Edge-Balanced $H^{(3)}$-Designs

Mario Gionfriddo

Dipartimento di Matematica e Informatica
Universitá di Catania, Italy

Abstract

Given an hypergraph $H^{(3)}$, uniform of rank 3, an $H^{(3)}$-decomposition of the complete hypergraph $\lambda K_v^{(3)}$, of order $v$ and index $\lambda$, is a collection of hypergraphs, all isomorphic to $H^{(3)}$, whose edge-sets partition the edge-set of $\lambda K_v^{(3)}$. An $H^{(3)}$-decomposition of $\lambda K_v^{(3)}$ is also called an $H^{(3)}$-design and the hypergraphs of the partition are said to be the blocks. An $H^{(3)}$-design is said to be balanced if the number of blocks containing any given vertex of $K_v^{(3)}$ is a constant. In this paper, we introduce the new concept of edge-balanced $H^{(3)}$-design, pointing out some research-problems and studying some cases.

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1 Introduction

Let $\lambda K_v^{(3)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank 3, defined in a vertex set $X = \{x_1, x_2, ..., x_v\}$ and having every $E \in E$ of multiplicity $\lambda$ [1]. This means that $\mathcal{E}$ is the family of all the triples contained in $X$ and that every triple $\{x, y, z\} \subseteq X$ is contained in exactly $\lambda$ subsets $E \in \mathcal{E}$. We will indicate such a family $\mathcal{E}$ by $\lambda P_3(X)$.

Let $H^{(3)}$ be subhypergraph of $\lambda \cdot K_v^{(3)}$, the complete 3-uniform hypergraph where every triple of vertices has multiplicity $\lambda$. An $H^{(3)}$-decomposition of $\lambda \cdot K_v^{(3)}$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of $\lambda \cdot K_v^{(3)}$.
into subsets all of which yield subhypergraphs all isomorphic to $H^{(3)}$. An $H^{(3)}$-decomposition $\Sigma = (X, \mathcal{B})$ of $\lambda \cdot K^{(3)}_v$ is also called an $H^{(3)}$-design of order $v$ and index $\lambda$ and the classes of the partition $\mathcal{B}$ of $P_3(X)$ are said to be the blocks of $\Sigma$ [1].

An $H^{(3)}$-design of order $v$ and index $\lambda$ can be also defined as a pair $\Sigma = (X, \mathcal{B})$, where $X$ is a finite set of $v$ elements (vertices) and $\mathcal{B}$ is a family of hypergraphs $H^{(3)}$ (blocks), such that every triple of distinct vertices forms an edge in exactly $\lambda$ blocks $H^{(3)}$ [7].

An $H^{(3)}$-design is said to be balanced if the degree of each vertex $x \in X$ (number of blocks of $\Sigma$ containing $x$) is a constant [16,20].

The concept of $H^{(3)}$-decomposition of $K^{(3)}_v$ is the natural generalization to hypergraph uniform of rank 3 of the more classical $G$-designs [2-6][8-26]. Much work about $G$-designs has been done in these last years, with many interesting results and open problems, which can be found in the literature. In the references some very recent results of the authors are cited, also regarding the determination of the spectrum for balanced $G$-designs, and observe that many of the problems examined there can be studied for $H^{(3)}$-designs [see all the references].

In what follows:

1) $P^{(3)}(2,4)$ will be the path-hypergraph having four vertices $x, y, z, t$ and edges $\{x, y, z\}, \{x, y, t\}$: we will indicate it by $[z, (x, y), t]$.

2) $P^{(3)}(1,5)$ will be the path-hypergraph having five vertices $x, y, z, t, w$ and edges $\{x, y, z\}, \{x, t, w\}$. we will indicate it by $[y, z, (x), t, w]$.

The spectrum of $P^{(3)}(2,4)$-designs and the spectrum of $P^{(3)}(1,5)$-designs have been determined in [7], together to other many results about $H^{(3)}$-designs.

**Theorem 1.1:** There exists a $P^{(3)}(2,4)$-design of order $v$ and index $\lambda$ if and only if $v \geq 4$ and $v \equiv 0 \mod 2$ or $\lambda(v-1) \equiv 0 \mod 4$.

