Characteristic Block-Centered

Finite Difference Method for Sobolev Equation

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Abstract

A characteristic block-centered difference method is presented for Sobolev equation in a bounded domain, the theoretical analysis show that the scheme is convergence, and we get the discrete $L^2$-norm errors in both the approximate solution and its first derivatives for all nonuniform grids. Meanwhile we have the same result in the numerical experiments.

Keywords: Sobolev equation; characteristic block-centered difference method; error estimates

1. Introduction

In mechanics, physics and other applications, we often encountered in the convection-dominated convection-diffusion equation, Sobolev equation is one such like that. The existence and uniqueness of its solution has been given in [1], [2], to show its hyperbolic features, in this block we use the characteristic block-centered difference method to construct the form, the calculation is simple and we can get the $O(\Delta t + \Delta th + h^2)$ error estimation. Numerical experiment is consistent with the theoretical analysis.

Consider the following initial-boundary value problem of Sobolev equation:

\begin{align*}
    u_t + \alpha(x,t)u_x - (\lambda u_{xx} + \theta u_{xxt}) &= f(x,t), \quad (x,t) \in \Omega \times (0, T], \\
    u(x,0) &= u_0(x), \quad x \in \Omega, \\
    u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times [0, T].
\end{align*}

(1)\hspace{6cm} (2)\hspace{6cm} (3)
where $\Omega$ is one dimension bounded area, assume that $\Omega = (0, 1)$. $\lambda, \theta$ are constants more than 0, let $J = [0, T]$, for any $(x, t) \in \Omega \times J$, make the following assumptions on these issues:

(i) $\forall (x, t) \in \Omega \times [0, T], |\alpha(x, t)| \leq C$.

(ii) $f, u_0$ are sufficiently smooth.

Note:

$\psi(x, t) = \sqrt{1 + \alpha^2(x, t)}, p = -\frac{\partial u}{\partial x}$.

We use $\tau$ to express the characteristic direction about $u_t + \alpha u_x$, then we have

$$\frac{\partial}{\partial \tau} = \frac{1}{\psi(x, t)} \frac{\partial}{\partial t} + \frac{\alpha(x, t)}{\psi(x, t)} \frac{\partial}{\partial x}.$$ 

And (1) – (3) is equivalent to:

$$\psi(x, t) \frac{\partial u}{\partial \tau} + \lambda \frac{\partial p}{\partial x} + \theta \frac{\partial^2 p}{\partial x \partial t} = f(x, t), \quad (x, t) \in \Omega \times (0, T),$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

$$u(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, T].$$

In addition, note

$$\|u\|_{C^s(\Omega \times J)} = \sup_{(x, t) \in \Omega \times J, |k| \leq s} |\frac{\partial u^k}{\partial x^{i_1} \partial t^{i_2}}|, k = i_1 + i_2.$$ 

2. PRELIMINARIES

We define the partition $\delta_x$ in $\Omega = (0, 1)$ as:

$$\delta_x : 0 = x_1 < x_\frac{1}{2} < \ldots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = x_{N+1} = 1$$

From $i = 2$ to $N$, define:

$$x_i = \frac{x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}}{2}, h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, h = \max_i h_i,$$

$$h_{i+\frac{1}{2}} = x_{i+1} - x_i = \frac{h_i + h_{i+1}}{2}, \Omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}).$$

$$\varphi_i = \frac{1}{2}(\varphi_{i-\frac{1}{2}} + \varphi_{i+\frac{1}{2}}), h = \min_i h_i.$$ 

In addition: $h_1 = 2(x_{\frac{1}{2}} - x_1)$.

Note $\Delta t$ as time step, $t^n = n \cdot \Delta t, n = 0, 1, \ldots, L; L = \left[\frac{T}{\Delta t}\right]$;

$$g^n_{i} = g(x_i, t^n), g_{i+\frac{1}{2}} = g(x_{i+\frac{1}{2}}),$$

$$[d_x g]_{i+\frac{1}{2}} = \frac{g_{i+1} - g_i}{h_{i+\frac{1}{2}}}, [D_x g]_i = \frac{g_{i+\frac{1}{2}} - g_{i-\frac{1}{2}}}{h_i}.$$
Characteristic block-centered finite difference method

