New Results and Matrix Representation for
Daehee and Bernoulli Numbers and Polynomials

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Abstract

In this paper, we derive new matrix representation for Daehee numbers and polynomials, the \( \lambda \)-Daehee numbers and polynomials and the twisted Daehee numbers and polynomials. This helps us to obtain simple and short proofs of many previous results on Daehee numbers and polynomials. Moreover, we obtained some new results for Daehee and Bernoulli numbers and polynomials.

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1 Introduction

The \( n \)-th Daehee polynomials are defined by the generating function, [6], [8], [11], [13] and [14],

\[
\left( \frac{\log (1 + t)}{t} \right) (1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.
\]  (1)
In the special case, \( x = 0 \), \( D_n = D_n(0) \) are called the Daehee numbers. The Stirling numbers of the first kind are defined by

\[
(x)_n = \prod_{i=0}^{n} (x - i) = \sum_{l=0}^{n} s_1(n, l)x^l,
\]

and the Stirling numbers of the second kind are given by the generating function to be, \([2, 3, 5]\).}

\[
(e^t - 1)^m = m! \sum_{l=m}^{\infty} s_2(l, m) t^l / l!.
\]

For \( \alpha \in \mathbb{Z} \), the Bernoulli polynomials of order \( \alpha \) are defined by the generating function to be, \([1, 2]\) and \([11]\),

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!},
\]

when \( x = 0 \), \( B_n^{(\alpha)} = B_n^{(\alpha)}(0) \) are called the Bernoulli numbers of order \( \alpha \).

## 2 Daehee Numbers and Polynomials

In this section we derive an explicit formula and recurrence relation for Daehee numbers and polynomials of the first and second kinds and the relation between these numbers and Nörlund numbers are given. D-S. Kim and T. Kim \([8]\), obtained some results for Daehee numbers and polynomial of both kinds. We introduce the matrix representation and investigate a simple and short proofs of these results.

From Eq. (1), when \( x = 0 \), the Daehee numbers of first kind are defined by

\[
\frac{\log(1 + t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!},
\]

by expanding the left hand side and equating the coefficients of \( t^n \) on both sides gives

\[
D_n = (-1)^n \frac{n!}{n + 1}, \quad n \geq 0.
\]

It is easy to show that \( D_n \) satisfy the recurrence relation

\[
(n + 1)D_n + n^2 D_{n-1} = 0, \quad n \geq 1.
\]
For $x = -1$, the Nörlund numbers of the second kind have the explicit formula, see Liu and Srivastava [10, Remark 4],

$$b_n^{-1} = \frac{(-1)^n}{n+1},$$  \hspace{1cm} (8)

thus we conclude that the relation between Daehee numbers and Nörlund numbers is given by

$$D_n = n!b_n^{-1}.$$  \hspace{1cm} (9)

**Theorem 2.1** For $n \geq 1$, $x \in \mathbb{Z}$, we have

$$D_n(1 + x) = D_n(x) + nD_{n-1}(x).$$  \hspace{1cm} (10)

**Proof.** From Eq. (1), replacing $x$ by $(1 + x)$, we get

$$\sum_{n=0}^{\infty} D_n(1 + x) \frac{t^n}{n!} = \left( \log \frac{1 + t}{t} \right) (1 + t)^{1+x} = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} (1 + t)$$

$$= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} D_n(x) \frac{t^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} (n+1)D_n(x) \frac{t^{n+1}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} + \sum_{n=1}^{\infty} nD_{n-1}(x) \frac{t^n}{n!}.$$  

Equating the coefficients of $t^n$ on both sides gives (10).

Similarly

$$D_n(1 - x) = D_n(-x) + nD_{n-1}(-x).$$  \hspace{1cm} (11)

**Corollary 2.2** For $1 \leq k \leq n$, we have

$$D_n(k) = \sum_{l=0}^{k} \left( \sum_{i=l}^{k} P(k,i) s_1(i,l) D_{n-i} \right) n^l, \quad k \leq n.$$  \hspace{1cm} (12)

**Proof.** From Eq. (10), setting $x = 0$, we have

$$D_n(1) = D_n + nD_{n-1}.$$  

If $x = 1$,

$$D_n(2) = D_n(1) + nD_{n-1}(1) = D_n + nD_{n-1} + n(D_{n-1} + (n-1)D_{n-2})$$

$$= D_n + 2nD_{n-1} + n(n-1)D_{n-2}.$$
If \( x = 2 \),
\[
D_n(3) = D_n(2) + nD_{n-1}(2)
\]
\[
= D_n + 2nD_{n-1} + n(n-1)D_{n-2} + n(D_{n-1} + 2(n-1)D_{n-2} + \ldots + (n-1)(n-2)D_{n-3})
\]
\[
= D_n + 3nD_{n-1} + 3n(n-1)D_{n-2} + n(n-1)(n-2)D_{n-3}.
\]

