Some New Inequalities Involving Generalized Erdélyi-Kober Fractional q -Integral Operator

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Abstract

During the past four decades and longer, the subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has provided several potentially useful tools for solving differential, integral and integro-differential equations, and various other problems involving special functions of mathematical physics as well as their extensions (q -extensions) and generalizations in one and more variables. Here, in this paper, we aim to establish some new and potentially useful inequalities involving generalized Erdélyi-Kober fractional q -integral operator of the two parameters of deformation q_1 and

q_2 due to Gaulué [12], by following the similar process used by Gaulué [13] and Dumitru and Agarwal [5]. Relevant connections of the results presented here with those earlier ones are also pointed out.

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1 Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$. The enormous success of the theory of integral inequalities involving various fractional integral operators has stimulated the development of a corresponding theory in q -fractional integral inequalities (see, *e.g.*, [3, 4, 6, 7, 8, 9, 10, 16, 18]). In this paper, we are aiming at presenting some new and potentially useful inequalities involving generalized Erdélyi-Kober fractional q -integral operator of the two parameters of deformation q_1 and q_2 due to Gaulué [12], by following the same lines used by Gaulué [13] and Baleanu and Agarwal [5]. We also point out relevant connections of the results presented here with those earlier ones.

For our purpose, we recall the following definitions (see, *e.g.*, [17, Section 6]) and some earlier works.

The q -shifted factorial $(a; q)_n$ is defined by

$$(a; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} (1 - a q^k) & (n \in \mathbb{N}), \end{cases} \quad (1)$$

where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ ($m \in \mathbb{N}_0$).

The q -shifted factorial for negative subscript is defined by

$$(a; q)_{-n} := \frac{1}{(1 - a q^{-1})(1 - a q^{-2}) \cdots (1 - a q^{-n})} \quad (n \in \mathbb{N}_0). \quad (2)$$

We also write

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - a q^k) \quad (a, q \in \mathbb{C}; |q| < 1). \quad (3)$$

It follows from (1), (2) and (3) that

$$(a; q)_n = \frac{(a; q)_\infty}{(a q^n; q)_\infty} \quad (n \in \mathbb{Z}), \quad (4)$$

which can be extended to $n = \alpha \in \mathbb{C}$ as follows:

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(a q^\alpha; q)_\infty} \quad (\alpha \in \mathbb{C}; |q| < 1), \quad (5)$$

where the principal value of q^α is taken.

It is noted that Jackson [14] was the first to develop q -calculus in a systematic way. The q -calculus was also developed in the recent monograph [15].

The q -derivative of a function $f(t)$ is defined by

$$D_q\{f(t)\} := \frac{d_q}{d_q t} \{f(t)\} = \frac{f(qt) - f(t)}{(q-1)t}. \quad (6)$$

For $q \rightarrow 1$, the q -derivative (6) is easily seen to yield the usual derivative, that is,

$$\lim_{q \rightarrow 1} D_q\{f(t)\} = \frac{d}{dt} \{f(t)\},$$

if we assume that $f(t)$ is a differentiable function.

The function $F(t)$ is a q -antiderivative of $f(t)$ if $D_q\{F(t)\} = f(t)$. It is denoted by

$$\int f(t) d_q t. \quad (7)$$

The Jackson integral of $f(t)$ is defined, formally, by

$$\int f(t) d_q t := (1-q)t \sum_{j=0}^{\infty} q^j f(q^j t), \quad (8)$$

which can be easily generalized in the Stieltjes sense as follows:

$$\int f(t) d_q g(t) = \sum_{j=0}^{\infty} f(q^j t) (g(q^j t) - g(q^{j+1} t)). \quad (9)$$

Suppose that $0 < a < b$. The definite q -integral is defined as follows:

$$\int_a^b f(t) d_q t := (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b) \quad (10)$$

and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \quad (11)$$

A more general version of (10) in the Stieltjes sense is given by

$$\int_0^b f(t) d_q g(t) = \sum_{j=0}^{\infty} f(q^j b) (g(q^j b) - g(q^{j+1} b)). \quad (12)$$

