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On Additive Decomposition GSA

Yong Uk Cho

Department of Mathematics
College of Education, Silla University
Pusan 617-736, Korea

Abstract

Throughout this paper, we will concerned with some concepts of R-groups, where R is a near-ring, and their related algebraic structures, that is, group semiautomata and the distributive group semiautomata. The author already studied the 1-st, 2-nd and the 3-rd isomorphic theorems of group semiautomata.

In this paper, we will investigate a characterization of additive decomposition group semiautomata.

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1. Introduction

The concept of GSA is introduced by J.R. Clay and Y. Pong [6]. The purpose of this paper is to study the algebraic theory of distributive group semiautomata, which is related with the following papers [1, 2, 3, 4, 5].

A *group semiautomaton (in short, GSA)* is a quadruple $(Q,\ +,\ X,\ \delta)$, where $(Q,\ +)$, as the set of states, is a group (not necessarily abelian), X is the set of inputs and $\delta : Q \times X \longrightarrow Q$ is the state transition function.

Now, we can define more strong version of GSA as following:

A GSA $(Q,\ +,\ X,\ \delta)$ is called *distributive* if for any $a,\ b \in Q$ and for each $x \in X$, $(a+b)x = ax + bx$.

On the other hand, for a GSA $(Q,\ +,\ X,\ \delta)$, an element $x_0 \in X$ is said to be a *zero-input element* if $0x_0 = 0$.

We can easily prove the fact that $\forall q \in Q,\ (-q)x_0 = -(qx_0)$.

A GSA $(Q,\ +,\ X,\ \delta)$ is called *additive decomposition* if there exists zero-input $x_0 \in X$ with the following properties:

(i) decomposition property: $qx = qx_0 + 0x$ for all $q \in Q$ and $x \in X$.

(ii) zero-input additivity: $(q_1 + q_2)x_0 = q_1x_0 + q_2x_0$ for all $q_1,\ q_2 \in Q$, equivalently, $(q_1 - q_2)x_0 = q_1x_0 - q_2x_0$ for all $q_1,\ q_2 \in Q$.

At first, this paper is concerned with some concepts of near-ring R, R-group and GSA, and then we introduce some properties of GSA, the fundamental isomorphic theorem and the 2-nd isomorphic theorem of GSA.

Next, we will study our main theorem, that is, the 3-rd isomorphic property in group semiautomata.

Let R be a (left) near-ring and G an additive group. G is called an *R-group* if there exists a mapping $\mu : G \times R \longrightarrow G$ defined by $\mu(x, a) = xa$ which satisfies (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (if R has a unity 1), for all $x \in G$ and $a,\ b \in R$. We denote it by G_R.

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that $(N + x)a - xa \subset N$ for all $x \in G,\ a \in R$. This will be denoted by $N \triangleleft G$.

For the remainder part of basic concepts and results on near-ring theory, we refer to the book [7].

2. The third isomorphic properties in GSA.

From now on, the input set X will be fixed for every GSA. So a GSA on the set Q of states will be denoted by $(Q,\ +,\ \delta)$, or sometimes briefly denoted by Q if there arises no confusions. Moreover, for the brief notation $\delta(q, x)$ we shall write simply qx.

We would like to characterize some algebraic properties of the ideal of GSA which is related with the concept of ideal in R-group. Also, we will study GSA homomorphisms and the properties of ideals in GSA.

Now we give some definitions. Let $(Q,\ +,\ \delta)$ and $(Q',\ +,\ \delta')$ be two GSA with the same input set X and $f : Q \longrightarrow Q'$ a group homomorphism. We call that f is a *GSA homomorphism* from $(Q,\ +,\ \delta)$ into $(Q',\ +,\ \delta')$ if f satisfies $(qx)f = (qf)x$ for all $q \in Q,\ x \in X$.

We can define monomorphism, epimorphism, isomorphism in GSA as following:

Let $f : (Q,\ +,\ \delta) \longrightarrow (Q',\ +,\ \delta')$ be a GSA homomorphism. Then (1) f is called a *GSA monomorphism* if f is injective. (2) f is called a *GSA epimorphism* if f is surjective. (3) f is called a *GSA isomorphism* if f is bijective. In this case we denote that $Q \cong Q'$.

A normal subgroup K of $(Q,\ +)$ is called an *ideal* of a GSA $(Q,\ +,\ \delta)$ if $(K+q)x - qx \subseteq K$ for all $q \in Q,\ x \in X$. This will be denoted by $K \triangleleft (Q,\ +,\ \delta)$.

Clearly, 0 and Q are trivial ideals of $(Q,\ +,\ \delta)$.

Let $(Q,\ +,\ \delta)$ be a GSA with the input set X and K an ideal of a GSA $(Q,\ +,\ \delta)$. Then the set $Q/K = \{q + K | q \in Q\}$ is a quotient group under addition as group theory. Define $\bar{\delta} : Q/K \times X \longrightarrow Q/K$ by $(q + K,\ x) \mapsto qx + K$. Then $\bar{\delta}$ becomes a state transition function. This GSA $(Q/K,\ +,\ \bar{\delta})$ is called a *quotient GSA.*

A subgroup S of $(Q,\ +)$ is called a *subgroup semiautomaton* or simply *SGSA* of a GSA $(Q,\ +,\ \delta)$ if $(S,\ +,\ \delta)$ is a GSA. This will be denoted by $S < (Q,\ +,\ \delta)$.