If \( x = 3 \),
\[
D_n(4) = D_n(3) + nD_{n-1}(3)
\]
\[
= D_n + 3nD_{n-1} + 3n(n-1)D_{n-2} + n(n-1)(n-2)D_{n-3} + n(D_{n-1} + 3(n-1)D_{n-2} + 3(n-1)(n-2)D_{n-3} + (n-1)(n-2)(n-3)D_{n-4})
\]
\[
= D_n + 4nD_{n-1} + 6n(n-1)D_{n-2} + 4n(n-1)(n-2)D_{n-3} + n(n-1)(n-2)(n-3)D_{n-4},
\]

hence by iteration, for \( x = k, \ 1 \leq k \leq n \), we have
\[
D_n(k) = \sum_{i=0}^{k} P(k,i)(n)_{i}D_{n-i} = \sum_{i=0}^{k} P(k,i) \sum_{i=0}^{k} s_{1}(i,l)n^l D_{n-i}
\]
\[
= \sum_{l=0}^{k} \left( \sum_{i=l}^{k} P(k,i)s_{1}(i,l)D_{n-i} \right) n^l, \ k \leq n.
\]

This completes the proof.

**Remark 1:** It is easy to show that \( D_n(k) \) satisfies the symmetric relation
\[
D_n(k) = D_n(n-k), \ 1 \leq k \leq n.
\]

We can write Eq. (12), in the matrix form as follows.
\[
D(k) = P_{k+1}D_{n-k}S_1N,
\]

where \( D(k) = (D_n(0) \ D_n(1) \ \cdots \ D_n(k))^T \), is the \((k + 1) \times 1\) matrix of the Dahee polynomials for \( x = k, \ k \in N \). \( P_{k+1} \) is the \((k + 1) \times (k + 1)\) lower triangular Pascal matrix, \( D_{n-k} \), is the diagonal matrix of the Dahee number with elements \( D_{n-i} \), for \( i = 0, 1, \cdots, k, \ 0 \leq k \leq n \), \( N = (1 \ n^1 \ n^2 \ \cdots \ n^k)^T \) is the \((k + 1) \times 1\) matrix and \( S_1 \) is the \((k + 1) \times (k + 1)\), lower triangular matrix for Stirling numbers of the first kind, see Comtet [3] and El-Desouky et al. [4], i.e.
\[
S_1 = \begin{pmatrix}
  s_{0,0} & 0 & 0 & \cdots & 0 \\
  s_{1,0} & s_{1,1} & 0 & \cdots & 0 \\
  s_{2,0} & s_{2,1} & s_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  s_{n,0} & s_{n,1} & s_{n,2} & \cdots & s_{n,n}
\end{pmatrix},
\]
and
\[
(P_{k+1})_{ij} = \begin{cases} 
{i \choose j}, & \text{if } i \geq j, \ i, j = 0, 1, \ldots, k, \\
0, & \text{otherwise}.
\end{cases}
\]

For example, if setting \( n = 4 \), \( 0 \leq k \leq n \), in (13), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
24/5 & 0 & 0 & 0 & 0 \\
0 & -3/2 & 0 & 0 & 0 \\
0 & 0 & 2/3 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
= \begin{pmatrix}
24/5 \\
-6/5 \\
4/5 \\
-6/5 \\
24/5
\end{pmatrix}.
\]

D-S. Kim and T. Kim [8, Eq. (2.10)] proved the following relation
\[
D_n = \sum_{l=0}^{n} s_1(n, l)B_l. \tag{14}
\]

We can write this relation in the matrix form as follows
\[
D = S_1B, \tag{15}
\]
where \( D = (D_0, D_1, \ldots, D_n)^T \) and \( B = (B_0, B_1, \ldots, B_n)^T \), respectively, are the matrices for Daehee numbers of the first kind and Bernoulli numbers.

For example, if \( 0 \leq n \leq 4 \), in (15), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1/2 \\
1/6 \\
0 \\
-1/30
\end{pmatrix}
= \begin{pmatrix}
1 \\
-1/2 \\
2/3 \\
-3/2 \\
24/5
\end{pmatrix}.
\]

D-S. Kim and T. Kim [8, Corollary 3] introduced the following result. For \( n \geq 0 \), we have
\[
D_n(x) = \sum_{l=0}^{n} s_1(n, l)B_l(x). \tag{16}
\]

Eq. (16) can be represented in the matrix form as follows
\[
D(x) = S_1B(x), \tag{17}
\]
where \( D(x) = (D_0(x), D_1(x), \ldots, D_n(x))^T \) and \( B(x) = (B_0(x), B_1(x), \ldots, B_n(x))^T \) are the matrices for Daehee and Bernoulli polynomials.
For example, if setting $0 \leq n \leq 4$ in (17), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
x - 1/2 \\
x^2 - x + 1/6 \\
x^3 - 3x^2/2 + x/2 \\
x^4 - 2x^3 + x^2 - 1/30
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
x - 1/2 \\
x^2 - 2x + 2/3 \\
x^3 - 9x^2/2 + 11x/2 - 3/2 \\
x^4 - 8x^3 + 21x^2 - 20x + 24/5
\end{pmatrix}.
\]

D-S. Kim and T. Kim [8, Theorem 4] introduced the following result. For $m \geq 0$, we have
\[
B_m = \sum_{n=0}^{m} s_2(m, n) D_n.
\]  
(18)

We can write Eq. (18) in the matrix form as follows
\[
B = S_2 D,
\]  
(19)

where $S_2$ is $(n + 1) \times (n + 1)$ lower triangular matrix for the Stirling numbers of the second kind.

