Observation on Sums of Powers of Integers

Divisible by $4k - 1$

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Abstract

Gulliver (2010) considered the sum of powers of odd integers of the forms $2k - 1$ and obtained a simple derivation of some well known sequences as well as construction of many new sequences. The observation of Gulliver has been continued by the author and Suwarno (2014). In this paper, we further observe the sum of powers of odd integers of the forms $4k - 1$ which were unknown before. We obtain a simple derivation of some well known sequences as well as construction of many new sequences. We also derive several properties of divisibility of the sequences.

Mathematics Subject Classification: 11Y55, 11B50

Keywords: integer sequences, divisibility

1 Introduction

The integer sequences have attracted many people, ranging from mathematicians, computer scientists, chemists, electrical engineers, geophysicists, astronomers, to epistemologists [3]. On the other hand, the sums of powers of odd integers have been the subject of research for many years. Among very recent results is the paper of Gulliver [1] where he introduced a new method to reconstructed several well
known integer sequences in an elementary way. His method is based on an observation to the following sum of the first $n$ of $m$-th powers

$$\sum_{k=1}^{n} (2k - 1)^m$$  (1)

and the property of its divisibility.

By looking at the sequences

$$\left\{ \sum_{k=1}^{n} (2k - 1)^m \right\}_{m=1}^{\infty},$$

Gulliver reconstructed several known sequences in an easy manner, as well as constructed many new sequences. For example, for $n = 2$, and $n = 3$ the sequences $\left\{ \sum_{k=1}^{2} (2k - 1)^m \right\}_{m=1}^{\infty}$ and $\left\{ \sum_{k=1}^{3} (2k - 1)^m \right\}_{m=1}^{\infty}$ are A034472 and A074507, in the Online Encyclopedia of Integer Sequences [2] respectively. Recently, the author and Suwarno [4] applied his method to observe the following sum of the first $n$ of $m$-th powers

$$\sum_{k=1}^{n} (3k - 1)^m$$  (2)

and the property of its divisibility.

By looking at the sequences

$$\left\{ \sum_{k=1}^{n} (3k - 1)^m \right\}_{m=1}^{\infty},$$

we succeed to reconstructed the sequences A000079, A074600, A074539, A005449, and A024394 in [2] in an easy manner, by fixing $n$. Moreover, by fixing $m$, we succeed to reconstructed the sequences A005449 and A024394 in [2]. We also constructed several new sequences. (The integer sequences related with the results in [4] are A005449, A034472, A074600, A074507, A024394, A074539).

The purpose of this paper is to continue discussions on the sequences of the form first questioned by Gulliver, namely the sequences generated by the sum of power of integers of the form

$$\sum_{k=1}^{n} (4k - 1)^m.$$

By considering sequences of the above form, we succeed to reconstruct several known sequences. We also constructed many new sequences which were unknown to exist in [2] before, besides several properties of its divisibility.
2 Results

In this section we consider the sum of the first \( n \) of \( m \)-th powers of \( 4k - 1 \) as follows

\[
\sum_{k=1}^{n} (4k - 1)^m. \tag{3}
\]

We divide our observation into two cases: \( n \) fixed and \( m \) fixed.

2.1 Case 1: \( n \) fixed

Consider the following table. The rows mean the sequences with fixed \( n \), namely

\[
\left\{ \sum_{k=1}^{n_0} (4k - 1)^m \right\}_{m=1}^{\infty}, \text{ for certain } n_0 \in \mathbb{Z}^+,
\]

and the columns mean the sequences with fixed \( m \), namely

\[
\left\{ \sum_{k=1}^{n} (4k - 1)^{m_0} \right\}_{n=1}^{\infty}, \text{ for certain } m_0 \in \mathbb{Z}^+.
\]

<table>
<thead>
<tr>
<th>value of ( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3, 9, 27, 81, 243, 729, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10, 58, 370, 2482, 17050, 118378, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>21, 179, 1701, 17123, 178101, 1889939, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>36, 404, 5076, 67748, 937476, 13280564, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>55, 765, 11935, 198069, 3413575, 60326445, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>78, 1294, 24102, 477910, 9849918, 20836445, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>105, 2023, 43785, 1009351, 24198825, 595782823, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>136, 2984, 73576, 1932872, 52827976, 1483286504, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>171, 4209, 116451, 3433497, 105349851, 3321552129, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>210, 5730, 175770, 5746938, 195574050, 6840295890, \ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Sequences of the forms (3) with fixed \( m \) and \( n \).