We note that there is no direct connections between ideals and SGSA as we shall see in the following examples.

Example 2.1. *Let us consider the integer group modulo 4:* $(Q,\ +) = (Z_4,\ +) = \{0,\ 1,\ 2,\ 3\}$ *as the set of states and* $X = \{x, y\}$ *the set of inputs.*

Define the state transition function δ *by the following rule:* $(a, x) \longmapsto 0, \forall a \in Q$, *and* $(a, y) \longmapsto 2, \forall a \in Q$.

It is easy to see that $S = \{0,\ 2\}$ *forms a SGSA of* $(Q,\ +,\ \delta)$ *with* $SX = \{0,\ 2\}$. *Also,* S *is an ideal of* $(Q,\ +,\ \delta)$, *for instance,* $(1+0)x-1x = 1x-1x = 0-0 = 0$, $(1+0)y-1y = 1y-1y = 2-2 = 0$, $(3+2)x-3x = 1x-3x = 0-0 = 0$ *and* $(3 + 2)y - 3y = 1y - 3y = 2 - 2 = 0$. *They are all elements of* S.

Example 2.2. *Let us consider the Klein 4-group* $(Q,\ +) = (K_4,\ +) = \{0,\ a,\ b,\ c\}$ *as the set of states and* $X = \{x\}$ *the set of inputs.*

Define the state transition function δ *by the following graph:*

$$a \longmapsto b \longmapsto c \longmapsto 0 \longmapsto 0$$

using x*-action. It is easy to see that* $S = \{0,\ c\}$ *forms a SGSA of* $(Q,\ +,\ \delta)$ *with* $SX = \{0\}$. *However* S *is not an ideal of* $(Q,\ +,\ \delta)$, *for* $(b + c)x - bx = ax - bx = b - c = a$. *This is not an element of* S.

Let $f : (Q,\ +,\ \delta) \longrightarrow (Q',\ +,\ \delta')$ be a GSA homomorphism. Then (1) $0f^{-1} = \{a \in Q | af = 0\}$ is called the *kernel of f*, which is denoted by $Kerf$ and (2) Qf is called the *image of f*, which is denoted by Imf. Obviously, $Kerf$ and Imf are GSA.

First, we introduce some basic properties of GSA which are proved usually as group and ring theory.

Theorem 2.3. [2] *Let* $(Q,\ +,\ \delta)$, $(Q',\ +,\ \delta')$ *be two GSA and* $f : (Q,\ +,\ \delta) \longrightarrow (Q',\ +,\ \delta')$ *be a GSA epimorphism. Then the set* $Kerf = \{q \in Q | qf = 0\}$ *is an ideal of* $(Q,\ +,\ \delta)$.

Conversely, if $K \lhd (Q,\ +,\ \delta)$, *then the canonical group homomorphism* $\pi : Q \to Q/K$ *by* $q\pi = q + K$ *is a GSA epimorphism from* $(Q,\ +,\ \delta)$ *onto* $(Q/K,\ +,\ \mu)$, *for all* $q \in Q$ *and* $Ker\pi = K$. *Where* $\mu : Q/K \times X \longrightarrow Q/K$ *via* $(q + K,\ x) \mapsto qx + K$.

Proposition 2.4. *Let* $(Q,\ +,\ \delta)$ *be a GSA. Then an ideal* K *of* $(Q,\ +,\ \delta)$ *is an SGSA if and only if* $\{0\}X \subseteq K$.

Proof. Suppose that an ideal K of $(Q,\ +,\ \delta)$ is an SGSA. Then for all $k \in K$, and $x \in X$, $kx - 0x = (0+k)x - 0x$, which is contained in K, so $0k \in K$. Hence $\{0\}X \subseteq K$. [Since K is an SGSA, $kx \in K, -0x \in K - kx \subseteq K$.]

Conversely, suppose that $\{0\}X \subseteq K$. Let $k \in K$, and $x \in X$. To show that $kx \in K$, consider $kx - 0x = (0+k)x - 0x \in K$, because that $K \lhd (Q,\ +,\ \delta)$. This equation implies $kx \in K + 0x \subseteq K$. □

Thus we obtain the following equivalent statements:

Corollary 2.5. *Let $(Q,\ +,\ \delta)$ be a GSA. Then the following statements are equivalent.*
(1) $\{0\}X = \{0\}$.
(2) $(\{0\},\ +,\ \delta)$ *is an SGSA of* $(Q,\ +,\ \delta)$.
(3) Any ideal K of $(Q,\ +,\ \delta)$ is an SGSA.

Obviously, we get the following important statement using the group theory.