For example, if $0 \leq n \leq 4$, in (19) we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 1 & 7 & 6 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1/2 \\
2/3 \\
-3/2 \\
24/5
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
-1/2 \\
1/6 \\
0 \\
-1/30
\end{pmatrix}.
\]

**Remark 2:** Using the matrix form (15), we easily derive a short proof of Theorem 4 in Kim D-S. Kim and T. Kim [8]. Multiplying both sides by $S_2$, as follows
\[
S_2 D = S_2 S_1 B = IB = B,
\]
where $I$ is the identity matrix of order $n + 1$.

The Daehee polynomials of the second kind by the generating function as follows, see [8]
\[
\frac{(1 + t) \log (1 + t)}{t} \frac{1}{(1 + t)^x} = \sum_{n=0}^{\infty} \hat{D}_n(x) \frac{t^n}{n!}.
\]  
(20)

In the special case, $x = 0$, $\hat{D}_n = \hat{D}_n(0)$ are called the Daehee numbers of the second kind
\[
\frac{(1 + t) \log (1 + t)}{t} = \sum_{n=0}^{\infty} \hat{D}_n \frac{t^n}{n!},
\]  
(21)

by expanding the left hand side and equating the coefficients of $t^n$ on both sides gives
\[
\hat{D}_n = (-1)^{n-1} \frac{n!}{n(n + 1)}, \quad n \geq 1, \quad \text{and} \quad \hat{D}_0 = 1.
\]  
(22)
It is easy to show that \( \hat{D}_n \) satisfy the recurrence relation
\[
(n + 1)\hat{D}_n + (n - 1)n\hat{D}(n - 1) = 0. \tag{23}
\]

The relation between Daehee numbers of the first and the second kinds can be obtained as
\[
\hat{D}_n = D_n + nD_{n-1}, \quad n \geq 1. \tag{24}
\]
and
\[
n\hat{D}_n + D_n = 0, \quad n \geq 1. \tag{25}
\]

D-S. Kim and T. Kim [8, Theorem 5 and 6] introduced the following results. For \( n \geq 0 \), we have
\[
\hat{D}_n = \sum_{l=0}^{n} s_1(n, l)(-1)^l B_l, \tag{26}
\]
and
\[
\hat{D}_n(x) = \sum_{l=0}^{n} s_1(n, l)(-1)^l B_l(x). \tag{27}
\]

We can write Eqs. (26, 27) in the matrix form as follows
\[
\hat{D} = S_1 I_1 B, \tag{28}
\]
and
\[
\hat{D}(x) = S_1 I_1 B(x), \tag{29}
\]
where \( \hat{D} = \left( \hat{D}_0 \; \hat{D}_1 \; \cdots \; \hat{D}_n \right)^T \), \( \hat{D}(x) = \left( \hat{D}_0(x) \; \hat{D}_1(x) \; \cdots \; \hat{D}_n(x) \right)^T \) are the \((n + 1) \times (n + 1)\) matrices for Daehee numbers and polynomials of the second kind, respectively. \( I_1 \) is the \((n + 1) \times (n + 1)\) diagonal matrix its elements \((I_1)_{ii} = (-1)^i, \quad 0 \leq i \leq n\), and \( B, B(x) \) are the \((n + 1) \times 1\), matrix for Bernoulli numbers and polynomials respectively.

For example, if setting, \( 0 \leq n \leq 4 \) in (29), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
x - 1/2 \\
x^2 - x + 1/6 \\
x^3 - 3x^2/2 + x/2 \\
x^4 - 2x^3 + x^2 - 1/30
\end{pmatrix} =
\begin{pmatrix}
1 \\
-x + 1/2 \\
-x^2 - 1/3 \\
-x^3 - 3x^2/2 + x/2 + 1/2 \\
x^4 + 4x^3 + 3x^2 - 2x - 6/5
\end{pmatrix}.
\]
D-S. Kim and T. Kim [8, Theorem 7] derived the following result. For \( m \geq 0 \), we have

\[
B_m(1 - x) = \sum_{n=0}^{m} s_2(m, n) \hat{D}_n(x),
\]

where \( B_n(1 - x) = (-1)^n B_n(x) \).

