is $g^*\mu^m_{\alpha b}$-open and $f(K)$ is $g^*\mu^m_{\alpha b}$-open. Moreover, $F_1 \subseteq f(V)$, $F_2 \subseteq f(K)$, and $f(V) \cap f(K) = \emptyset$. Therefore, $Y$ is a $g^*\alpha y\mu_{(m,n)}$-normal space.

**References**


Received: March 25, 2015; Published: May 7, 2015