The first and second rows are the sequences A000244 and A074608 in [2], respectively while the other rows are new. Moreover, the first column is the sequence A014105 in [2] while the other columns are new. Considering the fifth and last rows in Table 1, we find a regularity in the unit digits, namely

**Proposition 2.1** For any positive integer \( m \), the following hold:
1. \[ 5 \mid \sum_{k=1}^{5} (4k - 1)^m \iff m \not\equiv 0 \pmod{4}. \] \hspace{1cm} (4)

2. \[ 10 \mid \sum_{k=1}^{10} (4k - 1)^m \iff m \not\equiv 0 \pmod{4}. \] \hspace{1cm} (5)

**Proof.** The first part of the proposition is equivalent with the following three statements:

For any non-negative integer \( \alpha \), we have
\[
5 \mid \sum_{k=1}^{5} (4k - 1)^{(4\alpha+1)} \quad \text{or} \quad 5 \mid \sum_{k=1}^{5} (4k - 1)^{(4\alpha+2)} \quad \text{or} \quad 5 \mid \sum_{k=1}^{5} (4k - 1)^{(4\alpha+3)}.
\]

We prove one of the above statements since the proof of another statements are similar. By induction on \( \alpha \), suppose \( 5 \mid \sum_{k=1}^{5} (4k - 1)^{(4\alpha+1)} \). Then we have
\[
\sum_{k=1}^{5} (4k - 1)^{(4\alpha+1)+1} = \sum_{k=1}^{5} (4k - 1)^{4\alpha+5} \\
= 3^4 \cdot 3^{4\alpha+1} + 7^4 \cdot 7^{4\alpha+1} + 11^4 \cdot 11^{4\alpha+1} + 15^4 \cdot 15^{4\alpha+1} + 19^4 \cdot 19^{4\alpha+1} \\
\equiv (3^{4\alpha+1} + 7^{4\alpha+1} + 11^{4\alpha+1} + 15^{4\alpha+1} + 19^{4\alpha+1}) + (15^4 - 1)15^{4\alpha+1} \\
\equiv 0 \pmod{5},
\]
and the result follows. The proof of the second part is exactly the same. \( \square \)

We may generalized Proposition 2.1 to the following.

**Proposition 2.2** For any positive integer \( r \), we have

1. \[ 5 \mid \sum_{k=1}^{5r} (4k - 1)^m \iff m \not\equiv 0 \pmod{4}, m \in \mathbb{Z}^+. \] \hspace{1cm} (6)

2. \[ 10 \mid \sum_{k=1}^{10r} (4k - 1)^m \iff m \not\equiv 0 \pmod{4}, m \in \mathbb{Z}^+. \] \hspace{1cm} (7)

**Proof.** We prove only the first part of the proposition, since the second part can also be proved by exactly the same method. By strong induction on \( r \), suppose the following hold for all positive integer \( r' \) with \( 1 \leq r' \leq r \):
\[
5 \mid \sum_{k=1}^{5r'} (4k - 1)^m \iff m \not\equiv 0 \pmod{4}, m \in \mathbb{Z}^+.
\]
Then we have

\[
\sum_{k=1}^{5r+1} (4k - 1)^m = \sum_{k=1}^{5r} (4k - 1)^m + \sum_{k=5r+1}^{5r+5} (4k - 1)^m
\]

\[
\equiv 0 + (20r + 3)^m + (20r + 7)^m + (20r + 11)^m + (20r + 15)^m + (20r + 19)^m
\]

\[
\equiv \sum_{l=0}^{m} \binom{m}{l} (20r)^{3m-l} + \sum_{l=0}^{m} \binom{m}{l} (20r)^{7m-l} + \sum_{l=0}^{m} \binom{m}{l} (20r)^{11m-l}
\]

\[
+ \sum_{l=0}^{m} \binom{m}{l} (20r)^{15m-l} + \sum_{l=0}^{m} \binom{m}{l} (20r)^{19m-l}
\]

\[
\equiv 3^m + 7^m + 11^m + 15^m + 19^m \pmod{5}.
\]

From hypothesis of induction, we have \(3^m + 7^m + 11^m + 15^m + 19^m \equiv 0 \pmod{5}\). \(\square\)

Another interesting case is the residue modulo 3 which are given in Table 2.