We can write Eq. (30) in the matrix form as follows.

\[
B(1 - x) = S_2 \hat{D}(x).
\]

**Remark 3:** In fact we can prove Eq. (30), D-S. Kim and T. Kim [8, Theorem 7] by multiplying Eq. (29) by \( S_2 \) as follows.

\[
S_2 \hat{D}(x) = S_2 S_1 I_n B(x) = I_1 B(x) = B(1 - x).
\]

For example, if setting \( 0 \leq n \leq 4 \) in (31), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 1 & 7 & 6 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-x + 1/2 \\
x^2 - 1/3 \\
-x^3 - 3x^2/2 + x/2 + 3/2 \\
x^4 + 4x^3 + 3x^2 - 2x - 6/5
\end{pmatrix}
= \begin{pmatrix}
1/2 - x \\
x^2 - x + 1/6 \\
x^2 - x + 1/6 \\
-x(x - 1)(2x - 1)/2 \\
x^4 - 2x^3 + x^2 - 1/30
\end{pmatrix}.
\]

### 3 The lambda-Daehee Numbers and Polynomials

In this section we introduce the matrix representation for the lambda-Daehee polynomials. Kim et al. [9] introduced some results for \( \lambda \)-Daehee polynomial, we can derive these results in matrix representation and prove these results simply by using matrix forms. The \( \lambda \)-Daehee polynomials of the first kind can be defined by the generating function to be, Kim et al. [9],

\[
\frac{\lambda \log(1 + t)}{(1 + t)^\lambda - 1} (1 + t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}.
\]

When \( x = 0 \), \( D_{n,\lambda} = D_{n,\lambda}(0) \) are called the \( \lambda \)-Daehee numbers.

\[
\frac{\lambda \log(1 + t)}{(1 + t)^\lambda - 1} = \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!}.
\]

It is easy to see that \( D_n(x) = D_{n,1}(x) \) and \( D_n = D_{n,1} \).
Kim et al. [9, Theorem 2] obtained the following results. For $m \geq 0$, we have

\[
D_{m,\lambda}(x) = \sum_{l=0}^{m} s_1(m, l)\lambda^l B_l \left( \frac{x}{\lambda} \right),
\]  

(34)

and

\[
\lambda^m B_m \left( \frac{x}{\lambda} \right) = \sum_{n=0}^{m} s_2(m, n) D_{n,\lambda}(x),
\]  

(35)

we can write these results in the following matrix form

\[
D_\lambda(x) = S_1 \Lambda B \left( \frac{x}{\lambda} \right),
\]  

(36)

and

\[
\Lambda B \left( \frac{x}{\lambda} \right) = S_2 D_\lambda(x),
\]  

(37)

where, $D_\lambda(x) = (D_0(x) \ D_1(x) \ \cdots \ D_n(x))^T$, is the $(n+1) \times 1$, matrix for $\lambda$-Daehee polynomials of the first kind, $B \left( \frac{x}{\lambda} \right) = (B_0(\frac{x}{\lambda}) \ B_1(\frac{x}{\lambda}) \ \cdots \ B_n(\frac{x}{\lambda}))^T$, is the $(n+1) \times 1$, matrix for Bernoulli polynomials, when $x \to \frac{x}{\lambda}$ and $\Lambda$ is the $(n+1) \times (n+1)$, diagonal matrix with elements, $(\Lambda)_{ii} = \lambda^i$, $i = 0, 1, \cdots, n$. For example, if setting $0 \leq n \leq 4$, in (36), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & 0 \\
0 & 0 & 0 & 0 & \lambda^4
\end{pmatrix}
\begin{pmatrix}
1/(x-\lambda/2)/\lambda \\
(\lambda^2/6 - \lambda x + x^2)/\lambda^2 \\
x(\lambda - x)(\lambda - 2x)/(2\lambda^3) \\
(\lambda^2 x^2 - \lambda^4/30 - 2\lambda x^4 + x^4)/\lambda^4 \\
x - \lambda/2 \\
\lambda^2/6 - \lambda x + \lambda/2 + x^2 - x \\
-\lambda(2x+2)(\lambda - 2x + x^2 - \lambda x)/2 \\
\lambda^2 x^2 - \lambda^4/30 - 3\lambda^2 x + 11\lambda^2/6 - 2\lambda x^3 + 9\lambda x^2 - 11\lambda x + 3\lambda + x^4 - 6x^3 + 11x^2 - 6x
\end{pmatrix} =
\begin{pmatrix}
1 \\
(\lambda^2/6 - \lambda x + x^2)/\lambda^2 \\
x(\lambda - x)(\lambda - 2x)/(2\lambda^3) \\
(\lambda^2 x^2 - \lambda^4/30 - 2\lambda x^4 + x^4)/\lambda^4 \\
1 \\
\lambda^2/6 - \lambda x + \lambda/2 + x^2 - x \\
-\lambda(2x+2)(\lambda - 2x + x^2 - \lambda x)/2 \\
\lambda^2 x^2 - \lambda^4/30 - 3\lambda^2 x + 11\lambda^2/6 - 2\lambda x^3 + 9\lambda x^2 - 11\lambda x + 3\lambda + x^4 - 6x^3 + 11x^2 - 6x
\end{pmatrix}.
\]

**Remark 4:** In fact, we can prove Eq. (37) by multiplying Eq. (36) by $S_2$ as follows.

\[
S_2 D_\lambda(x) = S_2 S_1 \Lambda B \left( \frac{x}{\lambda} \right) = \Lambda B \left( \frac{x}{\lambda} \right).
\]

**Theorem 3.1** For $m \geq 0$, we have

\[
D_{m,\lambda}(\lambda x) = m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D_n(x)}{n!} \left( \frac{\lambda}{i_1} \right) \left( \frac{\lambda}{i_2} \right) \cdots \left( \frac{\lambda}{i_n} \right).
\]  