**Theorem 1.2:** There exists a $P^{(3)}(1,5)$-design of order $v$ and index $\lambda$ if and only if $v \geq 5$ and $v \equiv 0 \mod 2$ or $\lambda(v-1) \equiv 0 \mod 4$.

In this paper we give the new concept of edge-balanced $H^{(3)}$-design, where the balance regards the pairs of vertices contained in the edges of $H^{(3)}$, instead of the classical balance of the vertices.
In what follows, fixed an hypergraph $H^{(3)} = (V, \mathcal{E})$, we will call $3$-edges the triples of $V$ contained in the family $\mathcal{E}$ and edges the pairs of $V$ contained in the 3-edges of $\mathcal{E}$; such edges will be indicated by $[x, y]$.

## 2 Main Definitions

Let $H^{(3)}$ be an hypergraph uniform of rank 3, with $n$ vertices. An $H^{(3)}$-design $\Sigma = (X, B)$ is said to be balanced if the degree $d(x)$ of a vertex $x \in X$ is a constant.

Observe that, if $H^{(3)}$ is regular, then the correspondent $H^{(3)}$-designs are all balanced, hence the notion of balanced $H^{(3)}$-design becomes meaningful only for a non-regular hypergraphs $H^{(3)}$.

**Definition:** We say that an $H^{(3)}$-design is EDGE-balanced if, for every pair of distinct vertices $x, y \in X$, the number of 3-edges of $H^{(3)}$ containing both $x, y$ is a constant.

If $x, y \in X$, we will indicate by $d(x, y)$ the number of 3-edges of $H^{(3)}$ containing the pair $x, y$ and we will call it the degree of $\{x, y\}$. Further, if $B=[a, (b, c), d]$ or $B=[a, b, (c), d, e]$ is an hypergraph respectively of type $P^{(3)}(2, 4)$ or $P^{(3)}(1, 5)$, defined in $\mathbb{Z}_v$, we will say the translates of $B$ all the hypergraphs of type $B_i=[a+i, (b+i, c+i), d+i]$ or $B_i=[a+i, b+i, (c+i), d+i, e+i]$, for every $i \in \mathbb{Z}_v$.

The hypergraph $B$ will be said to be a base-block of the system having for translates all the hypergraphs $B_i$.

## 3 Edge-Balanced $H^{(3)}$-designs

In this section we determine necessary conditions for edge-balanced $H^{(3)}$-designs.

**Theorem 3.1 :** Let $\Sigma = (X, B)$ be an edge-balanced $H^{(3)}$-design of order $v$ and index $\lambda$. If $H^{(3)}$ has $n$ vertices, $p$ edges and $m$ 3-edges, then for every $x, y \in X$, $x \neq y$:

$$d_{x,y} = D = \frac{\lambda p(v - 2)}{3m}.$$

**Proof.** Let $\Sigma = (X, B)$ be an edge-balanced $H^{(3)}$-design of order $v$ and index $\lambda$. For every pair $x, y \in X$, the degree of the edge $[x, y]$ is a constant $d(x, y) = D$. Since the number of positions that a pair of vertices can occupy, as an
edge, in a block of $\Sigma$ is $p$, it follows: $p \cdot |B| = D \cdot (\frac{v}{2})$. From which, since $|B| = \lambda v(v - 1)(v - 2)/6m$, it follows that:

$$D = \frac{\lambda p(v - 2)}{3m}.$$ 

\[\square\]

4 Edge-Balanced $P^{(3)}(2, 4)$-designs

In this section we determine necessary conditions for edge-balanced $P^{(3)}(2, 4)$-designs. If $[z, (x, y), t]$ is any hypergraph $P^{(3)}(2, 4)$, we say that the edge $[x, y]$ occupies the central position of the $P^{(3)}(2, 4)$ and all the others edges occupy a lateral position. For every pair of vertices $x, y \in X$ of a $P^{(3)}(2, 4)$-design $\Sigma = (X, B)$, we will indicate by $C_{x,y}$ the central degree of the edge $[x, y]$, which is the number of 3-edges of the $P^{(3)}(2, 4)$-design containing $\{x, y\}$ as an edge in the central position, and by $L_{x,y}$ the lateral degree of $[x, y]$, which is the number of 3-edges of the $P^{(3)}(2, 4)$-design containing $\{x, y\}$ as an edge in a lateral position. Observe that in any $P^{(3)}(2, 4)$ it is: $n = 4$ (number of vertices, $p = 5$ (number of edges), $m = 2$ (number of 3-edges).