The notation $[z]_q$ is defined by

$$[z]_q := \frac{1 - q^z}{1 - q} = \frac{q^z - 1}{q - 1} \quad (z \in \mathbb{C}; \quad q \in \mathbb{C} \setminus \{1\}; \quad q^z \neq 1). \quad (13)$$

A special case of (13) when $z \in \mathbb{N}$ is

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \quad (n \in \mathbb{N}), \quad (14)$$

which is called the *q-analogue* (or *q-extension*) of $n \in \mathbb{N}$, since

$$\lim_{q \rightarrow 1} [n]_q = \lim_{q \rightarrow 1} (1 + q + \cdots + q^{n-1}) = n.$$

The classical Gamma function $\Gamma(z)$ (see, *e.g.*, [17, Section 1.1]) was introduced by Leonhard Euler in 1729 while he was trying to extend the factorial $n! = \Gamma(n+1)$ ($n \in \mathbb{N}_0$) to real numbers. The *q-analogue* of $n!$ is then defined by

$$[n]_q! := \begin{cases} 1 & \text{if } n = 0, \\ [n]_q [n-1]_q \cdots [2]_q [1]_q & \text{if } n \in \mathbb{N}, \end{cases} \quad (15)$$

from which the *q-binomial coefficient* or the *Gaussian polynomial* analogous to $\binom{n}{k}$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!} \quad (n, k \in \mathbb{N}_0; \quad 0 \leq k \leq n). \quad (16)$$

The $[n]_q!$ can be rewritten as follows:

$$(1-q)^{-n} \prod_{k=0}^{\infty} \frac{(1-q^{k+1})}{(1-q^{k+1+n})} = \frac{(q; q)_{\infty}}{(q^{n+1}; q)_{\infty}} (1-q)^{-n} := \Gamma_q(n+1) \quad (0 < q < 1). \quad (17)$$

Replacing n by $a-1$ in (17), Jackson [14] defined the *q-Gamma function* $\Gamma_q(a)$ by

$$\Gamma_q(a) := \frac{(q; q)_{\infty}}{(q^a; q)_{\infty}} (1-q)^{1-a} \quad (0 < q < 1). \quad (18)$$

The q -analogue of $(t - a)^n$ is defined by the polynomial

$$(t - a)_q^n := \begin{cases} 1 & (n = 0), \\ (t - a)(t - qa) \cdots (t - q^{n-1}a) & (n \in \mathbb{N}). \end{cases} \quad (19)$$

$$= t^n \left(\frac{a}{t}; q \right)_n \quad (n \in \mathbb{N}_0).$$

We recall more definitions which will be needed in the sequel.

Definition 1.1. Let f and g be two real-valued functions defined on an interval $I \subset \mathbb{R}$ which are integrable on I . Then we say that f and g are synchronous on I if, for each $x, y \in I$, the following inequality holds:

$$(f(x) - f(y))(g(x) - g(y)) \geq 0. \quad (20)$$

Similarly f and g are asynchronous on I if, for any $x, y \in I$, the inequality in (20) is reversed, that is,

$$(f(x) - f(y))(g(x) - g(y)) \leq 0. \quad (21)$$

Definition 1.2. A real-valued function $f(t)$ ($t > 0$) is said to be in the space C_λ ($\lambda \in \mathbb{R}$) if there exists a real number $p > \lambda$ such that $f(t) = t^p \phi(t)$, where $\phi(t) \in C(0, \infty)$. A function $f(t)$ ($t > 0$) is said to be in the space C_λ^n ($n \in \mathbb{R}$) if $f^{(n)} \in C_\lambda$.

Definition 1.3. Let $0 < q < 1$, $\Re(\beta), \Re(\mu) > 0$, and $\eta \in \mathbb{C}$. Then a q -analogue of generalized Erdélyi-Kober fractional integral $I_q^{\alpha, \beta, \eta}$ is defined by (see [12]):

$$\begin{aligned} I_q^{\eta, \mu, \beta} \{f(t)\} &= \frac{\beta t^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^t (t^\beta - \tau^\beta q)_{\mu-1} \tau^{\beta(\eta+1)-1} f(\tau) d_q \tau, \\ &= \beta(1 - q^{1/\beta})(1 - q)^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^\mu; q)_k}{(q; q)_k} q^{k(\eta+1)} f(tq^{k/\beta}), \end{aligned} \quad (22)$$

where

$$(x - yq)_\nu := x^\nu \prod_{n=0}^{\infty} \left[\frac{1 - (\frac{y}{x})q^n}{1 - (\frac{y}{x})q^{n+\nu}} \right]. \quad (23)$$

For $f(t) = t^\lambda$ in (22), we get the known result [13, p. 82, Eq. (31)]

$$I_q^{\eta, \mu, \beta} \{t^\lambda\} = \beta \left[\frac{1}{\beta} \right]_q \frac{\Gamma_q(\eta + 1 + \lambda/\beta)}{\Gamma_q(\mu + \eta + 1 + \lambda/\beta)} t^\lambda, \quad (24)$$

where $\Re(\beta) > 0$, $\Re(\mu) > 0$, $\Re(\eta + 1 + \lambda/\beta) > 0$, $\eta \in \mathbb{C}$, and $0 < q < 1$.

Definition 1.4. Let $0 < q < 1$, $\Re(\mu) > 0$ and $\eta \in \mathbb{C}$. Then the q -analogue of the Kober fractional integral operator is given by (see [1])

$$\begin{aligned} I_q^{\eta, \mu} \{f(t)\} &= \frac{t^{-\eta-\mu}}{\Gamma_q(\mu)} \int_0^t (t - \tau q)_{\mu-1} \tau^\eta f(\tau) d_q \tau \\ &= (1-q)^\mu \sum_{k=0}^{\infty} \frac{(q^\mu; q)_k}{(q; q)_k} q^{k(\eta+1)} f(tq^k). \end{aligned} \quad (25)$$

Remark 1.5. It is easy to see that

$$\Gamma_q(\mu) > 0 \quad \text{and} \quad (q^\mu; q)_k > 0 \quad (26)$$

for all $\mu > 0$ and $k \in \mathbb{N}_0$. If $f : [0, \infty) \rightarrow [0, \infty)$ is a continuous function. Then it is seen that, in the second equality in (22), each term being nonnegative,

$$I_q^{\eta, \mu, \beta} \{f(t)\} \geq 0 \quad (27)$$

for all $\beta, \mu > 0$ and $\eta \in \mathbb{R}$.

Likewise the Kober q -integral operator (25) is also nonnegative, that is,

$$I_q^{\eta, \mu} \{f(t)\} \geq 0 \quad (28)$$

for all $\mu > 0$ and $\eta \in \mathbb{R}$.

2 Generalized Erdélyi-Kober q -integral Inequalities

Here we present five q -integral inequalities involving the generalized Erdélyi-Kober q -integral (22) stated in Theorems 2.1 to 2.8 below.

Theorem 2.1. Let f, g be two synchronous functions on $[0, \infty)$ and $h, u : [0, \infty) \rightarrow (0, \infty)$ be continuous. Then the following inequality holds true:

$$\begin{aligned} &I_{q_2}^{\zeta, \nu, \delta} \{u(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) g(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} \\ &+ I_{q_2}^{\zeta, \nu, \delta} \{u(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) g(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t)\} \\ &\geq I_{q_2}^{\zeta, \nu, \delta} \{u(t) g(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} \\ &+ I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{u(t) g(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\}, \end{aligned} \quad (29)$$

for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, and $\zeta > -1$.

Proof. By using Definition 1.1 with $h > 0$, we get

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0 \quad \text{for all } \tau, \rho \in [0, \infty), \quad (30)$$

that is,

$$\begin{aligned} & f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\tau) + f(\tau)g(\tau)h(\rho) + f(\rho)g(\rho)h(\rho) \\ & \geq f(\tau)g(\rho)h(\tau) + f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho) + f(\tau)g(\rho)h(\rho). \end{aligned} \quad (31)$$

It is easy to see from (23) that

$$(t^\beta - \tau^\beta q_1)_{\mu-1} = t^{\beta(\mu-1)} \prod_{k=0}^{\infty} \left[\frac{1 - (\tau/t)^\beta q_1^k}{1 - (\tau/t)^\beta q_1^{k+\mu-1}} \right] > 0, \quad (32)$$

for all $\beta > 0$, $0 < q_1 < 1$, $\mu > 1$, and $\tau, t \in \mathbb{R}$ with $0 < \tau < t$. Let

$$F(\eta, \mu, \beta; t, \tau; q_1; u) := \frac{\beta t^{-\beta(\eta+\mu)}}{\Gamma_{q_1}(\mu)} (t^\beta - \tau^\beta q_1)_{\mu-1} \tau^{\beta(\eta+1)-1} u(\tau). \quad (33)$$

Then we find from (32) that $F(\eta, \mu, \beta; t, \tau; q_1; u) > 0$ under the conditions in (32) and the function $u : [0, \infty) \rightarrow (0, \infty)$.

Now, multiplying both sides of (31) by $F(\eta, \mu, \beta; t, \tau; q_1; u)$ and taking the q_1 integration of the resulting inequality with respect to τ from 0 to t , and using (22), we get

$$\begin{aligned} & I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} + f(\rho) g(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} + h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} \\ & + f(\rho) g(\rho) h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t)\} \geq g(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} + f(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} \\ & + f(\rho) h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} + g(\rho) h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\}. \end{aligned} \quad (34)$$

Next, multiply both sides of (34) by $F(\zeta, \nu, \delta; t, \tau; q_2; u)$ which is positive under the conditions in (29), and integrating the resulting inequality with respect to ρ from 0 to t , and applying Definition 1.3, we are led to the desired result (29). This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. Let f, g be two synchronous functions on $[0, \infty)$ and $h, u, l : [0, \infty) \rightarrow (0, \infty)$ be continuous. Then the following inequality holds true:

$$\begin{aligned} & I_{q_2}^{\zeta, \nu, \delta} \{l(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t) g(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} \\ & + I_{q_2}^{\zeta, \nu, \delta} \{l(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t) g(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t)\} \\ & \geq I_{q_2}^{\zeta, \nu, \delta} \{l(t) g(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} \\ & + I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{l(t) g(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\}, \end{aligned} \quad (35)$$

for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, and $\zeta > -1$.

Proof. To prove the above result, multiplying both sides of (34) by $F(\zeta, \nu, \delta; t, \tau; q_2; l)$, where $F(\zeta, \nu, \delta; t, \tau; q_2; l) > 0$ under the conditions in (32) and the function $l : [0, \infty) \rightarrow (0, \infty)$. Then integrating the resulting inequality with respect to ρ from 0 to t , and using (22), we are led to the desired result (35). This completes the proof of Theorem 2. $\square$

Remark 2.3. It may be noted that the inequalities in (29) and (35) are reversed if the involved functions are asynchronous. The special case of (35) when $u = l$ is easily seen to reduce to the result in Theorem 2.1.

Theorem 2.4. Let f, g, h be three continuous functions on $[0, \infty)$ and $u : [0, \infty) \rightarrow (0, \infty)$ be a continuous function satisfying the following inequality:

$$\psi \leq f(x) \leq \Psi, \quad \phi \leq g(x) \leq \Phi \text{ and } \omega \leq h(x) \leq \Omega \quad (36)$$

for some $\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}$, and for all $x \in [0, \infty)$.

Then, for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, and $\zeta > -1$, the following inequality holds true:

$$\begin{aligned} & \left| I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) g(t)\} \right. \\ & + I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) h(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\} I_{q_2}^{\zeta, \nu, \delta} \{g(t) h(t) u(t)\} \\ & - I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t)\} - I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) g(t)\} \\ & \left. - I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) h(t)\} - I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) g(t) h(t)\} \right| \\ & \leq I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t)\} (\Psi - \psi)(\Phi - \phi)(\Omega - \omega). \end{aligned} \quad (37)$$

Proof. We find from (36) that, for all $\tau, \rho \geq 0$,

$$|f(\tau) - f(\rho)| \leq \Psi - \psi, \quad |g(\tau) - g(\rho)| \leq \Phi - \phi, \quad |h(\tau) - h(\rho)| \leq \Omega - \omega.$$

This implies

$$|\mathcal{A}(\tau, \rho)| \leq (\Psi - \psi)(\Phi - \phi)(\Omega - \omega), \quad (38)$$

where

$$\begin{aligned} \mathcal{A}(\tau, \rho) &:= (f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) - h(\rho)) \\ &= f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\rho) \\ &\quad - f(\tau)g(\rho)h(\tau) - f(\rho)g(\rho)h(\rho) - f(\tau)g(\tau)h(\rho) - f(\rho)g(\tau)h(\tau), \end{aligned} \quad (39)$$

for all $\tau, \rho \in [0, \infty)$. Multiplying both sides of (39) by $F(\eta, \mu, \beta; t, \tau; q_1; u)$, where $F(\eta, \mu, \beta; t, \tau; q_1; u) > 0$ under the conditions in (32) and the function

$u : [0, \infty) \rightarrow (0, \infty)$, and taking q_1 -integration of the resulting identity with respect to τ from 0 to t and using (22), we get

$$\begin{aligned} & \int_0^t F(\eta, \mu, \beta; t, \tau; q_1; u) \mathcal{A}(\tau, \rho) d_{q_1} \tau \\ &= I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} + f(\rho) g(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} + g(\rho) h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\} \\ &+ f(\rho) h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} - h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} - g(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} \\ &- f(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} - f(\rho) g(\rho) h(\rho) I_{q_1}^{\eta, \mu, \beta} \{u(t)\}. \end{aligned} \quad (40)$$

Next, multiply both sides of (40) by $F(\zeta, \nu, \delta; t, \tau; q_2; u)$, where $F(\zeta, \nu, \delta; t, \tau; q_2; u) > 0$ under the conditions in (32) and the function $u : [0, \infty) \rightarrow (0, \infty)$, then integrating the resulting equality with respect to ρ from 0 to t , and using (22), we have

$$\begin{aligned} & \int_0^t \int_0^t F(\eta, \mu, \beta; t, \tau; q_1; u) F(\zeta, \nu, \delta; t, \tau; q_2; u) \mathcal{A}(\tau, \rho) d_{q_1} \tau d_{q_2} \rho \\ &= I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) g(t)\} \\ &+ I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) h(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\} I_{q_2}^{\zeta, \nu, \delta} \{g(t) h(t) u(t)\} \\ &- I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t)\} - I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) g(t)\} \\ &- I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) h(t)\} - I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t) f(t) g(t) h(t)\}. \end{aligned} \quad (41)$$

Now it is easy to see from the inequality (38) that the quantity in (41) is bounded by the resulting one obtained by applying the two integrations to the right-hand side of (38), that is, the last expression of (37). The proof is complete. $\square$

Theorem 2.5. Let f , g , and h be three continuous functions on $[0, \infty)$, and $u, l : [0, \infty) \rightarrow (0, \infty)$ be two continuous functions satisfying the following inequality:

$$\psi \leq f(x) \leq \Psi, \quad \phi \leq g(x) \leq \Phi \quad \text{and} \quad \omega \leq h(x) \leq \Omega \quad (42)$$

for some $\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}$ and for all $x \in [0, \infty)$.

Then, for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, and $\zeta > -1$, the following inequality holds true:

$$\begin{aligned} & \left| I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t) g(t)\} \right. \\ &+ I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t) h(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t)\} I_{q_2}^{\zeta, \nu, \delta} \{g(t) h(t) l(t)\} \\ &- I_{q_1}^{\eta, \mu, \beta} \{u(t) g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t)\} - I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t) g(t)\} \\ &- I_{q_1}^{\eta, \mu, \beta} \{u(t) f(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t) h(t)\} - I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t) f(t) g(t) h(t)\} \left. \right| \\ &\leq I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{l(t)\} (\Psi - \psi)(\Phi - \phi)(\Omega - \omega). \end{aligned} \quad (43)$$

Proof. Multiply both sides of (40) by $F(\zeta, \nu, \delta; t, \tau; q_2; l)$, where $F(\zeta, \nu, \delta; t, \tau; q_2; l) > 0$ under the conditions in (32) and the function $l : [0, \infty) \rightarrow (0, \infty)$ and then integrate the resulting inequality with respect to ρ from 0 to t , and use (22). Then a similar argument as in the proof of the inequality (37) is easily seen to yield the desired result (43). $\square$

Remark 2.6. The special case of (43) when $u = l$ is easily seen to reduce to the result in Theorem 2.4.

