Time-Dependent Single Server Markovian Queue with Catastrophe

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Abstract

A catastrophic single server queue with time-dependent arrival and service rates is considered in this paper. The empty system size probability \( P_0(t) \) is obtained through Volterra integral equation. Further, a busy period distribution is discussed for the catastrophic time-dependent queues.

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1 Introduction

The theory of queues has been applied to an embarrassingly large number of problems. We mention few areas and problems that have been studied by using the theory of queues which is elaborately discussed in [2] are (i) telephone traffic (ii) the landing of aircraft (iii) the scheduling of patients in clinic (iv) restaurant service (v) check-out stands in supermarkets. Most of the queueing literature discuss about the queueing models with arrival and service rates are
time independent. Considering such model is easy to analyze and reduce the transient solution in closed form. