A Note on a $q$-Analogue of
$\lambda$-Dahee Polynomials

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Abstract

In this paper, we introduce a new $q$-analogue of the $\lambda$-Dahee numbers and polynomials of the first kind and the second kind, and derive some new interesting identities.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{\log p}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1 - q^x}{1 - q}.$$ 

Note that $\lim_{q \to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$. 

Let $UD({\mathbb Z}_p)$ be the space of uniformly differentiable functions on $\mathbb Z_p$. For $f \in UD({\mathbb Z}_p)$, the $p$-adic invariant integral on $\mathbb Z_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb Z_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see } [6, 7, 8]). \tag{1.1}$$

Let $f_1$ be the translation of $f$ with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \quad \text{where } f'(0) = \left| \frac{df(x)}{dx} \right|_{x=0}. \tag{1.2}$$

As it is well-known fact, the Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n, l) x^l, \tag{1.3}$$

and the Stirling number of the second kind is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!}, \quad (\text{see } [3, 14]). \tag{1.4}$$

It is well known that the $(h, q)$-Bernoulli polynomials are defined by the generating function to be

$$\left( q - 1 + \frac{q-1}{\log q} (h \log q + t) \right) e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h,1)}(x) \frac{t^n}{n!}, \quad (\text{see } [10, 15, 16]), \tag{1.5}$$

where $h$ is an integer. When $x = 0$ and $h = -1$, $B_{n,q}^{(-1,1)}(0) = B_{n,q}^{(-1,1)}$ are called the ordinary $q$-Bernoulli numbers.

Recently, D. S. Kim and T. Kim introduced the Daehee polynomials of the first kind are defined by the generating function to be

$$\log(1 + t) \left( \frac{1 - q}{t} (1 + t)^x \right) = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \tag{1.6}$$

and the Daehee polynomials of the second kind are given by

$$\log(1 + t) \left( \frac{1 + q - q t}{1 - q - qt} (1 + t)^x \right) = \sum_{n=0}^{\infty} \hat{D}_n(x) \frac{t^n}{n!}, \quad (\text{see } [4, 7, 11, 13]), \tag{1.7}$$

and Cho et. al. defined the $q$-Daehee polynomials as follows.

$$\frac{1 - q + \frac{q - 1}{\log q} \log(1 + t)}{1 - q - qt} (1 + t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!}, \quad (\text{see } [2]). \tag{1.8}$$

In the viewpoint of generalization of the Dahee polynomials, we consider the $q$-analogue of the $\lambda$-Daehee polynomials are defined to be

$$\sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{t^n}{n!} = (1 + t)^x \frac{q - 1 + \frac{q-1}{\log q} (\lambda \log(1 + t) - \log q)}{(1 + t)^\lambda - 1}. \tag{1.9}$$
When $x = 0$, $D_{n,q}(\lambda) = D_{n,q}(0|\lambda)$ are called the $q$-analogue of the $\lambda$-Daehee numbers. 

In the past years, Kim et. al. have studied the various generalization of Daehee polynomials (see [2, 5, 9, 11, 12, 13]), and in [1], authors give new $q$-analogue of Changhee numbers and polynomials.

In this paper, we introduce a new $q$-analogue of the $\lambda$-Daehee numbers and polynomials of the first kind and the second kind, which are called the Witt-type formula for the $q$-analogue of $\lambda$-Daehee polynomials. We can derive some new interesting identities related to the $q$-Daehee polynomials.

2. Witt-type formula for the $q$-analogue of $n$th $\lambda$-Daehee polynomials

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

First, we consider the following integral representation associated with falling factorial sequences:

$$
\int_{\mathbb{Z}_p} q^{-y}(x + \lambda y)_n d\mu_q(y), \text{ where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.
$$

(2.1)

By (2.1),

$$
\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y}(x + \lambda y)_n d\mu_q(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y} \binom{x + \lambda y}{n} d\mu_q(y) t^n
$$

(2.2)

$$
= \int_{\mathbb{Z}_p} q^{-y} (1 + t)^{x+\lambda y} d\mu_q(y)
$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, we get

$$
\int_{\mathbb{Z}_p} q^{-y} (1 + t)^{x+\lambda y} d\mu_q(y) = (1 + t)^x \frac{q - 1 + \frac{q-1}{\log q} (\lambda \log(1 + t) - \log q)}{(1 + t)^\lambda - 1}
$$