<table>
<thead>
<tr>
<th>power</th>
<th>n (number of terms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>1 2 3 4 5 6 7 8 9 10 ...</td>
</tr>
<tr>
<td>1</td>
<td>0 1 0 0 1 0 0 1 0 0 ...</td>
</tr>
<tr>
<td>2</td>
<td>0 1 2 2 0 1 1 2 0 0 ...</td>
</tr>
<tr>
<td>3</td>
<td>0 1 0 0 1 0 0 1 0 0 ...</td>
</tr>
<tr>
<td>4</td>
<td>0 1 2 2 0 1 1 2 0 0 ...</td>
</tr>
<tr>
<td>5</td>
<td>0 1 0 0 1 0 0 1 0 0 ...</td>
</tr>
<tr>
<td>6</td>
<td>0 1 2 2 0 1 1 2 0 0 ...</td>
</tr>
<tr>
<td>7</td>
<td>0 1 0 0 1 0 0 1 0 0 ...</td>
</tr>
<tr>
<td>8</td>
<td>0 1 2 2 0 1 1 2 0 0 ...</td>
</tr>
<tr>
<td>9</td>
<td>0 1 0 0 1 0 0 1 0 0 ...</td>
</tr>
<tr>
<td>10</td>
<td>0 1 2 2 0 1 1 2 0 0 ...</td>
</tr>
</tbody>
</table>

Table 2: The residues modulo 3 of the powers of integers of the form \((4k - 1)^m\)

If we consider the columns in the above table, we conclude that

**Proposition 2.3**

1. For any positive integer \(m\) and \(n = 1, 9, \text{ and } 10\), we have

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^m.
\]

2. For any odd integer \(m\) and \(n = 3, 4, 6, \text{ and } 7\), we have

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^m.
\]
3. For any even integer \( m \) and \( n = 5 \), we have

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^m.
\]

**Proof.** We prove only one case for each part, since the proof of another cases is exactly the same. By induction on \( m \),

1. we have

\[
\sum_{k=1}^{9} (4k - 1)^{m+1} = 3^{m+1} + 7^{m+1} + 11^{m+1} + \cdots + 23^{m+1} + 27^{m+1} + 31^{m+1} + 35^{m+1}
\]

\[
= 3(1 + 2^m) \equiv 0 \pmod{3},
\]

and the result follows.

2. we have

\[
\sum_{k=1}^{6} (4k - 1)^{2\alpha+3} = 3^{2\alpha+3} + 7^{2\alpha+3} + 11^{2\alpha+3} + 15^{2\alpha+3} + 19^{2\alpha+3} + 23^{2\alpha+3}
\]

\[
= 2(1 + 2^{2\alpha+1}) \equiv 0 \pmod{3}, \text{ (since } 1 + 2^{2\alpha+1} \equiv 0 \pmod{3}),
\]

and the result follows.

3. we have

\[
\sum_{k=1}^{5} (4k - 1)^{2\alpha+2} = 3^{2\alpha+2} + 7^{2\alpha+2} + 11^{2\alpha+2} + 15^{2\alpha+2} + 19^{2\alpha+2}
\]

\[
= 2(1 + 2^{2\alpha+1}) \equiv 0 \pmod{3},
\]

and the result follows. \( \square \)

Again, we may generalized Proposition 2.3 to the following.