Theorem 2.6. *Let $(Q,\ +,\ \delta)$, $(Q',\ +,\ \delta')$ be two GSA and $f : (Q,\ +,\ \delta) \longrightarrow (Q',\ +,\ \delta')$ be a GSA homomorphism. Then the set $Kerf = \{0\}$ if and only if f is a GSA monomorphism.*

We, again, introduce the fundamental theorem for a homomorphism of GSA which was proved in [2], a useful result to study the sequent statements.

Theorem 2.7 (Fundamental Theorem)**.** [2] *Let $(Q,\ +,\ \delta)$, $(Q',\ +,\ \delta')$ be two GSA and $f : (Q,\ +,\ \delta) \longrightarrow (Q',\ +,\ \delta')$ be a GSA homomorphism. Then*
(1) $Kerf$ is an ideal of $(Q,\ +,\ \delta)$.
(2) There exists a unique GSA monomorphism $g : Q/Kerf \to Q'$ such that $f = \pi g$, where $\pi : Q \to Q/Kerf$ is a canonical GSA epimorphism. In particular, $Q/Kerf \cong Imf$

For further discussion, in a GSA $(Q,\ +,\ \delta)$, let $S,\ T$ be the subsets of Q. We define their sum as the set

$$S + T = \{s + t|\ s \in S,\ t \in T\}.$$

It is not hard to see that the sum of two SGSA is not a SGSA in general, from Example 2.2. For example, $S = \{0,\ c\}$ and $T = \{0,\ b\}$ are two SGSA of $(K_4,\ +,\ \delta)$, but $S + T$ is not a SGSA of $(K_4,\ +,\ \delta)$.

However, we have the following substructures:

Theorem 2.8. [4] *Let $(Q,\ +,\ \delta)$ be a GSA. Then*
(1) If S is an SGSA of $(Q,\ +,\ \delta)$ and I is an ideal of $(Q,\ +,\ \delta)$, then $S+I$ is an SGSA of $(Q,\ +,\ \delta)$.
(2) If S and I are ideals of $(Q,\ +,\ \delta)$, then $S + I$ is an ideal of $(Q,\ +,\ \delta)$.

Theorem 2.9 (Second Isomorphic Theorem)**.** [4] *Let $(Q,\ +,\ \delta)$ be a GSA, S is an SGSA of $(Q,\ +,\ \delta)$ and I is an ideal of $(Q,\ +,\ \delta)$. Then*
(1) $S + I = I + S$, $S + I$ is an SGSA of $(Q,\ +,\ \delta)$, $I \lhd (S + I,\ +,\ \delta)$ and $I \cap S \lhd (S,\ +,\ \delta)$.

(2) $S/(I \cap S) \cong (I + S)/I$ *as quotient GSA.*

Theorem 2.10 (Third Isomorphic Theorem)**.** [5] *Let* $(Q,\ +,\ \delta)$ *be a GSA,* $I,\ K \triangleleft (Q,\ +,\ \delta)$ *and* $K < (Q,\ +,\ \delta)$ *with* $I \subseteq K$. *Then*
(1) $K/I \triangleleft Q/I$, *as GSA.*
(2) $(Q/I)/(K/I) \cong Q/K$, *as GSA-isomorphism.*

We can prove that the Fundamental Theorem,Second Isomorphic Theorem and Third Isomorphic Theorem in GSA are true in additive decomposition GSA.

Now, we introduce the following valuable statement which is a characterization of additive decomposition GSA.

Theorem 2.11. *Let* $(Q,\ +,\ \delta)$ *be a GSA. Then* $(Q,\ +,\ \delta)$ *is additive decomposition if and only if for some zero-input* $x_0 \in X$, *we have the following two conditions:*
(i) there exists a group homomorphism $f : Q \to Q$.
(ii) there exists a mapping $h : X \to Q$ *with* $x_0 h = 0$ *such that* $qx = qf + xh$.

Proof. Suppose that $(Q,\ +,\ \delta)$ is additive decomposition with zero-input $x_0 \in X$. Define

$$f : Q \to Q \ via\ qf = qx_0$$

From this, we see that for any $a,\ b \in Q$, $(a + b)f = (a + b)x_0 = ax_0 + bx_0 = af + bf$. Hence $f : Q \to Q$ is a group homomorphism.

Next, to prove (ii), define

$$h : X \to Q \ via\ xh = 0x$$

Then h satisfies the requirement $x_0 h = 0x_0 = 0$. Moreover, by the definition of decomposition property, we get that $qx = qx_0 + 0x = qf + xh$, for all $q \in Q$ and $x \in X$, as required.

Conversely, suppose that, given zero-input $x_0 \in X$, there exist f and h as in (i) and (ii) such that $qx = qf + xh$.

Now, $qx = qf + x_0 h + 0f + xh = qx_0 + 0x$, for all $q \in Q$ and $x \in X$, which is the decomposition property.

Also, for any $a,\ b \in Q$, $(a+b)x_0 = (a+b)f + x_0 h = (a+b)f + 0 = af + bf = ax_0 + bx_0$, which is the zero-input additivity. Consequently, $(Q,\ +,\ \delta)$ is an additive decomposition GSA. □

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