(2.3)

$$
= \sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{t^n}{n!}.
$$

By (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For $n \geq 0$, we have

$$
D_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} q^{-y}(x + \lambda y)_n d\mu_q(y).
$$

In (2.3), by replacing $t$ by $(e^t - 1)$, we have

$$
\sum_{n=0}^{\infty} D_{n,q}(x|\lambda) (e^t - 1)^n \frac{t^n}{n!} = e^{xt} \frac{q - 1 + \frac{q-1}{\log q} (\lambda t - \log q)}{e^{\lambda t} - 1}
$$

(2.4)

$$
= \sum_{n=0}^{\infty} \lambda^n B^{(-1,1)}_{n,q} \left( \frac{x}{\lambda} \right) \frac{t^n}{n!}.
$$
and
\[ \sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{1}{n!} \left( e^t - 1 \right)^n = \sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} D_{m,q}(x|\lambda) S_2(m, n) \right) \frac{t^n}{n!}. \quad (2.5) \]

By (2.4) and (2.5), we obtain the following corollary.

**Corollary 2.2.** For \( n \geq 0 \), we have
\[ B_{n,q}^{(-1,1)} \left( \frac{x}{\lambda} \right) = \sum_{m=0}^{n} D_{m,q}(x|\lambda) \lambda^{-n} S_2(n, m). \]

By the Theorem 2.1,
\[ D_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} q^{-y}(x + \lambda y)_n \, d\mu_q(y) \]
\[ = \sum_{l=0}^{n} S_1(n, l) \int_{\mathbb{Z}_p} q^{-y}(x + \lambda y)^l \, d\mu_q(y). \quad (2.6) \]

By (1.2), we can derive easily that
\[ \int_{\mathbb{Z}_p} q^{-y} e^{(x+\lambda y)t} \, d\mu_q(y) = \frac{q - 1 + \frac{q^{-1}}{\log q} (\lambda t - \log q)}{e^{\lambda t} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(-1,1)} \left( \frac{x}{\lambda} \right) \frac{(\lambda t)^n}{n!} \]
\[ = \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y}(x + \lambda y)^l \, d\mu_q(y) \frac{t^l}{l!}, \quad (2.7) \]

and so
\[ \lambda^n B_{n,q}^{(-1,1)} \left( \frac{x}{\lambda} \right) = \int_{\mathbb{Z}_p} q^{-y} (x + \lambda y)^n \, d\mu_q(y), \quad (n \geq 0). \quad (2.8) \]

By (2.2), (2.7) and (2.8), we obtain the following corollary.

**Corollary 2.3.** For \( n \geq 0 \), we have
\[ D_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^l B_{l,q}^{(-1,1)} \left( \frac{x}{\lambda} \right) \]
\[ = \sum_{l=0}^{n} |S_1(l, n)| (-1)^{n-l} \lambda^l B_{l,q}^{(-1,1)} \left( \frac{x}{\lambda} \right). \]

From now on, we consider \( q \)-analogue of \( n \)th \( \lambda \)-Daehee polynomials of order \( k \) \((k \in \mathbb{N})\). \( q \)-analogue of \( nt \) \( \lambda \)-Daehee polynomials of order \( k \) are defined by the
multivariable $p$-adic invariant integral on $\mathbb{Z}_p$:

$$D_{n,q}^{(k)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^{k} x_i}(\lambda(x_1 + \cdots + x_k) + x)^{\frac{n}{t}} d\mu_q(x_1) \cdots d\mu_q(x_k)$$

(2.9)

where $n$ is an nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D_{n,q}^{(k)}(\lambda) = D_{n,q}^{(k)}(\lambda|0)$ are called the $\lambda$-Dahee numbers of order $k$ with $q$-parameter.

From (2.9), we can derive the generating function of $D_{n,q}^{(k)}(x)$ as follows:

$$\sum_{n=0}^{\infty} D_{n,q}^{(k)}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^{k} x_i}(\lambda(x_1 + \cdots + x_k) + x)^{\frac{n}{t}} d\mu_q(x_1) \cdots d\mu_q(x_k)t^n$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^{k} x_i}(1 + t)^{\lambda(x_1 + \cdots + x_k) + x} d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= (1 + t)^x \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^{k} x_i}(1 + t)^{\lambda(x_1 + \cdots + x_k)} d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= (1 + t)^x \left( q - 1 + \frac{q^{-1}}{\log q} \frac{\lambda \log(1 + t) - \log q}{(1 + t)^\lambda - 1} \right)^k.$$