For \(f(x), g(x)\), define the following two Grid function groups:

\[
S = \{ f_i | f_i = f(x_i), f_1 = f_{N+1} = 0, i = 1, \ldots, N + 1 \},
\]

\[
S^{(1)} = \{ f_{i - \frac{1}{2}} | f_{i - \frac{1}{2}} = f(x_{i - \frac{1}{2}}), i = 2, \ldots, N + 1 \}.
\]

and its inner product and norm:

\[
(f, g)_M = \sum_{i=2}^{N} h_i f_i g_i, \quad \|f\|_M^2 = (f, f)_M,
\]

\[
(f, g)_x = \sum_{i=2}^{N+1} h_{i - \frac{1}{2}} f_{i - \frac{1}{2}} g_{i - \frac{1}{2}}, \quad \|f\|_x^2 = (f, f)_x.
\]

Let \(f \in [0, 1]\), define

\[
\|f\|_l^2 = \sum_{i=2}^{N} h_i \max \{|f(x)| : |x - x_i| < \frac{h_i}{2}\},
\]

\[
\|f\|_l^2 = \sum_{i=2}^{N} h_i \max \{|f(x)| : |x - x_i| < \bar{C} \Delta t\}.
\]

where \(\bar{C} = \sup_{(x,t) \in \Omega \times J} |\alpha(x,t)|\).

If there exist a constant \(\sigma > 0\), s.t.

\[
\frac{h}{\bar{h}} \leq \sigma,
\]

then \(\Omega\) is called split regularly. Now assume that \(c_1 h^2 \leq \Delta t \leq c_2 h^2\), \(c_i > 0, i = 1, 2\) are constants, now if \(\Omega\) is split regularly, then when

\[
h \leq \frac{1}{2c_2 \bar{C} \sigma},
\]

we have \(\Delta t \leq \frac{h}{2c_2 \bar{C}}\).

**Lemma 2.1.** \([3]\) Let \((f, g) \in S \times S^{(1)}\), then we have:

\[
(f, D_x g)_M = -(d_x f, g)_x.
\]

**Lemma 2.2.** \([5]\) Suppose the following conditions hold:

(1): \(u|_{\partial \Omega} = 0\)

(2): \(\frac{\partial u^{s+2}}{\partial x^2} \) is Lipschitz continuous in \(\Omega\), \(s = 0, 1\), then there exist \(\frac{\partial^s Q}{\partial t^s} \in S; \frac{\partial^s V^n}{\partial t^s} \in S^{(1)}\), s.t.:

\[
\left| \frac{\partial^s}{\partial t^s} (u_i - Q_i) \right| \leq C_0^{(s)} h^2, \quad \left| \frac{\partial^s}{\partial t^s} (p_{i - \frac{1}{2}} - V_{i - \frac{1}{2}}) \right| \leq C_0^{(s)} h^2
\]

\[
V_{i - \frac{1}{2}} = -(d_x Q)_{i - \frac{1}{2}}
\]

where \(C_0^{(s)} > 0\) is constant.

**Lemma 2.3.** \([5]\) Let \(f \in S\), then we have

\[
\|\bar{f}\|_M \leq \|f\|_x
\]
3. CHARACTERISTIC BLOCK CENTRAL DIFFERENCE SCHEME AND ERROR ESTIMATES

The characteristic block central difference scheme of problem (1) – (3) is: find \((U^n, Z^n) \in S \times S^{(1)}\) s.t.

\[
\begin{align*}
\frac{U^n - U^{n-1}}{\Delta t} + [D_x(\lambda Z^n)]_i + [D_x(\theta U^n)]_i - [D_x(\theta Z^{n-1})]_i &= f_i^n, \\
Z_i^{n-\frac{1}{2}} &= -[d_x U^n]_{i-\frac{1}{2}}, \\
U_i^n &= u_0(x_i), i = 2, \ldots, N.
\end{align*}
\]

where:

\[
U^{n-1}(x) = U^{n-1} - (x - x_i)Z_i^{n-1}, x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}).
\]

\[
\tilde{U}_i^{n-1} = U^{n-1}(\tilde{x}_i), \tilde{x}_i = x_i - \alpha(x, t)\Delta t.
\]

Use (8) in (7) we get the linear algebraic equations about the coefficient matrix is irreducible diagonally dominant tridiagonal matrix, so its solution vector is exist and unique, and we can get it by pursuit method.

