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A Note on a q-Analogue of λ-Daehee Polynomials

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Abstract

In this paper, we introduce a new q-analogue of the λ-Daehee numbers and polynomials of the first kind and the second kind, and derive some new interesting identities.

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined by Kim as follows :

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [6, 7, 8])}. \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \text{ where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{1.2}$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!}, \text{ (see [3, 14])}. \tag{1.4}$$

It is well known that the (h,q)-*Bernoulli polynomials* are defined by the generating function to be

$$\left(\frac{q-1+\frac{q-1}{\log q}(h\log q + t)}{e^t-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h,1)}(x) \frac{t^n}{n!}, \text{ (see [10, 15, 16])}, \tag{1.5}$$

where h is an integer. When $x = 0$ and $h = -1$, $B_{n,q}^{(-1,1)}(0) = B_{n,q}^{(-1,1)}$ are called the *ordinary q-Bernoulli numbers*.

Recently, D. S. Kim and T. Kim introduced the *Daehee polynomials of the first kind* are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},$$

and the *Daehee polynomials of the second kind* are given by

$$\frac{\log(1+t)}{t+1}(1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n(x) \frac{t^n}{n!} \text{ (see [4, 7, 11, 13])},$$

and Cho et. al. defined the *q-Daehee polynomials* as follows.

$$\frac{1-q+\frac{1-q}{\log q}\log(1+t)}{1-q-qt}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!}, \text{ (see [2])}. \tag{1.6}$$

In the viewpoint of generalization of the Daehee polynomials, we consider the *q-analogue of the λ-Daehee polynomials* are defined to be

$$\sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{t^n}{n!} = (1+t)^x \frac{q-1+\frac{q-1}{\log q}(\lambda\log(1+t) - \log q)}{(1+t)^\lambda - 1}. \tag{1.7}$$

When $x = 0$, $D_{n,q}(\lambda) = D_{n,q}(0|\lambda)$ are called the *q-analogue of the λ-Daehee numbers*.

In the past years, Kim et. al. have studies the various generalization of Daehee polynomials (see [2, 5, 9, 11, 12, 13]), and in [1], authors give new q-analogue of Changhee numbers and polynomials.

In this paper, we introduce a new q-analogue of the λ-Daehee numbers and polynomials of the first kind and the second kind, which are called the Witt-type formula for the q-analogue of λ-Daehee polynomials. We can derive some new interesting identities related to the q-Daehee polynomials.

2. Witt-type formula for the q-analogue of nth λ-Daehee polynomials

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$. First, we consider the following integral representation associated with falling factorial sequences :

$$\int_{\mathbb{Z}_p} q^{-y}(x+\lambda y)_n d\mu_q(y), \text{ where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{2.1}$$

By (2.1),

$$\begin{aligned}\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y}(x+\lambda y)_n d\mu_q(y) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y} \binom{x+\lambda y}{n} d\mu_q(y) t^n \\ &= \int_{\mathbb{Z}_p} q^{-y}(1+t)^{x+\lambda y} d\mu_q(y)\end{aligned} \tag{2.2}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, we get

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{-y}(1+t)^{x+\lambda y} d\mu_q(y) &= (1+t)^x \frac{q-1+\frac{q-1}{\log q}(\lambda \log(1+t) - \log q)}{(1+t)^{\lambda}-1} \\ &= \sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} q^{-y}(x+\lambda y)_n d\mu_q(y).$$

In (2.3), by replacing t by $(e^t - 1)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{(e^t-1)^n}{n!} &= e^{xt} \frac{q-1+\frac{q-1}{\log q}(\lambda t - \log q)}{e^{\lambda t}-1} \\ &= \sum_{n=0}^{\infty} \lambda^n B_{n,q}^{(-1,1)}\left(\frac{x}{\lambda}\right) \frac{t^n}{n!},\end{aligned} \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} D_{n,q}(x|\lambda)\frac{1}{n!}\left(e^t-1\right)^n = \sum_{n=0}^{\infty} D_{n,q}(x|\lambda)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} D_{m,q}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.5}$$

By (2.4) and (2.5), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$B_{n,q}^{(-1,1)}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n} D_{m,q}(x|\lambda)\lambda^{-n}S_2(n,m).$$

By the Theorem 2.1,

$$D_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} q^{-y}(x+\lambda y)_n d\mu_q(y)$$
$$= \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p} q^{-y}(x+\lambda y)^l d\mu_q(y). \tag{2.6}$$

By (1.2), we can derive easily that

$$\int_{\mathbb{Z}_p} q^{-y}e^{(x+\lambda y)t}d\mu_q(y) = \frac{q-1+\frac{q-1}{\log q}(\lambda t-\log q)}{e^{\lambda t}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(-1,1)}\left(\frac{x}{\lambda}\right)\frac{(\lambda t)^n}{n!}$$
$$= \sum_{l=0}^{\infty}\int_{\mathbb{Z}_p} q^{-y}(x+\lambda y)^l d\mu_q(y)\frac{t^l}{l!}, \tag{2.7}$$

and so

$$\lambda^n B_{n,q}^{(-1,1)}\left(\frac{x}{\lambda}\right) = \int_{\mathbb{Z}_p} q^{-y}\left(x+\lambda y\right)^n d\mu_q(y),\ (n \geq 0). \tag{2.8}$$

By (**??**), (2.7) and (2.8), we obtain the following corollary.

Corollary 2.3. *For $n \geq 0$, we have*

$$D_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n,l)\lambda^l B_{l,q}^{(-1,1)}\left(\frac{x}{\lambda}\right)$$
$$= \sum_{l=0}^{n} |S_1(l,n)|(-1)^{n-l}\lambda^l B_{l,q}^{(-1,1)}\left(\frac{x}{\lambda}\right).$$

From now on, we consider *q-analogue of nth λ-Daehee polynomials of order $k(\in \mathbb{N})$*. q-analogue of nt λ-Daehee polynomials of order k are defined by the

multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$\begin{aligned} &D_{n,q}^{(k)}(x|\lambda) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_k} \left(\lambda(x_1+\cdots+x_k)+x\right)_n d\mu_q(x_1)\cdots d\mu_q(x_k) \end{aligned} \tag{2.9}$$

where n is an nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D_{nq}^{(k)}(\lambda) = D_{n,q}^{(k)}(\lambda|0)$ are called the *λ-Daehee numbers of order k with q-parameter.*

From (2.9), we can derive the generating function of $D_{n,q}^{(k)}(x)$ as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} D_{n,q}^{(k)}(x|\lambda)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^k x_i} \binom{\lambda(x_1+\cdots+x_k)+x}{n} d\mu_q(x_1)\cdots d\mu_q(x_k) t^n \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^k x_i} (1+t)^{\lambda(x_1+\cdots+x_k)+x} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= (1+t)^x \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^k x_i} (1+t)^{\lambda(x_1+\cdots+x_k)} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= (1+t)^x \left(\frac{q-1+\frac{q-1}{\log q}\left(\lambda\log(1+t)-\log q\right)}{(1+t)^\lambda - 1}\right)^k. \end{aligned} \tag{2.10}$$

Note that, by (2.9),

$$\begin{aligned} &D_{n,q}^{(k)}(x|\lambda) \\ &= \sum_{m=0}^{n} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_k} (\lambda(x_1+\cdots+x_k)+x)^m d\mu_q(x_1)\cdots d\mu_q(x_k), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h\sum_{i=1}^k x_i} e^{(x_1+\cdots+x_k+x)t} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \left(\frac{q-1+\frac{q-1}{\log q}(h\log q+t)}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h,k)}(x)\frac{t^n}{n!}, \end{aligned} \tag{2.12}$$

where $B_{n,q}^{(h,k)}(x)$ are (h,q)-Bernoulli polynomials of order k. By (2.12), we can derive easily

$$B_{n,q}^{(-1,k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^k x_k} (x_1+\cdots+x_k+x)^n d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{2.13}$$

Thus, by (2.11) and (2.13), we have

$$\begin{aligned} D_{n,q}^{(k)}(x|\lambda) &= \sum_{m=0}^{n} S_1(n,m)\lambda^m B_m^{(-1,k)}\left(\frac{x}{\lambda}\right) \\ &= \sum_{m=0}^{n} |S_1(n,m)|(-1)^{n-m} B_m^{(-1,k)}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{2.14}$$

In (2.10), by replacing t by (e^t-1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} D_{n,q}^{(k)}(x|\lambda)\frac{(e^t-1)^n}{n!} &= e^{xt}\left(\frac{\frac{q-1}{\log q}(\lambda t - \log q)}{e^{\lambda t}-1}\right)^k \\ &= \sum_{n=0}^{\infty} \lambda^n B_n^{(-1,k)}\left(\frac{x}{\lambda}\right)\frac{t^n}{n!}, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} D_{n,q}^{(k)}(x|\lambda)\frac{1}{n!}(e^t-1)^n &= \sum_{n=0}^{\infty} D_{n,q}^{(k)}(x|\lambda)\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_{n,q}^{(k)}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}. \end{aligned} \tag{2.16}$$

By (2.14), (2.15) and (2.16), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned} D_{n,q}^{(k)}(x|\lambda) &= \sum_{m=0}^{n} S_1(n,m)\lambda^m B_{m,q}^{(-1,k)}\left(\frac{x}{\lambda}\right) \\ &= \sum_{m=0}^{n} |S_1(n,m)|(-1)^{n-m} B_{m,q}^{(-1,k)}\left(\frac{x}{\lambda}\right), \end{aligned}$$

and

$$B_{n,q}^{(-1,k)}\left(\frac{x}{\lambda}\right) = \lambda^{-n}\sum_{m=0}^{n} D_{n,q}^{(k)}(x|\lambda)S_2(m,n)S_2(n,m).$$

Now, we consider the λ-*Daehee polynomials of the second kind.*

$$\widehat{D}_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} q^{-y}(-\lambda y + x)_n d\mu_q(y), \ (n \geq 0). \tag{2.17}$$

In the special case, $x = 0$, $\widehat{D}_{n,q}(\lambda) = \widehat{D}_{n,q}(\lambda|0)$ are called the λ-*Daehee numbers of the second kind.*

By (2.17), we can derive the generating function of $\widehat{D}_{n,q}(x)$ by (1.1) as follows :

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{D}_{n,q}(x|\lambda)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{-y}\left(-\lambda y+x\right)_n d\mu_q(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{-\lambda y+x}{n}d\mu_q(y)t^n\\ &= \int_{\mathbb{Z}_p} q^{-y}\left(1+t\right)^{-\lambda y+x} d\mu_q(y)\\ &= \frac{-\lambda\frac{q-1}{\log q}\log(1+t)}{1-(1+t)^{\lambda}}\left(1+t\right)^{x+\lambda}.\end{aligned} \tag{2.18}$$

From (1.5), (**??**) and (2.17), we get

$$\begin{aligned}\widehat{D}_{n,q}(x|\lambda) &= \int_{\mathbb{Z}_p}\left(-\lambda y+x\right)_n d\mu_q(y)\\ &= \int_{\mathbb{Z}_p}\sum_{l=0}^{n} S_1(n,l)(-\lambda y+x)^l d\mu_q(y)\\ &= \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l\int_{\mathbb{Z}_p}\left(y-\frac{x}{\lambda}\right)^l d\mu_q(y)\\ &= \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_{l,q}^{(-1,1)}\left(-\frac{x}{\lambda}\right)\\ &= (-1)^n\sum_{l=0}^{n}|S_1(n,l)|\lambda^l B_{l,q}^{(-1,1)}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{2.19}$$