**Theorem 4.1**: If $\Sigma = (X, B)$ is an edge-balanced $P^{(3)}(2, 4)$-design of order $v$, then:

1. $d_{x,y} = D = \frac{5\lambda(v-2)}{6},$ for every $x, y \in X, x \neq y$;
2. $C_{x,y} = C = \frac{\lambda(v-2)}{6},$ for every $x, y \in X, x \neq y$;
3. $L_{x,y} = L = \frac{2\lambda(v-2)}{3},$ for every $x, y \in X, x \neq y$.

**Proof.** Let $\Sigma = (X, B)$ be an edge-balanced $P^{(3)}(2, 4)$-design of order $v$.

1. From Theorem 3.1, since in any $P^{(3)}(2, 4)$ it is $p = 5, m = 2$, then:

$$d_{x,y} = D = \frac{5\lambda(v-2)}{6},$$

for every $x, y \in X, x \neq y$.

2. Since there is exactly one central position in every block of $\Sigma$, the total number of central positions is equal to $|B| = \lambda v(v - 1)(v - 2)/12$. Therefore,
it follows:

\[ C_{x,y} \cdot \frac{v(v-1)}{2} = \frac{\lambda v(v-1)(v-2)}{12}, \]

from which: \( C_{x,y} = C = \lambda(v-2)/6. \)

(3) Since there are exactly four lateral positions in every block of \( \Sigma \), the total number of lateral positions is equal to \( 4 \cdot |\mathcal{B}| = v(v-1)(v-2)/3. \) Therefore, it follows:

\[ L_{x,y} \cdot \frac{v(v-1)}{2} = \frac{\lambda v(v-1)(v-2)}{3}, \]

from which: \( L_{x,y} = L = 2\lambda(v-2)/3. \) \( \square \)

From the previous Theorem it follows that:

**Theorem 4.2**: If \( \Sigma = (X, \mathcal{B}) \) is an edge-balanced \( P(3)(2, 4) \)-design of order \( v \) and index \( \lambda = 1 \), then \( v \equiv 2, \mod 6, v \geq 4. \)

**Proof.** Since \( |\mathcal{B}| = v(v-1)(v-2)/12 \), necessarily \( v \) is even or \( v = 4h + 1 \) for any \( h \in N, v \geq 4. \) From 1) of the previous Theorem, \( v \) is even, and therefore:

\( v \equiv 2, \mod 6, v \geq 4. \) \( \square \)

**Theorem 4.3**: If \( \Sigma = (X, \mathcal{B}) \) is an edge-balanced \( P(3)(2, 4) \)-design of order \( v \) and index \( \lambda = 2 \), then \( v \equiv 2, \mod 3, v \geq 5. \)

**Proof.** Since \( |\mathcal{B}| = v(v-1)(v-2)/6 \), every positive integer \( v, v \geq 4 \), can be the order of \( \Sigma \). From 1) of the previous Theorem, it follows: \( v \equiv 2, \mod 3, v \geq 5. \) \( \square \)

Now, we construct *edge-balanced\( P(3)(2, 4)\)-designs* of order \( v = 8 \) and index \( \lambda = 1. \)

**Theorem 4.4**: There exist edge-balanced \( P(3)(2, 4) \)-designs of order \( v = 8 \) and of index \( \lambda = 1. \)

**Proof.** Consider the following blocks defined in \( X = \mathbb{Z}_8 \):

\[
B_1 = [2, (0, 1), 5], B_2 = [3, (1, 2), 4], B_3 = [4, (2, 3), 5], B_4 = [5, (3, 4), 7],
\]

\[
B_5 = [6, (4, 5), 7], B_6 = [0, (5, 6), 7], B_7 = [3, (6, 7), 4], B_8 = [2, (7, 0), 6],
\]
\[ B_9 = [3, (0, 2), 5], B_{10} = [0, (1, 3), 5], B_{11} = [5, (2, 4), 7], B_{12} = [6, (3, 5), 7], \]
\[ B_{13} = [0, (4, 6), 3], B_{14} = [0, (5, 7), 2], B_{15} = [1, (6, 0), 2], B_{16} = [0, (7, 1), 6], \]
\[ B_{17} = [4, (0, 3), 6], B_{18} = [3, (1, 4), 5], B_{19} = [1, (2, 5), 6], B_{20} = [2, (3, 6), 1], \]
\[ B_{21} = [0, (4, 7), 1], B_{22} = [4, (5, 0), 3], B_{23} = [2, (6, 1), 4], B_{24} = [1, (7, 2), 3], \]
\[ B_{25} = [1, (0, 4), 2], B_{26} = [6, (1, 5), 7], B_{27} = [4, (2, 6), 7], B_{28} = [0, (3, 7), 1]. \]