At the point \((x_i, t^n)\), (4) can be written as:

\[
\frac{u_i^n - \tilde{u}_i^{n-1}}{\Delta t} + [D_x(\lambda p^n)]_i + [D_x(\theta p^{n-1})]_i + R_i^n + A_i^n + B_i^n = f_i^n,
\]

where:

\[
R_i^n = \psi(x_i, t^n)(\frac{\partial u}{\partial x})_i^n - \frac{u_i^n - \tilde{u}_i^{n-1}}{\Delta t} \\
A_i^n = (\lambda \frac{\partial p}{\partial x})_i^n - [D_x(\lambda p^n)]_i \\
B_i^n = (\theta \frac{\partial p}{\partial x})_i^n - \frac{1}{\Delta t}([D_x(\theta p^n)]_i - [D_x(\theta p^{n-1})]_i)
\]

Launch \(u_i^n\) along the direction of characteristics we obtain:

\[
\begin{align*}
\frac{u_i^n - \tilde{u}_i^{n-1}}{\Delta t} &= \frac{1}{\Delta t} \left[ \frac{\partial u}{\partial x} \right] (x_i - \tilde{x}_i)^2 + (t^n - t^{n-1})^2 \\
&- \int_{(x_i, t^{n-1})}^{(x_i, t^n)} (x(\tau) - \tilde{x}_i)^2 + (t(\tau) - t^{n-1})^2 \frac{\partial u}{\partial x} d\tau \\
&= \frac{1}{\Delta t} \left[ \frac{\partial u}{\partial x} \right] (x_i - \tilde{x}_i)^2 + (t(t) - t^{n-1})^2 \frac{\partial u}{\partial x} d\tau
\end{align*}
\]

then:

\[
\begin{align*}
\|R^n\|_M^2 &= \|\left[ \frac{\partial u}{\partial x} \right] (x_i - \tilde{x}_i)^2 + (t^n - t^{n-1})^2 \|_M^2 \\
&\leq \| \frac{1}{\Delta t} \cdot \psi \Delta t \int_{(x_i, t^{n-1})}^{(x_i, t^n)} \frac{\partial u}{\partial x} d\tau\|_2^2
\end{align*}
\]

With integral remainder Taylor formula and schwanz inequality we obtain:

\[
\begin{align*}
(A_i^n)^2 &= \left( (\lambda \frac{\partial p}{\partial x})_i^n - [D_x(\lambda p^n)]_i \right)^2 \\
&= \left( -\int_{0}^{\frac{h_i}{2}} (\frac{h_i}{2} - z) \frac{\partial(\lambda p^n)}{\partial x} (x_i + z, t^n)dz - \int_{\frac{h_i}{2}}^{h_i} (\frac{h_i}{2} + z) \frac{\partial(\lambda p^n)}{\partial x} (x_i + z, t^n)dz \right)^2 \\
&\leq Ch^4 \left( \max \left| \frac{\partial^2(\lambda p^n)}{\partial x^2} \right| \right)^2
\end{align*}
\]