Note that, by (2.9),

$$D_{n,q}^{(k)}(x|\lambda) = \sum_{m=0}^{n} S_{1}(n, m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^{k} x_i}(\lambda(x_1 + \cdots + x_k) + x)^{m} d\mu_q(x_1) \cdots d\mu_q(x_k),$$

(2.11)

and

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{h \sum_{i=1}^{k} x_i e^{(x_1 + \cdots + x_k + x)t}} d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= \left( q - 1 + \frac{q^{-1}}{\log q} (h \log q + t) \right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h,k)}(x) \frac{t^n}{n!},$$

(2.12)

where $B_{n,q}^{(h,k)}(x)$ are $(h,q)$-Bernoulli polynomials of order $k$. By (2.12), we can derive easily

$$B_{n,q}^{(-1,k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-\sum_{i=1}^{k} x_i(x_1 + \cdots + x_k + x)^n} d\mu_q(x_1) \cdots d\mu_q(x_k).$$

(2.13)
Thus, by (2.11) and (2.13), we have

\[
D^{(k)}_{n,q}(x|\lambda) = \sum_{m=0}^{n} S_1(n,m)\lambda^m B_m^{(-1,k)} \left( \frac{x}{\lambda} \right)
\]

(2.14)

\[
= \sum_{m=0}^{n} |S_1(n,m)|(-1)^{n-m} B_m^{(-1,k)} \left( \frac{x}{\lambda} \right).
\]

In (2.10), by replacing \( t \) by \((e^t - 1)\), we get

\[
\sum_{n=0}^{\infty} D^{(k)}_{n,q}(x|\lambda) (e^t - 1)^n = e^{xt} \left( \frac{q^{-1} \log_q (\lambda t - \log q)}{e^t - 1} \right)^k
\]

(2.15)

\[
= \sum_{n=0}^{\infty} \lambda^n B_n^{(-1,k)} \left( \frac{x}{\lambda} \right) t^n \frac{n!}{n!},
\]

and

\[
\sum_{n=0}^{\infty} D^{(k)}_{n,q}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \sum_{n=0}^{\infty} D^{(k)}_{n,q}(x|\lambda) \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}
\]

(2.16)

\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} D^{(k)}_{n,q}(x|\lambda) S_2(m,n) \right) \frac{t^m}{m!}.
\]

By (2.14), (2.15) and (2.16), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \) and \( k \in \mathbb{N} \), we have

\[
D^{(k)}_{n,q}(x|\lambda) = \sum_{m=0}^{n} S_1(n,m)\lambda^m B^{(-1,k)}_{m,q} \left( \frac{x}{\lambda} \right)
\]

\[
= \sum_{m=0}^{n} |S_1(n,m)|(-1)^{n-m} B^{(-1,k)}_{m,q} \left( \frac{x}{\lambda} \right),
\]

and

\[
B^{(-1,k)}_{n,q} \left( \frac{x}{\lambda} \right) = \lambda^{-n} \sum_{m=0}^{n} D^{(k)}_{n,q}(x|\lambda) S_2(m,n) S_2(n,m).
\]

Now, we consider the \( \lambda \)-Daehee polynomials of the second kind.

\[
\hat{D}_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} q^{-y}(-\lambda y + x) d\mu_q(y), \quad (n \geq 0).
\]

(2.17)

In the special case, \( x = 0 \), \( \hat{D}_{n,q}(\lambda) = \hat{D}_{n,q}(\lambda|0) \) are called the \( \lambda \)-Daehee numbers of the second kind.
A note on a $q$-analogue of $\lambda$-Dahee polynomials

By (2.17), we can derive the generating function of $\hat{D}_{n,q}(x)$ by (1.1) as follows:

$$\sum_{n=0}^{\infty} \hat{D}_{n,q}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{-y} (-\lambda y + x)_n d\mu_q(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( -\lambda y + x \right)_n d\mu_q(y) t^n = \int_{\mathbb{Z}_p} q^{-y} (1 + t)^{-\lambda y + x} d\mu_q(y) = \frac{-\lambda^{n-1} \log(1 + t)}{1 - (1 + t)^\lambda} (1 + t)^{x+\lambda}. \quad (2.18)$$