If \( \mathcal{B} = \{ B_1, B_2, \ldots, B_{28}, \} \), it is possible to verify that \( \Sigma = (X, \mathcal{B}) \) is a \( P^{(3)}(2, 4) \)-design of order \( v = 8 \). Further, every pair \( x, y \) of distinct elements of \( X \) is contained in exactly one block as central edge and is contained in exactly two blocks as lateral edge. Therefore, for every pair \( x, y \in X, x \neq y \), it is \( d(x, y) = 3 \) and this implies that \( \Sigma \) is an edge-balanced \( P^{(3)}(2, 4) \)-design of order \( v = 8 \).

**Theorem 4.5**: There exist edge-balanced \( P^{(3)}(2, 4) \)-designs of order \( v = 5 \) and index \( \lambda = 2 \).

**Proof.** Consider the following base-blocks defined in \( X = Z_5 \):
\[ B = [2, (0, 1), 4], C = [3, (0, 2), 4]. \]
If \( \mathcal{B} \) is the collection of all the translates of \( B \) and \( C \), then we can verify that \( \Sigma = (X, \mathcal{B}) \) is edge-balanced \( P^{(3)}(2, 4) \)-designs of order \( v = 5 \) and index \( \lambda = 2 \).

5. **Edge-Balanced \( P^{(3)}(1, 5) \)-designs**

In this section we determine necessary conditions for edge-balanced \( P^{(3)}(1, 5) \)-designs. If \( [y, z, (x, w, t)] \) is any hypergraph \( P^{(3)}(1, 5) \), we say that the edges \( [x, y], [x, z], [x, w], [x, t] \) occupy one of the four central positions of the \( P^{(3)}(1, 5) \) and all the others edges occupy one of the two lateral positions. For every pair of vertices \( x, y \in X \) of a \( P^{(3)}(1, 5) \)-design \( \Sigma = (X, \mathcal{B}) \), we will indicate by \( C_{x,y} \) the central degree of the pair \( \{x, y\} \), which is the number of 3-edges of the \( P^{(3)}(1, 5) \) containing \( \{x, y\} \) as an edge in a central position, and by \( L_{x,y} \) the lateral degree of \( \{x, y\} \), which is the number of 3-edges of the \( P^{(3)}(1, 5) \) containing \( \{x, y\} \) as an edge in a lateral position. Observe that in any \( P^{(3)}(1, 5) \) it is: \( n = 5 \) (number of vertices, \( p = 6 \) (number of edges), \( m = 2 \) (number of 3-edges).
**Theorem 5.1**: If Σ = (X, B) is an edge-balanced $P^{(3)}(1, 5)$-design of order v and index λ, then $v \geq 5$ and:

1. $d_{x,y} = D = \lambda \cdot (v - 2)$, for every $x, y \in X$, $x \neq y$;
2. $C_{x,y} = C = \frac{2\lambda(v-2)}{3}$, for every $x, y \in X$, $x \neq y$;
3. $L_{x,y} = L = \frac{\lambda(v-2)}{3}$, for every $x, y \in X$, $x \neq y$.

**Proof.** Let Σ = (X, B) be an edge-balanced $P^{(3)}(1, 5)$-design of order v and index λ.

1. From Theorem 3.1, since in any $P^{(3)}(1, 5)$ it is $p = 6$, $m = 2$, then:
   \[ d_{x,y} = D = \lambda(v - 2), \]
   for every $x, y \in X$, $x \neq y$.

2. Since there are exactly four central positions in every block of Σ, the total number of central positions is equal to $4 \cdot |B| = \lambda \cdot v(v - 1)(v - 2)/3$. Therefore, it follows:
   \[ C_{x,y} \cdot \frac{v(v-1)}{2} = \frac{\lambda v(v-1)(v-2)}{3}, \]
   from which: $C_{x,y} = C = 2\lambda(v - 2)/3$.