(38)
Proof. From (1), replacing \((1 + t)\) by \((1 + t)^\lambda\), we have
\[
\frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} (1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{((1 + t)^\lambda - 1)^n}{n!},
\]
thus from (32), we get
\[
\sum_{m=0}^{\infty} D_{m,\lambda}(\lambda x) \frac{t^m}{m!} = \sum_{n=0}^{\infty} D_n(x) \left( \sum_{i=0}^{\lambda} (\frac{\lambda}{i}) t^i - 1 \right)^n.
\]
Using Cauchy rule of product of series, we obtain
\[
\sum_{m=0}^{\infty} D_{m,\lambda}(\lambda x) \frac{t^m}{m!} = \sum_{n=0}^{\infty} \frac{D_n(x)}{n!} \sum_{m=n}^{\infty} \sum_{i_1 + i_2 + \cdots + i_n = m} (\frac{\lambda}{i_1}) \cdots (\frac{\lambda}{i_n}) t^m.
\]
Equating the coefficients of \(t^m\) on both sides yields (38). This completes the proof.

Setting \(x = 0\), in (38), we have the following corollary as a special case.

**Corollary 3.2** For \(m \geq 0\), we have
\[
D_{m,\lambda} = m! \sum_{n=0}^{m} \sum_{i_1 + i_2 + \cdots + i_n = m} \frac{D_n(x)}{n!} (\frac{\lambda}{i_1}) (\frac{\lambda}{i_2}) \cdots (\frac{\lambda}{i_n}).
\]

Kim et al. [9], defined the \(\lambda\)-Dahee polynomials of the second kind as follows.
\[
\frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} (1 + t)^x = \sum_{n=0}^{\infty} \hat{D}_{n,\lambda}(x) \frac{t^n}{n!},
\]
Note that \(\hat{D}_{n,1}(x) = \hat{D}_n(x)\).

Kim et al. [9, Theorem 4] introduced the following results. For \(m \geq 0\), we have
\[
\hat{D}_{m,\lambda}(x) = \sum_{l=0}^{m} s_1(m, l) \lambda^l B_l \left( 1 + \frac{x}{\lambda} \right),
\]
and
\[
\lambda^m B_m \left( 1 + \frac{x}{\lambda} \right) = \sum_{n=0}^{m} s_2(m, n) \hat{D}_{n,\lambda}(x),
\]
we can write these results in the following matrix form

\[ \hat{D}_\lambda(x) = S_1 \Lambda B \left( 1 + \frac{x}{\lambda} \right), \]  

and

\[ \Lambda B \left( 1 + \frac{x}{\lambda} \right) = S_2 \hat{D}_\lambda(x), \]

where \( \hat{D}_\lambda(x) = \left( \hat{D}_{0,\lambda}(x) \hat{D}_{1,\lambda}(x) \cdots \hat{D}_{n,\lambda}(x) \right)^T \), is the \((n+1) \times 1\) matrix for \( \lambda \)-Daehee polynomials of the second kind and \( B \left( 1 + \frac{x}{\lambda} \right) = B_0 \left( 1 + \frac{x}{\lambda} \right) B_1 \left( 1 + \frac{x}{\lambda} \right) \cdots B_n \left( 1 + \frac{x}{\lambda} \right)^T \), is the \((n + 1) \times 1\) matrix for Bernoulli Polynomials, when \( x \to 1 + \frac{x}{\lambda} \).

For example, if setting \( 0 \leq n \leq 4 \), in (43), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
\lambda \\
0 \\
-\lambda/2 + x
\end{pmatrix}
\begin{pmatrix}
1 \\
\lambda^2/6 + \lambda x - \lambda/2 + x^2 - x \\
-\lambda + 2x - 2(\lambda + x - x^2) \\
\lambda^2 x^2 - \lambda^4/30 - 3\lambda^2 x + 11\lambda^2/6 + 2\lambda x^3 - 9\lambda x^2 + 11\lambda x - 3\lambda + x^4 - 6x^3 + 11x^2 - 6x
\end{pmatrix}
\]

Remark 5: In fact, we can prove Eq. (44) by multiplying Eq.(43) by \( S_2 \) as follows.

\[ S_2 \hat{D}_\lambda(x) = S_2 S_1 \Lambda B \left( 1 + \frac{x}{\lambda} \right) = \Lambda B \left( 1 + \frac{x}{\lambda} \right). \]

4 The Twisted Daehee Numbers and Polynomials

The \( n \)th twisted Daehee polynomials of the first kind defined by the generating function to be, see [12]

\[ \left( \frac{\log (1 + \xi t)}{\xi t} \right) (1 + \xi t)^x = \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{t^n}{n!}. \]  

In the special case, \( x = 0, D_{n,\xi} = D_{n,\xi}(0) \) are called the \( n \)th twisted Daehee numbers of the first kind.

\[ \left( \frac{\log (1 + \xi t)}{\xi t} \right) = \sum_{n=0}^{\infty} D_{n,\xi} \frac{t^n}{n!}. \]  

