Symmetric Identities for Carlitz’s Twisted
q-Bernoulli Polynomials Associated with
p-Adic q-Integral on $\mathbb{Z}_p$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract
In this paper, we establish some interesting symmetric identities for Carlitz’s twisted $q$-Bernoulli polynomials in $p$-adic field.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Bernoulli numbers and polynomials, $q$-Bernoulli numbers and polynomials, Carlitz’s twisted $q$-Bernoulli numbers and polynomials

1 Introduction

The Bernoulli numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Many mathematicians have studied in the area of the $q$-Bernoulli numbers and polynomials(see [1-5]). Recently, Y. Hu studied several identities of symmetry for Carlitz’s $q$-Bernoulli numbers and polynomials in complex field(see [1]). D. Kim et al.[3] derived some identities of symmetry for Carlitz’s $q$-Bernoulli numbers and polynomials by using the $p$-adic $q$-integrals on $\mathbb{Z}_p$ in $p$-adic field. In this paper, we establish some interesting symmetric identities for Carlitz’s twisted $q$-Bernoulli polynomials in $p$-adic field. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. 
\( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{Z} \) denotes the ring of rational integers, \( \mathbb{Q} \) denotes the field of rational numbers, \( \mathbb{C} \) denotes the set of complex numbers, and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-\frac{1}{p^m}} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \).

Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad (\text{cf. [1-5]}).
\]

Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case. For \( g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\} \), the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} g(x)q^x \quad (\text{cf. [2]}). \tag{1.1}
\]

Let

\[
T_p = \cup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},
\]

where \( C_{p^m} = \{\zeta | \zeta^{p^m} = 1\} \) is the cyclic group of order \( p^m \). For \( \zeta \in T_p \), we denote by \( \phi_{\zeta} : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \mapsto \zeta^x \). If we take \( f_1(x) = f(x + 1) \) in (1.1), then we easily see that

\[
qI_q(f_1) = I_q(f) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0). \tag{1.2}
\]

From (1.2), we obtain

\[
q^n I_q(f_n) = I_q(f) + (q - 1) \sum_{l=0}^{n-1} q^l f(l) + \frac{q - 1}{\log q} \sum_{l=0}^{n-1} f'(l), \tag{1.3}
\]

where \( f_n(x) = f(x + n) \) (cf. [2-3]).

### 2 Symmetric identities for Carlitz’s twisted \( q \)-Bernoulli numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz’s twisted \( q \)-Bernoulli numbers \( \beta_{n,q,\zeta} \) and polynomials \( \beta_{n,q,\zeta}(x) \). By using the similar method of [3], expect for obvious modifications, we are going to obtain
the main results of Carlitz’s twisted $q$-Bernoulli polynomials. For $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-\frac{1}{p-1}}$, twisted $q$-Bernoulli numbers $\beta_{n,q,\zeta}$ are defined by

$$\beta_{n,q,\zeta} = \int_{\mathbb{Z}_p} \phi_\zeta(x)[x]_q^nd\mu_q(x). \quad (2.1)$$

By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain,

$$\int_{\mathbb{Z}_p} \phi_\zeta(x)[x]_q^nd\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q^{N-1}} \sum_{x=0}^{p^N-1} \zeta^x[x]_q^n q^x$$

$$= \left( \frac{1}{1 - q} \right)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1 + l}{1 - \zeta q^{1+l}}. \quad (2.2)$$

By (2.1), we have

$$\beta_{n,q,\zeta} = \left( \frac{1}{1 - q} \right)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1 + l}{1 - \zeta q^{1+l}}.\quad (2.3)$$

We set

$$F_{q,\zeta}(t) = \sum_{n=0}^{\infty} B_{n,q,\zeta} \frac{t^n}{n!}.$$

By using above equation and (2.2), we have

$$F_{q,\zeta}(t) = \sum_{n=0}^{\infty} \beta_{n,q,\zeta} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \zeta^x e^{[x]_q t} d\mu_q(x)$$

Next, we introduce twisted $q$-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$. The twisted $q$-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ are defined by

$$\beta_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)[x + y]_q^n d\mu_q(y).$$

By using $p$-adic $q$-integral, we obtain

$$\beta_{n,q,\zeta}(x) = \left( \frac{1}{1 - q} \right)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1 + l}{1 - \zeta q^{1+l}}. \quad (2.4)$$

We set

$$F_{q,\zeta}(t, x) = \sum_{n=0}^{\infty} \beta_{n,q,\zeta}(x) \frac{t^n}{n!}.$$
Since \([x + y]_q = [x]_q + q^x[y]_q\), we easily obtain that

\[
\beta_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)[x + y]_q^n d\mu_q(y)
= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{x l} \beta_{l,q,\zeta}
= ([x]_q + q^x \beta_{q,\zeta})^n
\]

(2.5)

Let \(w_1\) and \(w_2\) be natural numbers. Then we have

\[
\frac{1}{[w_1]_q} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q} d\mu_{q^{w_1}}(y)
= \lim_{N \to \infty} \frac{1}{[w_1]_q} \frac{1}{[p^N]_q^{w_1}} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_2 j + w_1 y]_q} \zeta^{w_1 y} q^{w_1 y}
\]

(2.6)

From (2.6), we can derive the following equation (2.7):

\[
\frac{1}{[w_1]_q} \sum_{j=0}^{w_1-1} \zeta^{w_2 j} q^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q} d\mu_{q^{w_1}}(y)
= \lim_{N \to \infty} \frac{1}{[w_1 w_2 p^N]_q} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_2 j + w_1 i + w_1 w_2 y]_q} q^{w_1 i} q^{w_1 y} 
\]

(2.7)

By the same method as (2.7), we have

\[
\frac{1}{[w_2]_q} \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 y]_q} d\mu_{q^{w_2}}(y)
= \lim_{N \to \infty} \frac{1}{[w_1 w_2 p^N]_q} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} e^{[w_1 w_2 x + w_1 j + w_2 i + w_1 w_2 y]_q} q^{w_2 i} q^{w_1 y}
\]

(2.8)

Therefore, by (2.7) and (2.8), we have the following theorem.