From (1.5), (2.17) and (2.18), we get

$$\hat{D}_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} (\lambda y + x)_n d\mu_q(y) = \int_{\mathbb{Z}_p} \sum_{l=0}^{n} S_1(n,l)(-\lambda y + x)^l d\mu_q(y) = \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l \int_{\mathbb{Z}_p} \left( y - \frac{x}{\lambda} \right)^l d\mu_q(y) \quad (2.19)$$

$$= \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_{l,q}^{(-1,1)} \left( -\frac{x}{\lambda} \right) = (-1)^n \sum_{l=0}^{n} |S_1(n,l)| \lambda^l B_{l,q}^{(-1,1)} \left( -\frac{x}{\lambda} \right).$$

By replacing $qt$ to $e^t - 1$ in the equation (2.18), we have

$$\sum_{n=0}^{\infty} \hat{D}_{n,q}(x|\lambda) \frac{1}{n!} \left( e^t - 1 \right)^n = \frac{q^{-1} \lambda t}{(e^t - 1)^{e^t}} e^{(x+\lambda)t} \quad (2.20)$$

and, by (1.4),

$$\sum_{n=0}^{\infty} \hat{D}_{n,q}(x|\lambda) \frac{1}{n!} \left( e^t - 1 \right)^n = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \hat{D}_{m,q}(x|\lambda) S_2(n,m) \right) \frac{t^n}{n!}. \quad (2.21)$$

Thus, from (2.19), (2.20) and (2.21), we have the following theorem.
Theorem 2.5. For \( n \geq 0 \), we have

\[
\widehat{D}_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n, l)(-\lambda)^l B_{l,q}^{(-1,1)} \left(-\frac{x}{\lambda}\right)
\]

\[
= (-1)^n \sum_{l=0}^{n} |S_1(n, l)|\lambda^l B_{l,q}^{(-1,1)} \left(-\frac{x}{\lambda}\right),
\]

and

\[
\lambda^n B_{n,q}^{(-1,1)} \left(1 + \frac{x}{\lambda}\right) = \sum_{m=0}^{n} \widehat{D}_{m,q}(x|\lambda) S_2(n, m).
\]

By (1.5), it is easy to show that \( B_{n,q}^{(-1,1)}(-x) = (-1)^n B_{n,q}^{(-1,1)}(1 + x) \). Hence, by Theorem 2.5, we obtain the following corollary.

Corollary 2.6. For \( n \geq 0 \),

\[
\widehat{D}_{n,q}(x|\lambda) = \sum_{l=0}^{n} \sum_{m=0}^{l} \widehat{D}_{m,q}(x|\lambda) S_1(n, l) S_2(l, m).
\]

Now, we observe that

\[
(-1)^n \frac{D_{n,q}(x|\lambda)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \left(\frac{x + \lambda y}{n}\right) d\mu_q(y)
\]

\[
= \int_{\mathbb{Z}_p} \left(\frac{-x - \lambda y + n - 1}{n}\right) d\mu_q(y)
\]

\[
= \sum_{m=1}^{n} \left(\frac{n - 1}{n - m}\right) \int_{\mathbb{Z}_p} \left(\frac{-x - \lambda y}{m}\right) d\mu_q(y)
\]

\[
= \sum_{m=1}^{n} \left(\frac{n - 1}{n - m}\right) \frac{\widehat{D}_{m,q}(-x|\lambda)}{m!},
\]

and, by the similar method to (2.23), we have

\[
(-1)^n \frac{\widehat{D}_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \left(\frac{n - 1}{n - m}\right) \frac{D_{n,q}(-x|\lambda)}{m!}.
\]

Hence, by (2.23) and (2.24), we obtain the following theorem.

Theorem 2.7. For \( n \geq 1 \), we have

\[
(-1)^n \frac{D_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \left(\frac{n - 1}{n - m}\right) \frac{\widehat{D}_{m,q}(\lambda - x)}{m!},
\]

and

\[
(-1)^n \frac{\widehat{D}_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \left(\frac{n - 1}{n - m}\right) \frac{D_{n,q}(\lambda - x)}{m!}.
\]
Now, we consider \( q \)-analogue of higher-order \( \lambda \)-Daehee polynomials of second kind. \( q \)-analogue of higher-order \( \lambda \)-Daehee polynomials of second kind are defined by the multivariant \( p \)-adic invariant integral on \( \mathbb{Z}_p \):

\[
\hat{D}_{n,q}^{(k)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_k} \left(-\lambda(x_1 + \cdots + x_k) + x\right) \mu_q(x_1) \cdots \mu_q(x_k)
\]

where \( n \) is an nonnegative integer and \( k \in \mathbb{N} \). In the special case, \( x = 0 \), \( \hat{D}_{n,q}^{(k)}(\lambda) = \hat{D}_{n,q}^{(k)}(\lambda|0) \) are called the \( q \)-analogue of higher-order \( \lambda \)-Daehee numbers of second kind.