In fact, we can obtain the relation between Daehee polynomials and twisted Daehee polynomials of the first kind as follows.
Replacing \( t \) with \( \xi t \) in Eq. (1), we have
\[
\left( \frac{\log (1 + \xi t)}{\xi t} \right) (1 + \xi t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{\xi^n t^n}{n!}.
\]
(47)

From Equations (47) and (45), equating the coefficients of \( t^n \) on both sides, we obtain
\[
D_{n,\xi}(x) = \xi^n D_n(x), \quad n \geq 0.
\]
(48)

Setting \( x = 0 \) in (48) we have
\[
D_{n,\xi} = \xi^n D_n, \quad n \geq 0.
\]
(49)

We can write Eq. (48) in the matrix form as follows.
\[
D_{\xi}(x) = \Xi D(x).
\]
(50)

where \( D_{\xi}(x) = (D_{0,\xi}(x) D_{1,\xi}(x) \cdots D_{n,\xi}(x))^T \), is the \((n+1) \times 1\), matrix for the twisted Dahee polynomials, \( \Xi \) is the \((n+1) \times (n+1)\) diagonal matrix with elements \((\Xi)_{ii} = \xi^i, \quad i = 0, 1, \cdots, n\) and \( D(x) \), is the \((n+1) \times 1\), matrix for Dahee polynomials.

For example, if setting \( 0 \leq n \leq 4 \), in (50), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 & 0 \\
0 & 0 & \xi^2 & 0 & 0 \\
0 & 0 & 0 & \xi^3 & 0 \\
0 & 0 & 0 & 0 & \xi^4
\end{pmatrix}
\begin{pmatrix}
x - 1/2 \\
x^2 - 2x + 2/3 \\
x^3 - 9x^2/2 + 11x/2 - 3/2 \\
x^4 - 8x^3 + 21x^2 - 20x + 24/5
\end{pmatrix}
=
\begin{pmatrix}
1 \\
\xi(x - 1/2) \\
\xi^2(x^2 - 2x + 2/3) \\
\xi^3(2x - 3)(x^2 - 3x + 1)/2 \\
\xi^4(x^4 - 8x^3 + 21x^2 - 20x + 24/5)
\end{pmatrix}
\]

From Eq. (49) in Equations (6) and (7), we have
\[
D_{n,\xi} = (-1)^n \frac{n!}{n+1} \xi^n, \quad n \geq 0.
\]
(51)

and
\[
(n+1)D_{n,\xi} + n^2\xi D_{n-1,\xi} = 0, \quad n \geq 1.
\]
(52)

Substituting from Eq. (49) in Eq. (10), we have
\[
D_{n,\xi}(1 + x) = D_{n,\xi}(x) + n\xi^{-1}D_{n-1,\xi}(x).
\]
(53)

Park et al. [12, Eq. 17] proved the following relation
\[
D_{n,\xi} = \xi^n \sum_{l=0}^{n} s_1(n, l) B_l,
\]
(54)

which can be written in the following matrix form
\[
D_{\xi} = \Xi S_1 B,
\]
(55)
where \( \mathbf{D}_\xi = (D_{0,\xi} \, D_{1,\xi} \, \cdots \, D_{n,\xi})^T \) is the \((n + 1) \times 1\) matrix for the twisted Daehee numbers.

For example, if setting \(0 \leq n \leq 4\), in (55), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 & 0 \\
0 & 0 & \xi^2 & 0 & 0 \\
0 & 0 & 0 & \xi^3 & 0 \\
0 & 0 & 0 & 0 & \xi^4
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1/2 \\
1/6 \\
0 \\
-1/30
\end{pmatrix}
= \begin{pmatrix}
1 \\
-\xi/2 \\
2\xi^2/3 \\
-3\xi^3/2 \\
24\xi^4/5
\end{pmatrix}.
\]

Park et al. [12, Corollary 3] proved the following result for twisted Daehee polynomials of the first kind.

For \(n \geq 0\), we have
\[
D_{n,\xi}(x) = \xi^n \sum_{l=0}^{n} s_1(n, l) B_l(x),
\]
we can write Eq. (56) in the matrix form as follows.
\[
\mathbf{D}_\xi(x) = \Xi S_1 \mathbf{B}(x),
\]

For example, if setting \(0 \leq n \leq 4\), in (57), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 & 0 \\
0 & 0 & \xi^2 & 0 & 0 \\
0 & 0 & 0 & \xi^3 & 0 \\
0 & 0 & 0 & 0 & \xi^4
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
\begin{pmatrix}
x - 1/2 \\
x^2 - x + 1/6 \\
x^3 - 3x^2/2 + x/2 \\
x^4 - 2x^3 + x^2 - 1/30
\end{pmatrix}
= \begin{pmatrix}
1 \\
\xi(x - 1/2) \\
\xi^2(x^2 - 2x + 2/3) \\
\xi^3(2x - 3)(x^2 - 3x + 1)/2 \\
\xi^4(x^4 - 8x^3 + 21x^2 - 20x + 24/5)
\end{pmatrix}.
\]

The twisted Bernoulli polynomials are defined as, Kim [7],
\[
\left(\frac{t}{\xi e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \xi \in T_p,
\]
and the twisted Bernoulli numbers \(B_{n,\xi}\) are defined as \(B_{n,\xi} = B_{n,\xi}(0)\).
\[
\frac{t}{\xi e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi} \frac{t^n}{n!}, \xi \in T_p.
\]

We introduce the relation between the twisted Daehee polynomials of the first kind and the Bernoulli polynomials in the following Theorem.