**Theorem 2.1** For \(w_1, w_2 \in \mathbb{N}\), we have

\[
\frac{1}{[w_1]_q} \sum_{j=0}^{w_1-1} \zeta^{w_2 j} q^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q} d\mu_{q^{w_1}}(y)
= \frac{1}{[w_2]_q} \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 y]_q} d\mu_{q^{w_2}}(y).
\]

(2.9)
By substituting Taylor series of $e^{xt}$ into (2.9) and after elementary calculations, we obtain the following corollary.

**Corollary 2.2** For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

\[
[w_1]_{q}^{n-1} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{w_2j} \int_{\mathbb{Z}_p} \zeta^{w_1y} \left[ w_2x + \frac{w_2}{w_1}j + y \right]^{n}_{q^{w_1}} d\mu_{q^{w_1}}(y) = [w_2]_{q}^{n-1} \sum_{j=0}^{w_2-1} \zeta^{w_1j} q^{w_1j} \int_{\mathbb{Z}_p} \zeta^{w_2y} \left[ w_1x + \frac{w_1}{w_2}j + y \right]^{n}_{q^{w_2}} d\mu_{q^{w_2}}(y).
\]

(2.10)

By (2.5) and Corollary 2.2, we have the following theorem.

**Theorem 2.3** For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

\[
[w_1]_{q}^{n-1} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{w_2j} \beta_{n,q^{w_1},\zeta^{w_1}} \left( w_2x + \frac{w_2}{w_1}j \right) = [w_2]_{q}^{n-1} \sum_{j=0}^{w_2-1} \zeta^{w_1j} q^{w_1j} \beta_{n,q^{w_2},\zeta^{w_2}} \left( w_1x + \frac{w_1}{w_2}j \right).
\]

By (2.5), we can derive the following equation (2.11):

\[
\int_{\mathbb{Z}_p} \zeta^{w_1y} \left[ w_2x + \frac{w_2}{w_1}j + y \right]^{n}_{q^{w_1}} d\mu_{q^{w_1}}(y) = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \left[ \frac{[w_2]_{q}}{[w_1]_{q}} \right]^{i} [j]^{i}_{q^{w_2}} q^{w_2(n-i)j} \beta_{n-i,q^{w_1},\zeta^{w_1}} (w_2x).
\]

(2.11)

By (2.11), and Theorem 3, we have

\[
[w_1]_{q}^{n-1} \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{w_2j} \int_{\mathbb{Z}_p} \zeta^{w_1y} \left[ w_2x + \frac{w_2}{w_1}j + y \right]^{n}_{q^{w_1}} d\mu_{q^{w_1}}(y) = \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{w_2j} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]_{q}^{i} [w_1]_{q}^{n-i-1} [j]^{i}_{q^{w_2}} q^{w_2(n-i)j} \beta_{n-i,q^{w_1},\zeta^{w_1}} (w_2x)
\]

(2.12)

\[
= \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]_{q}^{i} [w_1]_{q}^{n-i-1} \beta_{n-i,q^{w_1},\zeta^{w_1}} (w_2x) \sum_{j=0}^{w_1-1} \zeta^{w_2j} q^{w_2(n-i+1)j} [j]^{i}_{q^{w_2}}
\]

\[
= \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_2]_{q}^{i} [w_1]_{q}^{n-i-1} \beta_{n-i,q^{w_1},\zeta^{w_1}} (w_2x) T_{n,i}(w_1, \zeta^{w_2}, q^{w_2}),
\]

where

\[
T_{n,i}(w_1, \zeta, q) = \sum_{j=0}^{w_1-1} \zeta^{j} q^{(n-i+1)j} [j]^{i}_{q}.
\]
By the same method as (2.12), we get

\[
[w_2]^n q^{-1} n \sum_{j=0}^{w_2-1} \zeta^{w_1 j} q^{w_1 j} \int_{Z_p} \zeta^{w_2 y} \left[ w_1 x + \frac{w_1}{w_2} j + y \right]^n d\mu_q (y)
\]

(2.13)

By (2.12) and (2.13), we have the following theorem.

**Theorem 2.4** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
\sum_{i=0}^{n} \binom{n}{i} [w_1]^i [w_2] q^{n-i-1} \beta_{n-i,q,w_2,\zeta} \left( w_1 x \right) T_{n,i}(w_2, \zeta^{w_1}, q^{w_1}).
\]

By (2.5) and Theorem 2.4, we have the following corollary.

**Corollary 2.5** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
\sum_{i=0}^{n} \binom{n}{i} [w_1] q^{n-i-1} \beta_{n-i,q,w_2,\zeta} \left( w_1 x \right) T_{n,i}(w_2, \zeta^{w_1}, q^{w_1}).
\]

**References**


Received: March 23, 2015; Published: April 27, 2015