3. Since there are exactly two lateral positions in every block of Σ, the total number of lateral positions is equal to $2 \cdot |B| = \lambda v(v - 1)(v - 2)/6$. Therefore, for every $x, y \in X$, $x \neq y$, it follows:
   \[ L_{x,y} \cdot \frac{v(v-1)}{2} = \frac{\lambda v(v-1)(v-2)}{6}, \]
   from which: $L_{x,y} = L = \lambda(v - 2)/3$.

From the previous Theorem it follows that:

**Theorem 5.2**: If Σ = (X, B) is an edge-balanced $P^{(3)}(1, 5)$-design of order v and index $\lambda = 1$, then $v \geq 5$ and $v \equiv 2, \mod 6$ or $v \equiv 5, \mod 12$.

**Proof.** Since $|B| = v(v - 1)(v - 2)/12$, necessarily v is even or $v = 4h + 1$ for any $h \in \mathbb{N}$, $v \geq 5$. From 1) of the previous Theorem, it is $v \equiv 2, \mod 3$, $v \geq 5$. Therefore, for v even it follows $v \equiv 2, \mod 6$, and for $v = 4h + 1$ it
follows $v \equiv 5, \mod 12$.

**Theorem 5.3**: If $\Sigma = (X, \mathcal{B})$ is an edge-balanced $P^{(3)}(1, 5)$-design of order $v$ and index $\lambda = 2$, then $v \equiv 2, \mod 3, v \geq 5$.

**Proof.** Since $|\mathcal{B}| = v(v-1)(v-2)/6$, every positive integer $v, v \geq 5$, can be the order of $\Sigma$. From 1) of the previous Theorem, it follows: $v \equiv 2, \mod 3, v \geq 5$.

Now, we see the only edge-balanced $P^{(3)}(1, 5)$-design of order $v = 5$ and index $\lambda = 1$ and construct an edge-balanced $P^{(3)}(1, 5)$-design of order $v = 14$ and index $\lambda = 2$.

To construct the following $H^{(3)}$-designs we apply the technique given in []. We will use the matrices $\mathcal{M}(5)$ and $\mathcal{M}(17)$ (see Appendix), which give all the triples that can be define in $\mathbb{Z}_5$ and in $\mathbb{Z}_{17}$, by ordered pairs. This technique uses the difference method, for which given a base-block $B$ defined in $\{x, y, z, \ldots\} \subseteq \mathbb{Z}_v$, its translates are all the blocks having elements $\{x + i, y + i, z + i, \ldots\}$, for every $i \in \mathbb{Z}_v$. Of course, for every base-block there are $v$ blocks (translates) of the system.

**Theorem 5.4**: There exist edge-balanced $P^{(3)}(1, 5)$-designs of index $\lambda = 1$ and order $v = 5$.

**Proof.** Let $X = \mathbb{Z}_5$. Consider in $X$ the base-block $B = [1, 4, (0), 2, 3]$. If $\mathcal{B}$ is the collection of all the translates $B_i = [1 + i, 4 + 1, i, 2 + i, 3 + i]$ of $B$, then we can verify that $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(1, 5)$-design of order $v = 5$ and index $\lambda = 1$. We can see that $C = 2, L = 1, D = 3$, for every pair $x, y \in X, x \neq y$. Therefore, $\Sigma$ is edge-balanced.

**Theorem 5.5**: There exist edge-balanced $P^{(3)}(1, 5)$-designs of index $\lambda = 2$ and order $v = 17$.

**Proof.** Let $X = \mathbb{Z}_{17}$. Observe that in any $P^{(3)}(1, 5)$-design, of order $v = 17$ and index $\lambda = 2$, there are $b = 680 = 17 \cdot 40$ blocks. Therefore, such a system could be defined by 40 base-blocks.