then: \(\|A^n\|_M^2 \leq Ch^4 \left| \frac{\partial^2(\lambda p^n)}{\partial x^2} \right|_2^2\)
Similarly:
\[
(B^n_i)^2 = \left(\frac{\partial^2 p^n_i}{\partial x^2}\right) - [D_x(\theta^n p^n)]_i + \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n}(s - t^{n-1})D_x(\theta^n p^n)ds)^2
\]
\[
\leq \left(\frac{\partial^2 p^n_i}{\partial x^2}\right) - [D_x(\theta^n p^n)]_i + \Delta t \frac{1}{2} \left(\int_{t_{n-1}}^{t_n}(D_x(\theta^n p^n))^2ds\right)^{\frac{3}{2}}
\]
\[
\leq Ch^4 \left(\max\{\frac{\partial^2(p^n_i)}{\partial x^2}, |x_i| \leq \frac{h_i}{2}\}\right)^2 + \Delta t \int_{t_{n-1}}^{t_n} \left(\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(\theta^n p^n)dx\right)^2 ds
\]
we get:
\[
\|B^n\|_L^2 \leq C h^4 \left(\max\{\frac{\partial^2(p^n_i)}{\partial x^2}, |x_i| \leq \frac{h_i}{2}\}\right)^2 + \Delta t \cdot h^2 \|\theta^n p^n\|_{L^2(t^{n-1}, t^n)}^2
\]
Let:
\[
Q^{n-1}(x) = Q_{1}^{n-1} - (x - x_i)\bar{V}_{i}^{n-1}, x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}).
\]
\[
\xi = Z - V, \mu = V - p, \eta = U - Q, \gamma = Q - u.
\]
(7) minus (10) we get:
\[
\begin{align*}
\frac{\eta^n - \eta^{n-1}}{\Delta t} + [D_x(\lambda \xi^n)]_i + \frac{1}{\Delta t}[D_x(\theta(\xi^n - \xi^{n-1}))]_i - \frac{1}{\Delta t}[D_x(\theta^{n-1} - \gamma^n)]_i - [D_x(\mu^n)]_i + \frac{1}{\Delta t}[D_x(\theta^n - \gamma^n)]_i &
\end{align*}
\]
(11)
Make M inner product on both sides of (11):
\[
\begin{align*}
(D^n(\eta^n) - \eta^{n-1} - \eta^n) + M(D^n(\lambda \xi^n, \eta^n) + (\frac{1}{\Delta t}D_x(\theta(\xi^n - \xi^{n-1})), \eta^n))_M &
\end{align*}
\]
(12)
Now we will estimate one by one.
Use schwartz inequality and Lemma 1 we get:
\[
\begin{align*}
(D^n(\eta^n) - \eta^{n-1} - \eta^n)_M = \frac{1}{\Delta t}(\|\eta^n\|^2_\Delta - (\eta^n, \eta^{n-1}))-M &
\end{align*}
\]
(13)
\[
\begin{align*}
(D^n(\lambda \xi^n, \eta^n)_M = \lambda(\xi^n, \xi^n)_x &
\end{align*}
\]
(14)
\[
\begin{align*}
\frac{\partial}{\partial x}(D^n(\theta(\xi^n - \xi^{n-1}), \xi^n))_x = \frac{\partial}{\partial x}(\theta(\xi^n - \xi^{n-1}, \xi^n)_x &
\end{align*}
\]
(15)
According to \(\alpha(x,t)\) is bounded we have:
\[
\begin{align*}
|\tilde{V}_{i}^{n-1} - Q_i^{n-1} - U_i^{n-1} + Q_i^{n-1}| &
\end{align*}
\]
From Lemma 3
\[
\begin{align*}
|\tilde{V}_{i}^{n-1} - Q_i^{n-1}, \eta^n)_M &
\end{align*}
\]
(17)
\[ \tilde{z}_i^{n-1} = \gamma_i^n + (\tilde{x}_i - x_i)\frac{\partial \gamma}{\partial x}(x_i, t^n) - \Delta t \frac{\partial \gamma}{\partial t}(x_i, t^n) + O(\Delta t^2) \]
\[ \frac{\partial \gamma}{\partial x}(x_i, t^n) = \frac{\partial Q}{\partial x}(x_i, t^n) - \frac{\partial Q}{\partial x}(x_i, t^n) = -V_i^n + \tilde{p}_i^n + \frac{1}{8} h_i^2 \frac{\partial^2 u}{\partial x^2}(\tilde{x}_i, t^n) \]
\[ \dot{x}_i = \xi x_i - \frac{1}{2} (1 - \xi) x_{i+\frac{1}{2}}, 0 < \xi < 1 \]

According to Lemma 2,
\[ \frac{\partial \gamma}{\partial t}(x_i, t^n) = O(h^2) \]
\[ \left| \left( \frac{\tilde{z}_i^{n-1} - \gamma_i^n}{\Delta t}, \gamma_i^n \right) \right| \leq \frac{1}{2} C \left\{ \Delta t^2 + h^4 + \| \eta^n \|^2_M \right\} \]
\[ \left| (D_x(\lambda \mu \eta^n), \eta^n) \right| = |(\lambda \mu)^n, \xi^n) | \]
\[ \leq \frac{1}{2} C (h^4 + \| \xi^n \|^2_2) \]