From (2.25), we can derive the generating function of \( \hat{D}_{n,q}^{(k)}(x|\lambda) \) as follows:

\[
\sum_{n=0}^{\infty} \hat{D}_{n,q}^{(k)}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_k} \left(\lambda(x_1 + \cdots + x_k) + x\right) \mu_q(x_1) \cdots \mu_q(x_k)t^n
\]

\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_k} (1 + t)^{-\lambda(x_1 + \cdots + x_k) + x} \mu_q(x_1) \cdots \mu_q(x_k)
\]

\[
= \left(\frac{-\lambda q^{-1} \log(1 + t)}{1 - (1 + t)^{\lambda}} (1 + t)^{\lambda}\right)^k (1 + t)^x.
\]

(2.26)

By (2.25),

\[
\hat{D}_{n,q}^{(k)}(x|\lambda) = \sum_{m=0}^{n} S_1(n, m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda(x_1 + \cdots + x_k) + x)^m \mu_q(x_1) \cdots \mu_q(x_k)
\]

\[
= \sum_{m=0}^{n} S_1(n, m)(-\lambda)^m \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - \frac{x}{\lambda})^m \mu_q(x_1) \cdots \mu_q(x_k)
\]

\[
= \sum_{m=0}^{n} S_1(n, m)(-\lambda)^m B_{m,q}^{(-1,k)} \left(-\frac{x}{\lambda}\right)
\]

\[
= (-1)^n \sum_{m=0}^{n} \lambda^m |S_1(n, m)| B_{m,q}^{(-1,k)} \left(-\frac{x}{\lambda}\right).
\]

(2.27)

From (1.5), we know that \( B_{n,q}^{(-1,k)}(-x) = (-1)^n B_{n,q}^{(-1,k)}(k + x) \). Hence, by (2.27), we obtain the following theorem.
Theorem 2.8. For \( n \geq 0 \), we have
\[
\hat{D}^{(k)}_{n,q}(x|\lambda) = \sum_{m=0}^{n} S_1(n,m)(-\lambda)^m B^{(-1,k)}_{m,q}\left(-\frac{x}{\lambda}\right)
\]
\[
= (-1)^n \sum_{m=0}^{n} (-\lambda)^m |S_1(n,m)| B^{(-1,k)}_{m,q}\left(k + \frac{x}{\lambda}\right).
\]

In (2.26), by replacing \( t \) by \( e^t - 1 \), we get
\[
\sum_{n=0}^{\infty} \hat{D}^{(k)}_{n,q}(x|\lambda) \frac{(e^t - 1)^n}{n!} = e^{(x+\lambda k)} \left( \frac{q-1}{\log q} \frac{\lambda t}{e^{\lambda t} - 1} \right)^k
\]
\[
= \sum_{n=0}^{\infty} \lambda^n B^{(-1,k)}_{n,q} \left( \frac{x}{\lambda} + k \right) \frac{t^n}{n!},
\]
and
\[
\sum_{n=0}^{\infty} \hat{D}^{(k)}_{n,q}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \sum_{n=0}^{\infty} \hat{D}^{(k)}_{n,q}(x|\lambda) \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}^{(k)}_{n,q}(x|\lambda) S_2(n,m) \right) \frac{t^m}{m!}. \tag{2.29}
\]

By (2.28) and (2.29), we obtain the following theorem.

Corollary 2.9. For \( n \geq 0 \) and \( k \in \mathbb{N} \), we have
\[
B^{(-1,k)}_{n,q} \left( \frac{x}{\lambda} + k \right) = \lambda^{-n} \sum_{m=0}^{n} \hat{D}^{(k)}_{m,q}(x|\lambda) S_2(n,m).
\]

By Theorem 2.8 and Corollary 2.9, we obtain the following corollary.

Corollary 2.10. For \( n \geq 0 \), we have
\[
\hat{D}^{(k)}_{n,q}(x|\lambda) = \sum_{m=0}^{n} \sum_{l=0}^{m} (-1)^m \hat{D}^{(k)}_{l,q}(x|\lambda) q^{n-l} S_1(n,m) S_2(m,l).
\]

References:
A note on a $q$-analogue of $\lambda$-Daehee polynomials


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