Consider the following $P^{(3)}(1, 5)$:

- $B_{1, (1, 8)} = [1, 2, (0), 7, 15]$, $B_{2, (1, 8)} = [1, 2, (0), 3, 11]$, $B_{3, (1, 8)} = [2, 3, (0), 7, 15]$,
- $B_{4, (1, 8)} = [2, 3, (0), 4, 12]$, $B_{5, (1, 8)} = [3, 4, (0), 6, 14]$,
- $B_{6, (1, 8)} = [3, 4, (0), 1, 9]$, $B_{7, (1, 8)} = [4, 5, (0), 3, 11]$, $B_{8, (1, 8)} = [4, 5, (0), 6, 14]$,
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$B_{3,(1,8)} = [5, 6, (0, 1), 9], B_{10,(1,8)} = [5, 6, (0), 4, 12],$

$B_{1,(2,7)} = [1, 3, (0), 5, 12], B_{2,(2,7)} = [1, 3, (0), 8, 12], B_{3,(2,7)} = [2, 4, (0), 1, 8],$

$B_{4,(2,7)} = [2, 4, (0), 7, 14], B_{5,(2,7)} = [3, 5, (0), 1, 8],$

$B_{6,(2,7)} = [3, 5, (0), 9, 16], B_{7,(2,7)} = [5, 7, (0), 8, 15], B_{8,(2,7)} = [5, 7, (0), 9, 16],$

$B_{9,(2,7)} = [6, 8, (0), 7, 14], B_{10,(2,7)} = [6, 8, (0), 5, 12],$

$B_{1,(3,6)} = [1, 4, (0), 2, 8], B_{2,(3,6)} = [1, 4, (0), 10, 16], B_{3,(3,6)} = [2, 5, (0), 1, 7],$

$B_{4,(3,6)} = [2, 5, (0), 4, 10], B_{5,(3,6)} = [3, 6, (0), 4, 10],$

$B_{6,(3,6)} = [3, 6, (0), 7, 13], B_{7,(3,6)} = [4, 7, (0), 2, 8], B_{8,(3,6)} = [4, 7, (0), 10, 16],$

$B_{9,(3,6)} = [5, 8, (0), 1, 7], B_{10,(3,6)} = [5, 8, (0), 7, 13],$

$B_{1,(4,5)} = [1, 5, (0), 3, 8], B_{2,(4,5)} = [1, 5, (0), 6, 11], B_{3,(4,5)} = [2, 6, (0), 3, 8],$

$B_{4,(4,5)} = [2, 6, (0), 4, 9], B_{5,(4,5)} = [3, 7, (0), 1, 6],$

$B_{6,(4,5)} = [3, 7, (0), 6, 11], B_{7,(4,5)} = [4, 8, (0), 1, 6], B_{8,(4,5)} = [4, 8, (0), 2, 7],$

$B_{9,(4,5)} = [11, 15, (0), 2, 7], B_{10,(4,5)} = [11, 15, (0), 4, 9].$

If $B$ is the collection of all the translates of these 40 base-blocks $B_{j,(a,b)}$, then we can verify that $\Sigma = (X, B)$ is a $P^{(3)}(1, 5)$-design of order $v = 17$ and index $\lambda = 2$. We can see that $C = 20, L = 10, D = 30$, for every pair $x, y \in X, x \neq y$. Therefore, $\Sigma$ is an edge-balanced.

To prove that $\Sigma = (X, B)$ is a $P^{(3)}(1, 5)$-design of order $v = 17$ and index $\lambda = 2$, we can see that, in the base-blocks $B_{j,(a,b)}$, all the ordered pairs described in the matrix $M(17)$ are used, one for every row, with molteplicity two. This means that $\Sigma$ has index $\lambda = 2$. Further, we can verify that the pair $(a, b)$ indicates the differences in the edges of the blocks which occupy the two lateral positions. Since in $Z_{17}$ the set of difference is $D = \{1, 2, 3, 4, 5, 6, 7, 8\}$, we can see that in the base-blocks it is $(a, b) = (1, 8), (2, 7), (3, 6), (4, 5)$, with molteplicity 10. This means that $L = 10$, for every pair $x, y \in X, x \neq y$, and that $C = 20$. $\square$
6 Appendix and Problems

We see some remark about the construction used in the previous Theorems, which can be useful to construct other systems. In Theorems 4.5 and 5.4 we have used the matrix \( M(5) \), in the Theorem 5.5 the matrix \( M(17) \). Both matrices are described at the last of this section. Further, for the construction of such a type of matrices see [XXX].