By replacing qt to e^t-1 in the equation (2.18), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{D}_{n,q}(x|\lambda)\frac{1}{n!}\left(e^t-1\right)^n &= \frac{\frac{q-1}{\log q}\lambda t}{(e^{\lambda t}-1)}e^{(x+\lambda)t}\\ &= \sum_{n=0}^{\infty} B_{n,q}^{(-1,1)}\left(1+\frac{x}{\lambda}\right)\lambda^n\frac{t^n}{n!},\end{aligned} \tag{2.20}$$

and, by (1.4),

$$\sum_{n=0}^{\infty}\widehat{D}_{n,q}(x|\lambda)\frac{1}{n!}\left(e^t-1\right)^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{D}_{m,q}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.21}$$

Thus, from (2.19), (2.20) and (2.21), we have the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\begin{aligned}\widehat{D}_{n,q}(x|\lambda) =& \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_{l,q}^{(-1,1)}\left(-\frac{x}{\lambda}\right)\\ =&(-1)^n \sum_{l=0}^{n} |S_1(n,l)|\lambda^l B_{l,q}^{(-1,1)}\left(-\frac{x}{\lambda}\right),\end{aligned}$$

and

$$\lambda^n B_{n,q}^{(-1,1)}\left(1+\frac{x}{\lambda}\right) = \sum_{m=0}^{n} \widehat{D}_{m,q}(x|\lambda) S_2(n,m).$$

By (1.5), it is easy to show that $B_{n,q}^{(-1,1)}(-x) = (-1)^n B_{n,q}^{(-1,1)}(1+x)$. Hence, by Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *For $n \geq 0$,*

$$\widehat{D}_{n,q}(x|\lambda) = \sum_{l=0}^{n}\sum_{m=0}^{l} \widehat{D}_{m,q}(x|\lambda) S_1(n,l) S_2(l,m). \tag{2.22}$$

Now, we observe that

$$\begin{aligned}(-1)^n \frac{D_{n,q}(x|\lambda)}{n!} =&(-1)^n \int_{\mathbb{Z}_p} \binom{x+\lambda y}{n} d\mu_q(y)\\ =& \int_{\mathbb{Z}_p} \binom{-x-\lambda y+n-1}{n} d\mu_q(y)\\ =& \sum_{m=1}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{-x-\lambda y}{m} d\mu_q(y)\\ =& \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{\widehat{D}_{m,q}(-x|\lambda)}{m!},\end{aligned} \tag{2.23}$$

and, by the similar method to (2.23), we have

$$(-1)^n \frac{\widehat{D}_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{D_{n,q}(-x|\lambda)}{m!}. \tag{2.24}$$

Hence, by (2.23) and (2.24), we obtain the following theorem.

Theorem 2.7. *For $n \geq 1$, we have*

$$(-1)^n \frac{D_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{\widehat{D}_{m,q}(\lambda|-x)}{m!},$$

and

$$(-1)^n \frac{\widehat{D}_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{D_{n,q}(\lambda|-x)}{m!}.$$

Now, we consider *q-analogue of higher-order λ-Daehee polynomials of second kind*. q-analogue of higher-order λ-Daehee polynomials of second kind are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$\begin{aligned}&\widehat{D}_{n,q}^{(k)}(x|\lambda)\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{-x_1-\cdots-x_k}(-\lambda(x_1+\cdots+x_k)+x)_n d\mu_q(x_1)\cdots d\mu_q(x_k)\end{aligned}\tag{2.25}$$

where n is an nonnegative integer and $k\in\mathbb{N}$. In the special case, $x=0$, $\widehat{D}_{n,q}^{(k)}(\lambda)=\widehat{D}_{n,q}^{(k)}(\lambda|0)$ are called the *q-analogue of higher-order λ-Daehee numbers of second kind*.