We recall the following definition to give more inequalities involving the q -integral operator.

Definition 2.7. If a real-valued function f has domain $D(f)$ contained in $\mathbb{R}$, we say that f satisfies a Lipschitz condition if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad (44)$$

for all points x, y in $D(f)$.

It is noted that the Lipschitz condition is an important tool in obtaining various famous inequalities in the literature and involves many other inequalities (for a very recent work, see [11] and the references therein).

Theorem 2.8. Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ satisfy Lipschitz conditions with constants L_1 and L_2 , respectively, and $u : [0, \infty) \rightarrow (0, \infty)$ be a continuous function. Then the following inequality holds true:

$$\begin{aligned} & \left| I_{q_1}^{\eta, \mu, \beta} \{f(t) g(t) u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{f(t) g(t) u(t)\} \right. \\ & \quad \left. - I_{q_1}^{\eta, \mu, \beta} \{f(t) u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{g(t) u(t)\} - I_{q_1}^{\eta, \mu, \beta} \{g(t) u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{f(t) u(t)\} \right| \\ & \leq L_1 L_2 \left[I_{q_1}^{\eta, \mu, \beta} \{t^2 u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{u(t)\} \right. \\ & \quad \left. - 2 I_{q_1}^{\eta, \mu, \beta} \{t u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t u(t)\} + I_{q_1}^{\eta, \mu, \beta} \{u(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^2 u(t)\} \right] \end{aligned} \quad (45)$$

for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, and $\zeta > -1$.

Proof. We find from the assumption that, for all $\tau, \rho \in [0, \infty)$,

$$-L_1 L_2 (\tau - \rho)^2 \leq \mathcal{B}(\tau, \rho) \leq L_1 L_2 (\tau - \rho)^2, \quad (46)$$

where

$$\begin{aligned} \mathcal{B}(\tau, \rho) &:= (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \\ &= f(\tau) g(\tau) - f(\tau) g(\rho) - f(\rho) g(\tau) + f(\rho) g(\rho) \end{aligned}$$

and

$$(\tau - \rho)^2 = \tau^2 - 2\tau\rho + \rho^2.$$

Now, multiplying both sides of (46) by $F(\eta, \mu, \beta; t, \tau; q_1; u)$ (which is positive under the given conditions) and applying the q_1 -integration to the resulting double inequality with respect to τ from 0 to t . Then multiplying both sides of the last resulting double inequality by $F(\zeta, \nu, \delta; t, \rho; q_2; u)$ (which is positive under the given conditions) and applying the q_2 -integration to the resulting double inequality with respect to ρ from 0 to t , after a little simplification, we are led to the desired result. $\square$

3 Special Cases and Concluding Remarks

We conclude our present investigation by remarking further that we can present a large number of special cases of our main inequalities in Theorems 1-5. We first illustrate the special cases of Theorems 1, 3, and 5 when $u(t) = t^\lambda$ as in Corollaries 1, 2, and 3.

Corollary 3.1. *Let f, g be two synchronous functions on $[0, \infty)$, $h : [0, \infty) \rightarrow (0, \infty)$. Then the following inequality holds true:*