We can see that in every row of \( M(17) \) there are three ordered pairs. Each of them defines the same triple of differences among elements of a triple of \( Z_{17} \): fixed a triple \( \{x, y, z\} \subseteq Z_{17} \), with \( x < y < z \), if \( y - x = a, z - y = b, x - z = c \), then there is a row of \( M(17) \) containing the ordered pairs \( (a, b), (b, c), (c, a) \). Therefore, we can try to construct an \( H^{(3)} \)-design, defining the base-blocks associating exactly one pair with every 3-edge of \( H^{(3)} \), if the index is \( \lambda = 1 \), or two pairs (not necessarily distinct) if the index is \( \lambda = 2, \ldots \) .

\[
M(5) = \begin{bmatrix}
(1, 1) & (1, 3) & (3, 1) \\
(1, 2) & (2, 2) & (2, 1)
\end{bmatrix}
\]

In the case of Theorems 5.5, the matrix \( M(5) \) has only two rows. In the Theorem 5.5, the index of the system to construct is \( \lambda = 1 \), therefore we define the only base-block taking a pair in the first row and the other pair in the second row. While, in the case of Theorems 4.5, since the index of the system to construct is \( \lambda = 2 \), we define two base-blocks taking two pairs in the first row and the other two pairs in the second row.

In the case of Theorem 5.6, the matrix \( M(17) \) has exactly 40. Since the index of the system is \( \lambda = 2 \), necessarily we can try to define 20 base-blocks. In the construction used there, we have taken a pair in every row of \( M(17) \), repeating it two times.

The techniques used can be useful to determine the spectrum of \( H^{(3)} \)-designs. In particular we point out the following:

**Problem n.1** Determine the spectrum of edge-balanced \( P^{(3)}(2, 4) \)-designs of order \( v \) and index \( \lambda \). At first consider the case \( \lambda = 1 \).

**Problem n.2** Determine the spectrum of edge-balanced \( P^{(3)}(1, 5) \)-designs of order \( v \) and index \( \lambda \). At first consider the case \( \lambda = 1 \).
$$\mathcal{M}(17) = \begin{bmatrix}
(1, 1) & (1, 15) & (15, 1) \\
(1, 2) & (2, 14) & (14, 1) \\
(1, 3) & (3, 13) & (13, 1) \\
(1, 4) & (4, 12) & (12, 1) \\
(1, 5) & (5, 11) & (11, 1) \\
(1, 6) & (6, 10) & (10, 1) \\
(1, 7) & (7, 9) & (9, 1) \\
(1, 8) & (8, 8) & (8, 1) \\
(1, 9) & (9, 7) & (7, 1) \\
(1, 10) & (10, 6) & (6, 1) \\
(1, 11) & (11, 5) & (5, 1) \\
(1, 12) & (12, 4) & (4, 1) \\
(1, 13) & (13, 3) & (3, 1) \\
(1, 14) & (14, 2) & (2, 1) \\
(2, 2) & (2, 13) & (13, 2) \\
(2, 3) & (3, 12) & (12, 2) \\
(2, 4) & (4, 11) & (11, 2) \\
(2, 5) & (5, 10) & (10, 2) \\
(2, 6) & (6, 9) & (9, 2) \\
(2, 7) & (7, 8) & (8, 2) \\
(2, 8) & (8, 7) & (7, 2) \\
(2, 9) & (9, 6) & (6, 2) \\
(2, 10) & (10, 5) & (5, 2) \\
(2, 11) & (11, 4) & (4, 2) \\
(2, 12) & (12, 3) & (3, 2) \\
(3, 3) & (3, 11) & (11, 3) \\
(3, 4) & (4, 10) & (10, 3) \\
(3, 5) & (5, 9) & (9, 3) \\
(3, 6) & (6, 8) & (8, 3) \\
(3, 7) & (7, 7) & (7, 3) \\
(3, 8) & (8, 6) & (6, 3) \\
(3, 9) & (9, 5) & (5, 3) \\
(3, 10) & (10, 4) & (4, 3) \\
(4, 4) & (4, 9) & (9, 4) \\
(4, 5) & (5, 8) & (8, 4) \\
(4, 6) & (6, 7) & (7, 4) \\
(4, 7) & (7, 6) & (6, 4) \\
(4, 8) & (8, 5) & (5, 4) \\
(5, 5) & (5, 7) & (7, 5) \\
(5, 6) & (6, 6) & (6, 5)
\end{bmatrix}$$
References


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