**Theorem 4.1** For \(m \geq 0\), we have
\[
B_m(x) = \sum_{n=0}^{m} s_2(m, n) \xi^{-n} D_{n,\xi}(x).
\]
Proof. Replace \( t \) by \((e^t - 1)/\xi\), in (45), we have

\[
\left( \frac{t}{e^t - 1} \right) e^{tx} = \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{(e^t - 1)^n}{n!} \xi^{-n} = \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{\xi^{-n}}{n!} \sum_{m=n}^{\infty} \frac{s_2(m,n)}{m!} t^m \xi^{-n} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{\xi^{-n} D_{n,\xi}(x) s_2(m,n)}{m!}. \quad (61)
\]

From (4), when \( \alpha = 1 \), we get

\[
\left( \frac{t}{e^t - 1} \right) e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}. \quad (62)
\]

From (61) and (62), we have

\[
B_m(x) = \sum_{n=0}^{m} \xi^{-n} D_{n,\xi}(x) s_2(m,n).
\]

This completes proof (60).

**Remark 6:** In fact, it seems that there is something not correct in Theorem 4, Park et al. [12]. Moreover, we can represent Equation (60), in the following matrix form.

\[
B(x) = S_2 \Xi^{-1} D_\xi(x). \quad (63)
\]

**Remark 7:** We can prove (56), Park et al. [12, Corollary 3], easily using the matrix form as follows. Multiplying the both sides of (63) by \( S_1 \), we get

\[
S_1 B(x) = S_1 S_2 \Xi^{-1} D_\xi(x) = I \Xi^{-1} D_\xi(x) = \Xi^{-1} D_\xi(x)
\]

and

\[
\Xi S_1 B(x) = \Xi \Xi^{-1} D_\xi(x) = D_\xi(x).
\]

This completes the proof of (56). For example, if setting \( 0 \leq n \leq 4 \), in (63), we have

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 3 & 1 & 0 \\
0 & 1 & 7 & 6 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 1 & 7 & 6 & 1
\end{pmatrix}
\begin{pmatrix}
\xi^{-1} & 0 & 0 & 0 & 0 \\
0 & \xi^{-2} & 0 & 0 & 0 \\
0 & 0 & \xi^{-3} & 0 & 0 \\
0 & 0 & 0 & \xi^{-4} & 0 \\
0 & 0 & 0 & 0 & \xi^{-5}
\end{pmatrix}
\begin{pmatrix}
1 \\
\xi(x - 1/2) \\
\xi^2(2x - 3)|x^2 - 3x + 1/2|/2 \\
\xi^3(x^4 - 8x^3 + 21x^2 - 20x + 24)/5 \\
\xi^4(x^2 - 2x + 1/6)
\end{pmatrix}
= \begin{pmatrix}
1 \\
x^2 - x + 1/6 \\
x(x - 1/2)/2 \\
x^3 - 2x^2 + x^2 - 1/30
\end{pmatrix}.
\]

Setting \( x = 0 \) in Theorem 4.1, we have the following Corollary.

**Corollary 4.2** For \( m \geq 0 \), we have

\[
B_m = \sum_{n=0}^{m} s_2(m,n) \xi^{-n} D_{n,\xi}. \quad (64)
\]
We can investigate a new relation between the twisted Bernoulli polynomials and Bernoulli polynomials as follows.

**Theorem 4.3** For \( n \geq 0 \), we have

\[
\xi^x B_{n, \xi}(x) + \xi^x \ln(\xi) B_{n+1, \xi}(x) \left(\frac{1}{n+1}\right) = \sum_{j=n}^{\infty} B_j(x) \frac{(\ln(\xi))^{j-n}}{(j-n)!}. \tag{65}
\]

**Proof.** From (4), when \( \alpha = 1 \), we get

\[
\left(\frac{t}{e^t - 1}\right)e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!},
\]

and replace \( t \) by \( (t + \ln(\xi)) \), we have

\[
\left(\frac{t + \ln(\xi)}{\xi e^t - 1}\right)e^{xt} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} \sum_{i=0}^{m} (\ln(\xi))^{m-i} t^i,
\]

\[
\xi^x \frac{t}{\xi e^t - 1} e^{xt} + \frac{\xi^x \ln(\xi)}{t} \left(\frac{t}{\xi e^t - 1}\right) e^{xt} = \sum_{i=0}^{\infty} \frac{B_m(x)}{m!} \sum_{i=0}^{m} (\ln(\xi))^{m-i} t^i,
\]

\[
\xi^x \sum_{n=0}^{\infty} B_{n, \xi}(x) \frac{t^n}{n!} + \frac{\xi^x \ln(\xi)}{t} \sum_{n=0}^{\infty} B_{n, \xi}(x) \frac{t^n}{n!} = \sum_{i=0}^{\infty} \frac{B_m(x)}{(m-i)!} (\ln(\xi))^{m-i} t^i.
\]

By equating the coefficients of \( t^n \) on both sides gives

\[
\xi^x B_{n, \xi}(x) + \left(\frac{\xi^x \ln(\xi)}{n+1}\right) B_{n+1, \xi}(x) = \sum_{m=n}^{\infty} \frac{(\ln(\xi))^{m-n}}{(m-n)!} B_m(x), \ n \geq 0.
\]

This completes the proof.