$$\begin{aligned} & \delta \left[\frac{1}{\delta} \right]_{q_2} \frac{\Gamma_{q_2}(\zeta + 1 + \lambda/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + \lambda/\delta)} t^\lambda I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) g(t) h(t)\} \\ & + I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) g(t)\} I_{q_1}^{\eta, \mu, \beta} \{t^\lambda h(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda h(t)\} I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) g(t)\} \\ & + \beta \left[\frac{1}{\beta} \right]_{q_1} \frac{\Gamma_{q_1}(\eta + 1 + \lambda/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + \lambda/\beta)} t^\lambda I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) g(t) h(t)\} \\ & \geq I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda g(t)\} I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) h(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t)\} I_{q_1}^{\eta, \mu, \beta} \{t^\lambda g(t) h(t)\} \\ & + I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{t^\lambda g(t)\} + I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda g(t) h(t)\} I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t)\}, \end{aligned} \quad (47)$$

for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, $\zeta > -1$, $\eta + 1 + \lambda/\beta > 0$ and $\zeta + 1 + \lambda/\delta > 0$.

Corollary 3.2. *Let f, g, h be three continuous functions on $[0, \infty)$ and satisfying the following inequality:*

$$\psi \leq f(x) \leq \Psi, \quad \phi \leq g(x) \leq \Phi \quad \text{and} \quad \omega \leq h(x) \leq \Omega \quad (48)$$

for some $\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}$, and for all $x \in [0, \infty)$.

Then, for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, $\zeta > -1$, $\eta + 1 + \lambda/\beta > 0$, and $\zeta + 1 + \lambda/\delta > 0$, the following inequality

holds true:

$$\begin{aligned}
& \left| \delta \left[\frac{1}{\delta} \right]_{q_2} \frac{\Gamma_{q_2}(\zeta + 1 + \lambda/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + \lambda/\delta)} t^\lambda I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) g(t) h(t)\} \right. \\
& + I_{q_1}^{\eta, \mu, \beta} \{t^\lambda h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) g(t)\} \\
& + I_{q_1}^{\eta, \mu, \beta} \{t^\lambda g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) h(t)\} + I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda g(t) h(t)\} \\
& - I_{q_1}^{\eta, \mu, \beta} \{t^\lambda g(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t)\} - I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) h(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda g(t)\} \\
& - I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda h(t)\} \\
& \left. - \beta \left[\frac{1}{\beta} \right]_{q_1} \frac{\Gamma_{q_1}(\eta + 1 + \lambda/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + \lambda/\beta)} t^\lambda I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) g(t) h(t)\} \right| \\
& \leq \beta \delta \left[\frac{1}{\beta} \right]_{q_1} \left[\frac{1}{\delta} \right]_{q_2} \frac{\Gamma_{q_1}(\eta + 1 + \lambda/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + \lambda/\beta)} \\
& \quad \times \frac{\Gamma_{q_2}(\zeta + 1 + \lambda/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + \lambda/\delta)} t^{2\lambda} (\Psi - \psi)(\Phi - \phi)(\Omega - \omega).
\end{aligned} \tag{49}$$

Corollary 3.3. *Let $f, g : [0, \infty) \rightarrow \mathbb{R}$ satisfy Lipschitz conditions with constants L_1 and L_2 , respectively. Then the following inequality holds true:*

$$\begin{aligned}
& \left| \delta [1/\delta]_{q_2} \frac{\Gamma_{q_2}(\zeta + 1 + \lambda/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + \lambda/\delta)} t^\lambda I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t) g(t)\} \right. \\
& + \beta [1/\beta]_{q_1} \frac{\Gamma_{q_1}(\eta + 1 + \lambda/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + \lambda/\beta)} t^\lambda I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t) g(t)\} \\
& - I_{q_1}^{\eta, \mu, \beta} \{t^\lambda f(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda g(t)\} - I_{q_1}^{\eta, \mu, \beta} \{t^\lambda g(t)\} I_{q_2}^{\zeta, \nu, \delta} \{t^\lambda f(t)\} \left. \right| \\
& \leq L_1 L_2 \beta \delta [1/\beta]_{q_1} [1/\delta]_{q_2} t^{2\lambda+2} \\
& \quad \times \left[\frac{\Gamma_{q_1}(\eta + 1 + (\lambda + 2)/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + (\lambda + 2)/\beta)} \frac{\Gamma_{q_2}(\zeta + 1 + \lambda/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + \lambda/\delta)} \right. \\
& - 2 \frac{\Gamma_{q_1}(\eta + 1 + (\lambda + 1)/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + (\lambda + 1)/\beta)} \frac{\Gamma_{q_2}(\zeta + 1 + (\lambda + 1)/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + (\lambda + 1)/\delta)} \\
& \left. + \frac{\Gamma_{q_1}(\eta + 1 + \lambda/\beta)}{\Gamma_{q_1}(\mu + \eta + 1 + \lambda/\beta)} \frac{\Gamma_{q_2}(\zeta + 1 + (\lambda + 2)/\delta)}{\Gamma_{q_2}(\nu + \zeta + 1 + (\lambda + 2)/\delta)} \right]
\end{aligned} \tag{50}$$