(18)

Use value theorem and Lemma 1 we get:
\[ \left| \left( \frac{D_x(\theta \mu^n) - D_x(\theta \mu^n^{-1})}{\Delta t}, \eta^n \right) \right| \]
\[ = \left| \left( \theta \mu^n - \theta \mu^n^{-1}, \xi^n \right) \right| \]
\[ \leq C \left\{ \left( \frac{\| \theta \mu^n \|_L^2}{\Delta t} \right)^2 + \| \xi^n \|^2_2 \right\} \leq C \left\{ h^4 + \| \xi^n \|^2_2 \right\} \]

(19)

Where \( t^* \in (t^{n-1}, t^n) \).

Use \( (13) - (19) \) in \( (12) \) we get:
\[ \frac{1}{2 \Delta t} \left\{ \| \eta^n \|^2_M - \| \eta^{n-1} \|^2_M \right\} + \lambda \| \xi^n \|^2_M + \frac{\theta}{2 \Delta t} \left\{ \| \xi^n \|^2_M - \| \xi^{n-1} \|^2_M \right\} \]
\[ \leq C \left\{ \left( \frac{\Delta t}{2 \Delta t} \right)^2 \| \theta \mu^n \|^2_L (0, T: \Omega^2) + \Delta t h^2 \| \theta \frac{\partial \mu}{\partial x} \|^2_L^2 (0, t^n: \Omega^2) \right\} \]
\[ + h^2 \| \frac{\partial (\theta \mu^n)}{\partial x} \|^2_L^2 + \| \eta^n \|^2_M + \| \xi^{n-1} \|^2_2 + \Delta t^2 + h^4 \]

Both sides of the above multiplied by \( 2 \Delta t \), sum \( n \) from \( 1 \) to \( L \) we obtain:
\[ \| \eta^L \|^2_M - \| \eta^0 \|^2_M + \theta (\| \xi^L \|^2_x - \| \xi^0 \|^2_x) \]
\[ \leq C \left( \Delta t^2 \| \frac{\partial \mu}{\partial t} \|^2_L^2 (0, T: \Omega^2) + \Delta t^2 h^2 \| \theta \frac{\partial \mu}{\partial x} \|^2_L^2 \right) + \Delta t^2 + h^4 + \Delta t \sum_{n=1}^{L-1} (\| \eta^n \|^2_M + \theta \| \xi^n \|^2_2) \]

Choose \( C \Delta t \leq \frac{1}{2} \), then:
\[ \| \eta^L \|^2_M + \theta \| \xi^L \|^2_x \]
\[ \leq \| \eta^0 \|^2_M + \theta \| \xi^0 \|^2_x + C (\Delta t^2 + \Delta t^2 h^2 + h^4 + \Delta t \sum_{n=1}^{L-1} (\| \eta^n \|^2_M + \theta \| \xi^n \|^2_2)) \]

(20)

Use discrete Gronwall inequality in (20) we obtain:
\[ \| \eta^L \|^2_M + \theta \| \xi^L \|^2_x \leq C (\Delta t^2 + \Delta t^2 h^2 + h^4) + C (\| \eta^0 \|^2_M + \theta \| \xi^0 \|^2_x) \]

From the choosing initial value we know: \( \| \eta^0 \|_M = 0, \| \xi^0 \|_x = 0 \).

Thus:
\[ \| \eta^L \|^2_M + \theta \| \xi^L \|^2_x \leq C (\Delta t^2 + \Delta t^2 h^2 + h^4) \]

**Theorem 3.1.** Let \( u \) and \( p \) are the solution of (4), \( U^n \) and \( Z^n \) are the solution of (7) – (9), assume that the grid in the solving area is splited regularly, then
there exist constant $C$ which is nothing to do with $h, \Delta t$, s.t. for any $0 \leq n \leq L(L \leq \lceil \frac{T}{\Delta t} \rceil)$ we have:

$$
\| Z^n - p^n \|_x \leq \| \xi^n \|_x + \| \mu^n \|_x \leq C(\Delta t + \Delta th + h^2),
$$

$$
\| U^n - u^n \|_M \leq \| \eta^n \|_M + \| \gamma^n \|_M \leq C(\Delta t + \Delta th + h^2).
$$

References


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