**Proposition 2.4** 1. For any non-negative integer \( r \) and \( n = 1 + 9r, n = 9 + 9r, \) and \( n = 10 + 9r \), we have

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^m, \text{ with } m \in \mathbb{Z}^+.
\]

2. For any non-negative integer \( r \) and \( n = 3 + 9r, 4 + 9r, 6 + 9r, \) and \( 7 + 9r \), we have

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^m, \text{ with } m \in 2\mathbb{Z}^+ + 1.
\]
3. For any positive integer r, we have

$$3 \bigg| \sum_{k=1}^{5+9r} (4k-1)^m, \text{ with } m \in 2\mathbb{Z}^+. $$

**Proof.** We prove only one case in the first part, since the proof for another cases in each part are exactly the same. By induction on $r$, suppose $3 \bigg| \sum_{k=1}^{9+9r} (4k-1)^m$, with $m \in \mathbb{Z}^+$. Then we have

$$\sum_{k=1}^{9+9(r+1)} (4k-1)^m = \sum_{k=1}^{9+9r} (4k-1)^m + \sum_{k=10+9r}^{18+9r} (4k-1)^m$$

$$\equiv 0 + (36r+39)^m + (36r+43)^m + \cdots + (36r+71)^m$$

$$\equiv 3(0 + 1 + 2^m) \equiv 0 \pmod{3},$$

and the result follows. □

**Remark 2.1**
1. The more accurate statement of Conjecture 2.1 in [4] should be as follows. "For any positive integer r, we have

\begin{equation}
(a) \quad 5 \bigg| \sum_{k=1}^{5r} (4k-1)^m \iff m \not\equiv 0 \pmod{4}, \ m \in \mathbb{Z}^+. \tag{8}
\end{equation}

\begin{equation}
(b) \quad 10 \bigg| \sum_{k=1}^{10r-5} (4k-1)^m \iff m \not\equiv 0 \pmod{4}, \ m \in \mathbb{Z}^+.\tag{9}
\end{equation}

2. The more accurate statement of Conjecture 2.2 in [4] should be as follows. "For any positive integer r, we have

\begin{equation}
3 \bigg| \sum_{k=1}^{3r} (4k-1)^m, \text{ with } m \in \mathbb{Z}^+. \tag{10}
\end{equation}

3. Conjecture 2.1 and 2.2 above can be proved by a similar method as above.

**2.2 Cases 2: m fixed**

Finally, we consider the sequences from the columns in (2.1). As mentioned above, for $m = 1$ we obtained

$$\left\{ \sum_{k=1}^{n} (4k-1) \right\}_{n=1}^{\infty} = (n(2n+1))_{n=1}^{\infty}.$$
which is the sequence A014105 in the *On-line Encyclopedia of Integer Sequences* [2].

For \( m = 2, 3, \) and 4 we obtained, respectively the following sequences

\[
\left\{ \frac{1}{3}n(16n^2 + 12n - 1) \right\}_{n=1}^{\infty},
\]

\[
\left\{ n(16n^3 + 16n^2 - 2n - 3) \right\}_{n=1}^{\infty},
\]

and

\[
\left\{ \frac{1}{15}(768n^4 + 960n^3 - 160n^2 - 360n + 7) \right\}_{n=1}^{\infty},
\]

which are all new.

We can also investigate the divisibility of the above sequences. For example, we can show easily that for any positive integer \( n = 4a, 5a, \) and \( 7a, a \geq 1, \) we have

\[
4 \mid \sum_{k=1}^{n} (4k - 1)^2, \quad 5 \mid \sum_{k=1}^{n} (4k - 1)^2, \quad \text{and} \quad 7 \mid \sum_{k=1}^{n} (4k - 1)^2,
\]

respectively. We have also

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^3,
\]

for \( n = 3a \) and \( n = 3a - 2, a \geq 1, \) while for \( n = 5a \) and \( n = 5a - 3, a \geq 1, \) we have

\[
5 \mid \sum_{k=1}^{n} (3k - 1)^3.
\]

Finally, for \( n = 4a, 7a, \) and \( 9a, a \geq 1, \) we have

\[
3 \mid \sum_{k=1}^{n} (4k - 1)^4.
\]

By a very similar observation, we may also have properties of divisibility of another sequences in an easy way.

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References


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