The Method of Determining the Locations of Reinforcing Elements in a Composite Orthotropic Plate Undergoing Dynamic Impact.

Part 1. Wave Problem

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Abstract

This investigation is devoted to modification of the wave approach to the problem of determining the dynamic characteristics of deformation of the plane target due to the impact, as well as application of this method to determination of the points of interaction between wave fronts after their multiple reflections from edges of the plane element. In the points where various wave surfaces converge, the extreme values of the voltage of different signs can be observed. In this work, a framework for reinforcing plane orthotropic elements made of composite materials in the points of interaction of elastic waves is proposed. The method of
constructing a wave portrait in the plate is based on the principle of superposition of two separate tasks: the contact problem of the impact of the initial dynamic load and the wave problem of the target’s deformation over time, including deformation due to propagation of elastic waves having a finite velocity. In this work, the wave problem is investigated and the solution is obtained in an analytical form for the functions of the displacements.

**Keywords:** Dynamic impact, Uflyand-Mindlin-Reissner plate, Elastic waves, Orthotropic properties, Reflected waves, Spherical functions, Hyperbolic equations

1 Introduction

Currently, the increasing interest in application of composite materials in mechanical and construction engineering cause researchers to move towards more complex rheological models of the materials used, as well as towards a more detailed description of the design of new composites and structural elements made of them. Thus the results for the determination of reinforcement positions in a composite anisotropic plate presented in this study can be considered quite relevant and timely. In order to determine the positions of reinforcement of plate elements, we investigate the dynamic behavior of the target’s points under time-varying load. It is a well-known fact that substantial material stresses occur in areas located close to the surface of plane elements, thus in theory of reinforced concrete structures there is a corresponding notion of "splitting thickness" determining the depth of the valve, i.e. thickness of the protective layer of concrete. In this work, to determine the locations of the reinforcement of the plane element along its thickness it is proposed to constructing a picture of the propagation of the fronts of elastic waves and separately find the points where the reflected longitudinal and transverse waves meet.

In this investigation, the method for the calculation of dynamic characteristics determining the behavior of the plate under impact loading is modified based on the representation of the unknown quantities in the form of expansions in time and spatial coordinate along the direction of propagation of elastic waves.

2 Governing equations

In the world of engineering practice, there have been numerous studies where each of them individually addressed different aspects taking into account the nature of application of the external dynamic load [1-4], properties of the load application area [3-5], rheological properties of the plate material [4,5,6-8], different conditions of fixing of the plane element [9,10]. However, all of them studied certain aspects of dynamic effects on a plate separately and not as the whole set of simultaneously influencing parameters of the construction and applied load. Moreover, a single computational algorithm as a software application has not yet been developed and implemented. In this paper, the dyna-
mic behavior of the plate is described by wave equations of Uflyand-Mindlin-Reissner type, which exhibit the hyperbolic structure and take into account shear deformations and inertial components of the rotation of cross sections \([4, 5, 11]\)

\[
D_r \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) - \frac{1}{r} \left[ -D_r \frac{\partial}{\partial r} \left( \frac{\partial \varphi}{\partial \theta} \right) + D_k \frac{1}{r} \left( \frac{\partial^3 w}{\partial \theta^2 \partial r} - \frac{\partial^2 \varphi}{\partial \theta^2} \right) + D_\theta \frac{1}{r} \frac{\partial \varphi}{\partial \theta} + \right.
\]

\[
+(D_\theta \sigma_r + D_k) \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial \theta} \right) - \frac{\partial^2 \psi}{\partial r \partial \theta} \right) + D_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} - D_k \frac{1}{r} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \right) \right] + KhG_{rz} \left( \frac{\partial w}{\partial r} - \varphi \right) = -\frac{h^3 \partial^2 \varphi}{12 \partial r^2},
\]

\[
KG_{rz} \left( \frac{\partial^2 w}{\partial r^2} - \frac{\partial \varphi}{\partial r} \right) + KG_{rz} \frac{1}{r} \left( \frac{\partial w}{\partial r} - \varphi \right) + KG_{\theta z} \frac{1}{r} \left( \frac{\partial^2 w}{\partial r \partial \theta} - \frac{\partial \psi}{\partial \theta} \right) = \rho \frac{\partial^2 w}{\partial t^2},
\]

\[
C_r \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + C_k \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + C_\theta \frac{u}{r^2} + (C_\theta \sigma_r + C_k) \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \rho \frac{\partial^2 u}{\partial t^2},
\]

\[
C_\theta \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + C_k \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + (C_\theta \sigma_r + C_k) \frac{1}{r} \frac{\partial \psi}{\partial \theta} + (C_\theta + C_k) \frac{1}{r} \frac{\partial u}{\partial \theta} = \rho \frac{\partial^2 v}{\partial t^2},
\]

\[
D_r \frac{1}{r \partial r^2} \frac{\partial \varphi}{\partial \theta} = \frac{1}{r} \left[ -D_\theta \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} + D_k \left( \frac{\partial^3 w}{\partial \theta^2 \partial r} - \frac{\partial^2 \varphi}{\partial \theta^2} \right) + \right.
\]

\[
+(D_\theta \sigma_r + D_k) \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{\partial \varphi}{\partial \theta} \right) - D_\theta \frac{1}{r} \frac{\partial \varphi}{\partial \theta} + D_k \frac{1}{r} \left( \frac{\partial^2 w}{\partial \theta \partial r} - \frac{\partial \varphi}{\partial r} \right) \right] + KhG_{\theta z} \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \psi \right) = -\frac{h^3 \partial^2 \psi}{12 \partial r^2},
\]

where \(D_r = \frac{h^3}{12} B_r, D_\theta = \frac{h^3}{12} B_\theta, D_k = \frac{h^3}{12} B_k, C_r = hB_r, C_\theta = hB_\theta, K = 5, D_r, D_\theta, C_r, C_\theta, E_r, E_\theta, \sigma_r, \sigma_\theta, \nu(r, \theta) – the functions of rotational angles in the two tangential directions.\]
3 Method of solution

The calculation of the force and kinematic factors involves solving the system of equations (1) – (5). When taking into account the existence of wave phenomena in the form of elastic waves expanding with velocities that do not change during the entire deformation process of the target, one can use equations (1) to (5), however in order to construct the solution far from the place of occurrence of the wave fronts, these equations must be rewritten in a slightly different form using following substitutions

\[
\tau = \frac{t \sqrt{c_1}}{h}, \quad c_1 = \frac{E_r}{(1-\sigma_r,\sigma_\theta)} \rho, \quad w = \frac{w}{h}, \quad u = \frac{u}{h}, \quad v = \frac{v}{h}, \quad r = \frac{r}{h}. \tag{6}
\]

These expressions allow us to change from the use of traditional displacements and mechanical characteristics of the material in the determining equations to the use of dimensionless displacements and velocities of elastic waves in various directions of anisotropy, which represent the ratio of certain parameters of the material [4,5,10].

For solving equations (1) – (5) obtained after substitution of relations (6) we suggest to transfer to the Laplace space, and for that one needs to change from the values \( \varphi, \psi, w, u, v, q_1 \) and \( M \) to the corresponding representation in the imaginary space of Laplace - \( \tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{u}, \tilde{v}, \tilde{q}_1 \) and \( \tilde{M} \)

\[
\frac{\tilde{c}_4}{c_1} \left( \tilde{w}_{rr} - \tilde{\varphi}_r + \frac{1}{r} \tilde{w}_r - \frac{\tilde{\varphi}}{r} + \frac{1}{r^2} \tilde{w}_{\theta\theta} - \frac{1}{r} \tilde{\varphi}_{\theta} \right) = \tilde{\varphi} p^2 + \tilde{q}_1 \sin \alpha_1, \tag{7}
\]

\[
\frac{c_2}{c_1 r^2} \tilde{v}_{rr} + \frac{c_3}{c_1 r^2} \tilde{v}_{r\theta} - \frac{c_4}{c_1 r} \tilde{v}_r + \frac{c_5}{c_1 r} \tilde{v}_\theta = \tilde{v} p^2 + \tilde{q}_1 \cos \alpha_1 \cos \alpha_2,
\]

\[
\frac{c_3}{c_1 r^2} \tilde{\varphi}_{rr} + \frac{c_4}{c_1 r^2} \tilde{\varphi}_{r\theta} - \frac{c_5}{c_1 r} \tilde{\varphi}_r + \frac{c_6}{c_1 r} \tilde{\varphi}_\theta = \tilde{\varphi} p^2 + \tilde{q}_1 \cos \alpha_1 \sin \alpha_2,
\]

here \( p \) – the parameter of Laplace space, the subscripts \( r \) and \( \theta \) define the private derivative by the respective variable, \( c_2 = E_r/(1-\sigma_r,\sigma_\theta) \rho, \quad c_3 = G_{r\theta}/\rho, \quad c_4 = KG_{r\zeta}/\rho, \quad c_5 = KG_{\theta\zeta}/\rho, \quad q_1 = qh/\rho c_1, \quad \rho \) – the density of the target’s material, \( q \) – the external load (in this case from impact), \( R_1 \) – the radius of the spherical striker hitting the body.

The magnitudes \( c_1, c_2, c_3, c_4, c_5 \) used in expressions (7) determine the squares of the velocities of the fronts of elastic waves, the lower indices indicate the type of the wave surface: 1, 2 define quasi-longitudinal wave surfaces of tension-com-
pression propagating along the directions \( r \) and \( \theta \), index 3 determines the quasi-transverse wave surface of shear in the plane \( r\theta \), indices 4, 5 define quasi-transverse wave surfaces of shear in directions normal to the planes \( rz, \theta z \). Values of the velocity of wave surfaces are directly dependent on mechanical anisotropic properties of the target’s material in the respective directions. It can be assumed that the set of values of the velocity of wave surfaces determines the elastic properties of the plate.

All linear and angular displacements in equations (7), as well as the external load \( q(t,r,\theta) \) (a function that is associated with the contact force \( P(t) \)) are proposed to be written in the form of expansions in power series with spherical functions using Legendre polynomials \([8,10]\), such a representation of the unknown quantities allows us to link the desired functions and their derivatives of various orders with accuracy of unknown coefficients

\[
x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_{2n+m} P_{2n+1} \left( \cos \frac{\pi r}{2R} \right) \cos (m\theta),
\]

\[
\tilde{\alpha}_1 = \frac{P(p)}{\pi R^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (4n+3) P_{2n+1} \left( \cos \frac{\pi r}{2R} \right) \cos (m\theta),
\]

in these expressions, \( R \) – the radius of the plane circular element modeling the target; \( r_1 \) – the coordinate of the point of dynamic impact of the striker on the target; \( x \) – the magnitude that can take the values of the basic displacements \( \phi, \psi, w, u, v \); these magnitudes are represented in the imaginary space indicated by the line above, \( P(p) \) represents the dependence of the interaction force on the parameter of the imaginary space.

To determine the coefficients of the series (8) and (9) we will use their representation in the form of Laurent series near the investigated point of the target \([8,9]\)

\[
x_{2n+m} = x_{2n+m}^0 \varepsilon^0 + x_{2n+m}^1 \varepsilon^1 + x_{2n+m}^2 \varepsilon^2 + x_{2n+m}^3 \varepsilon^3, \text{ where } \varepsilon = p^{-2}
\]

Substituting the series (8), (9) with consideration of the expressions (10) into (7) and equating coefficients of the same degree \( \varepsilon \), we obtain the system of linear algebraic equations, from which we can find the coefficients of the expressions (10)

\[
\psi_{2n+m}^3 = P(p) P_s \cdot \varphi_{2n+m}^3 = P(p) \left( \frac{c_1}{c_4} \frac{R_i \cos \alpha_i}{A_i^3} - P_s \right),
\]

\[
\psi_{2n+m}^2 = P(p) P_s \left( 1 + D_1 \frac{B_1^3}{A_i^3} - D_2 \right) - \frac{P(p) R_i \cos \alpha_i}{A_i^3} \frac{c_1}{c_4} D_1,
\]

\[
\varphi_{2n+m}^2 = P(p) \frac{B_3^3}{A_i^3} P_s \left[ -2 - \frac{D_1}{A_i^3} + D_2 \right] + \frac{P(p) R_i \cos \alpha_i}{A_i^3} \frac{c_1}{c_4} \left( 1 + \frac{D_3}{A_i^3} \right),
\]

\[
\varphi_{2n+m}^1 = P(p) \frac{B_3^3}{A_i^3} P_s \left[ -2 - \frac{D_1}{A_i^3} + D_2 \right] + \frac{P(p) R_i \cos \alpha_i}{A_i^3} \frac{c_1}{c_4} \left( 1 + \frac{D_3}{A_i^3} \right),
\]

\[
\varphi_{2n+m}^0 = P(p) \frac{B_3^3}{A_i^3} P_s \left[ -2 - \frac{D_1}{A_i^3} + D_2 \right] + \frac{P(p) R_i \cos \alpha_i}{A_i^3} \frac{c_1}{c_4} \left( 1 + \frac{D_3}{A_i^3} \right),
\]

\[
\varphi_{2n+m}^0 = P(p) \frac{B_3^3}{A_i^3} P_s \left[ -2 - \frac{D_1}{A_i^3} + D_2 \right] + \frac{P(p) R_i \cos \alpha_i}{A_i^3} \frac{c_1}{c_4} \left( 1 + \frac{D_3}{A_i^3} \right),
\]

\[
\varphi_{2n+m}^0 = P(p) \frac{B_3^3}{A_i^3} P_s \left[ -2 - \frac{D_1}{A_i^3} + D_2 \right] + \frac{P(p) R_i \cos \alpha_i}{A_i^3} \frac{c_1}{c_4} \left( 1 + \frac{D_3}{A_i^3} \right),
\]
\[ \psi'_{2n+m} = \left[ P(p) \frac{c_1}{c_4} \left( A_1^3 - A_3^2R_i \cos \alpha_1 \right) - \psi'_{2n+m} \left( A_1^3 - A_2^2 \right) - \psi'_{2n+m} \left( A_1^3 B_2^2 - A_3^2 B_1^2 \right) \right] - \psi'_{2n+m} B_3^2, \]

\[ \phi_{2n+m} = \frac{P(p) R_i \cos \alpha_i c_4 - B_4}{A_4} \left( \psi'_{2n+m} \left( A_1^3 - A_3^2 R_i \cos \alpha_1 \right) \right) - \psi'_{2n+m} \left( A_1^3 A_2^2 - A_1^3 A_2^2 \right) \left( B_1^2 A_1^3 - B_1^3 A_1^3 \right) - \psi'_{2n+m} \left( A_1^3 B_2^2 - A_3^2 B_1^2 \right) - \psi'_{2n+m} A_3^2 B_1^2, \]

\[ w_{2n+m}^3 = -C_3 \psi'_{2n+m}, \quad w_{2n+m}^j = C_2 \psi'_{2n+m} - C_1 \psi'_{2n+m}, \]

\[ v_{2n+m}^3 = \frac{\cos \alpha_1 P(p) S_p \left( T_2^3 \cos \alpha_2 - T_1^3 \sin \alpha_2 \right)}{T_1^3 S_2^3 - S_1^3 T_2^3}, \quad u_{2n+m}^3 = \cos \alpha_1 \sin \alpha_2 \frac{P(p) S_p}{S_2^3} - \frac{v_{2n+m}^3 T_2^3}{S_2^3}, \]

\[ u_{2n+m}^j = \frac{u_{2n+m}^j \cos (m \theta) + \cos \alpha_1 \cos \alpha_2 P(p) S_p}{S_2^3} - \frac{v_{2n+m}^j T_2^3}{S_2^3}, \]

\[ v_{2n+m}^j = \frac{u_{2n+m}^j \cos (m \theta) + \cos \alpha_1 \cos \alpha_2 P(p) S_p}{T_1^3 S_2^3 - S_1^3 T_2^3}, \]

where the magnitudes \( A_1^3, A_2^3, B_1^3, B_3^3, A_1^2, A_2^2, B_1^2, B_2^2, B_3^2, D_1, D_2, D_3, P_2, C_1^3, S_1^3, T_1^3, S_2^3, T_2^3, S_p \) are determined by the ratios of velocities of different elastic waves, as well as by other initially known parameters of the plate element and by the character of the load application to it.

Expressions (11) are calculated for a particular point of the plate that belongs to the cross-section in which it is necessary to calculate the reinforcement, \( r \) and \( \theta \) take certain values.

After the inverse Laplace transform, the displacements can be written in the space of originals as a function of time, the two coordinates and the force of interaction on the plate

\[ \psi(x, \theta, \tau) = \left( 1 - \sigma_3 \right) \int_0^\infty \int_0^\infty \left( 4n + 3 \right) P(p) P_{n+1} \left( \cos \left( \frac{\pi r}{2 R} \right) \right) P_{n+1} \left( \cos \left( \frac{\pi r}{2 R} \right) \right) \cos (m \theta) \times \]

\[ \left[ x_{2n+m}^0 \text{Dirac} (\tau - \tau_1) + x_{2n+m}^1 (\tau - \tau_1) + \frac{x_{2n+m}^2}{6} (\tau - \tau_1)^3 + \frac{x_{2n+m}^3}{120} (\tau - \tau_1)^5 \right] d\tau_1, \]

(12)

where \( \text{Dirac} (\tau - \tau_1) \) – the Delta-function from the initial time interval.

This expression relates the displacements with the force of dynamic interaction on the target. For a full determination of kinematic and dynamic characteristics,
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It is necessary to use functional equations linking the contact force and displacements of the points of the plate [2,4,8] and boundary conditions [3,5,7].

4 Numerical investigations and conclusions

The proposed approach allows us to present a picture of the wave process in the form of normals to the fronts of elastic waves (in fact, these are the rays along which the wave surfaces propagate after application of dynamic load), taking into account their reflections from the upper and lower surface of the plate [12]. The scheme of the propagation of wave surfaces in the elastic isotropic plate, the dynamic behavior of which is described as a particular case of equations (1) to (5), after application of elastic contact force and upon fulfillment of the conditions of the total internal waves’ reflection from the edges of the target is represented in Fig.1. Figure 1 shows time-dependence of the coordinate of the wave front displacement along the thickness of the plate with the following initial characteristics: $E_r = 250$ GPa, $E_\theta = 150$ GPa, $\sigma_r = 0.3$, $\sigma_\theta = 0.3$, $\rho = 6000$ kg/m$^3$, $h = 0.1$ m, $t = 25$ ms, $l = 0.5$.

The proposed algorithm for the solution of the wave problem allows us to determine the position of the front of each of the elastic waves generated in an orthotropic plate after application of the dynamic load to its surface at any moment of time based on multiple construction’s parameters and external influences.

![Fig. 1 Time-dependence of the coordinate of the wave front displacements](image)

References


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