From (2.25), we can derive the generating function of $\widehat{D}_{n,q}^{(k)}(x|\lambda)$ as follows:

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{D}_{n,q}^{(k)}(x|\lambda)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{-x_1-\cdots-x_k}\binom{\lambda(x_1+\cdots+x_k)+x}{n}d\mu_q(x_1)\cdots d\mu_q(x_k)t^n\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{-x_1-\cdots-x_k}\,(1+t)^{-\lambda(x_1+\cdots+x_k)+x}\,d\mu_q(x_1)\cdots d\mu_q(x_k)\\&=\left(\frac{-\lambda\frac{q-1}{\log q}\log(1+t)}{1-(1+t)^{\lambda}}\,(1+t)^{\lambda}\right)^k(1+t)^x.\end{aligned}\tag{2.26}$$

By (2.25),

$$\begin{aligned}&\widehat{D}_{n,q}^{(k)}(x|\lambda)\\&=\sum_{m=0}^{n}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda(x_1+\cdots+x_k)+x)^m d\mu_q(x_1)\cdots d\mu_q(x_k)\\&=\sum_{m=0}^{n}S_1(n,m)(-\lambda)^m\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+\cdots+x_k-\frac{x}{\lambda}\right)^m d\mu_q(x_1)\cdots d\mu_q(x_k)\\&=\sum_{m=0}^{n}S_1(n,m)(-\lambda)^m B_{m,q}^{(-1,k)}\left(-\frac{x}{\lambda}\right)\\&=(-1)^n\sum_{m=0}^{n}\lambda^m|S_1(n,m)|B_{m,q}^{(-1,k)}\left(-\frac{x}{\lambda}\right).\end{aligned}\tag{2.27}$$

From (1.5), we know that $B_{n,q}^{(-1,k)}(-x)=(-1)^n B_{n,q}^{(-1,k)}(k+x)$. Hence, by (2.27), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$\begin{aligned}\widehat{D}_{n,q}^{(k)}(x|\lambda) =& \sum_{m=0}^{n} S_1(n,m)(-\lambda)^m B_{m,q}^{(-1,k)}\left(-\frac{x}{\lambda}\right)\\ =&(-1)^n \sum_{m=0}^{n}(-\lambda)^m |S_1(n,m)| B_{m,q}^{(-1,k)}\left(k+\frac{x}{\lambda}\right).\end{aligned}$$

In (2.26), by replacing t by $e^t - 1$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_{n,q}^{(k)}(x|\lambda)\frac{(e^t-1)^n}{n!} =& e^{(x+\lambda k)}\left(\frac{\frac{q-1}{\log q}\lambda t}{e^{\lambda t}-1}\right)^k\\ =& \sum_{n=0}^{\infty}\lambda^n B_{n,q}^{(-1,k)}\left(\frac{x}{\lambda}+k\right)\frac{t^n}{n!},\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_{n,q}^{(k)}(x|\lambda)\frac{1}{n!}\left(e^t-1\right)^n =& \sum_{n=0}^{\infty} \widehat{D}_{n,q}^{(k)}(x|\lambda)\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}\\ =& \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_{n,q}^{(k)}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.29}$$

By (2.28) and (2.29), we obtain the following theorem.

Corollary 2.9. *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$B_{n,q}^{(-1,k)}\left(\frac{x}{\lambda}+k\right) = \lambda^{-n}\sum_{m=0}^{n}\widehat{D}_{m,q}^{(k)}(x|\lambda)S_2(n,m).$$

By Theorem 2.8 and Corollary 2.9, we obtain the following corollary.

Corollary 2.10. *For $n \geq 0$, we have*

$$\widehat{D}_{n,q}^{(k)}(x|\lambda) = \sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^m \widehat{D}_{l,q}^{(k)}(x|\lambda)q^{n-l}S_1(n,m)S_2(m,l).$$

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