for all $t > 0$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\mu > 1$, $\nu > 1$, $\beta > 0$, $\delta > 0$, $\eta > -1$, $\zeta > -1$, $\eta + 1 + \lambda/\beta > 0$, and $\zeta + 1 + \lambda/\delta > 0$.

The formula (24) is further specialized as follows (see [13, p. 83, Eq. (32)]):

$$I_q^{\eta, \mu, \beta} \{K\} = \beta \left[\frac{1}{\beta} \right]_q \frac{\Gamma_q(\eta + 1)}{\Gamma_q(\mu + \eta + 1)} K, \tag{51}$$

where K is a constant, $\Re(\beta) > 0$, $\Re(\mu) > 0$, $\Re(\eta) > -1$, and $0 < q < 1$.

Setting $u(t) = 1$ and $q_1 = q_2$ in those results of Theorem 1 and Corollary 1 and using (51) is seen to yield the known results [13, p. 84, Eq. (37)] and [13, p. 86, Eq. (42)], respectively.

Also we briefly consider some other consequences of the results derived in the previous sections. Following Gaulué [13], the operator (22) would reduce immediately to the extensively investigated Riemann-Liouville and Kober type fractional integral operators, respectively, given by the following relationships (see also [2] and [1]):

$$I_q^{\eta, \mu} \{f(t)\} = I_q^{\eta, \mu, 1} \{f(t)\} = \frac{t^{-(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^t (t - \tau q)_{\mu-1} \tau^\eta f(\tau) d_q \tau \quad (52)$$

$$(\eta, \mu \in \mathbb{C}; \Re(\mu) > 0)$$

and

$$I_q^\mu \{f(t)\} = t^\mu I_q^{0, \mu, 1} \{f(t)\} = \frac{1}{\Gamma_q(\mu)} \int_0^t (t - \tau q)_{\mu-1} f(\tau) d_q \tau \quad (53)$$

$$(\mu \in \mathbb{C}; \Re(\mu) > 0).$$

Setting $u(t) = 1$, $\beta = \delta = 1$, $q_1 = q_2 = q$ in the result of Theorem 1 and using (52) is seen to give the known fractional q -integral inequalities [13, p. 86, Eq. (43)].

The special cases of the results of Theorems 1-5 when $\beta = \delta = 1$ and the limit $q \rightarrow 1-$ is taken are seen to provide, respectively, the known inequalities due to Baleanu and Agarwal [5].

Setting $\beta = \delta = 1$, $\lambda = \eta = \zeta = 0$ and $q_1 = q_2 = q$ in the result of Corollary 1 and using (53) is seen to yield the known fractional q -integral inequalities due to Gaulué [13, p. 87, Eq. (44)] and Öğüzmez and Özkam [16, p. 5, Eq. (3.11)]. Further taking the limit $q \rightarrow -1$ in those results just obtained gives the known result Belarbi and Dahmani [6, p. 188, Eq. 16].

The special case of Theorem 1 when $u(t) = 1$, $\beta = \delta = 1$, $\zeta = \eta = 0$ and $q_1 = q_2 = q$ and formulas (51) and (53) are used yields the known result Gaulué [13, p. 87, Eq. (45)] and Sulaiman [18, p. 456, Eq. (3.2)].

As we have seen in this section, the results presented here are of general character and potentially useful in deriving various q -inequalities in the theory of fractional q -integral operators and inequalities in the theory of fractional integral operators.

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