**Corollary 4.4** For \( n \geq 0 \), we have

\[
B_{n, \xi} + \left(\frac{\ln(\xi)}{n+1}\right) B_{n+1, \xi} = \sum_{m=n}^{\infty} \frac{(\ln(\xi))^{m-n}}{(m-n)!} B_m, \ n \geq 0. \tag{66}
\]

Equation (66) gives a new connection between twisted Bernoulli numbers and Bernoulli numbers.

The \( n \)th twisted Dahee polynomials of the second kind defined as follows, see [12]

\[
\left(\frac{(1 + \xi t) \log(1 + \xi t)}{\xi t}\right) \frac{1}{(1 + \xi t)^x} = \sum_{n=0}^{\infty} \hat{D}_{n, \xi}(x) \frac{t^n}{n!}. \tag{67}
\]

In special, if \( x = 0 \), \( \hat{D}_{n, \xi} = \hat{D}_{n, (0)} \), we have the twisted Dahee numbers of second kind.

\[
\frac{(1 + \xi t) \log(1 + \xi t)}{\xi t} = \sum_{n=0}^{\infty} \hat{D}_{n, \xi} \frac{t^n}{n!}. \tag{68}
\]
For example, if setting $0 \leq \xi \leq 1$, we can write Eq. (73) in the matrix form as follows.

$$
\hat{D}_{n, \xi} = \xi^n \sum_{l=0}^{n} s_1(n, l)(-1)^l B_l.
$$

(69)

and

$$
\hat{D}_{n, \xi}(x) = \xi^n \sum_{l=0}^{n} s_1(n, l)(-1)^l B_l(x).
$$

(70)

We can write Eqs. (69, 70) in the following matrix form.

$$
\hat{D}_\xi = \Xi S_1 I_1 B,
$$

(71)

and

$$
\hat{D}_\xi(x) = \Xi S_1 I_1 B(x),
$$

(72)

where $\hat{D}_\xi = (\hat{D}_{0, \xi} \hat{D}_{1, \xi} \cdots \hat{D}_{n, \xi})^T$ and $\hat{D}_\xi(x) = (\hat{D}_{0, \xi}(x) \hat{D}_{1, \xi}(x) \cdots \hat{D}_{n, \xi}(x))^T$ are the $(n+1) \times 1$ matrices for twisted Dahee numbers and polynomials of the second kind, respectively.

For example, if setting $0 \leq n \leq 4$, in (72), we have

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 & 0 \\
0 & 0 & \xi^2 & 0 & 0 \\
0 & 0 & 0 & \xi^3 & 0 \\
0 & 0 & 0 & 0 & \xi^4
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
-x + 1/2 \\
x^2 - x + 1/6 \\
x^3 - (3x^2)/2 + x/2 \\
x^4 - 2x^3 + x^2 - 1/30
\end{pmatrix}.
$$

We can obtain the relation between Dahee polynomials and twisted Dahee polynomials of the second kind as follows.

**Corollary 4.5** For $n \geq 0$, we have

$$
\hat{D}_{n, \xi}(x) = \xi^n \hat{D}_n(x), \ n \geq 0.
$$

(73)

**Proof.** Replacing $t$ with $\xi t$ in (19), we have

$$
\frac{(1 + \xi t) \log (1 + \xi t)}{\xi t} \frac{1}{(1 + \xi t)^2} = \sum_{n=0}^{\infty} \hat{D}_n(x) \frac{\xi^n t^n}{n!}.
$$

(74)

From Equations (74) and (67), equating the coefficients of $t^n$ on both sides, we obtain (73).

We can write Eq. (73) in the matrix form as follows.

$$
\hat{D}_\xi(x) = \Xi \hat{D}(x).
$$

(75)

For example, if setting $0 \leq n \leq 4$, in (75), we have

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 & 0 \\
0 & 0 & \xi^2 & 0 & 0 \\
0 & 0 & 0 & \xi^3 & 0 \\
0 & 0 & 0 & 0 & \xi^4
\end{pmatrix}
\begin{pmatrix}
1 \\
-x + 1/2 \\
x^2 - 1/3 \\
-x^3 - 3x^2/2 + x/2 + 1/2 \\
x^4 + 4x^3 + 3x^2 - 2x - 6/5
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
-\xi(x - 1/2) \\
\xi^2(x^2 - 1/3) \\
-(\xi^3/2)(2x + 1)(x^2 + x - 1) \\
\xi^4(x^4 + 4x^3 + 3x^2 - 2x - 6/5)
\end{pmatrix}.
$$

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References


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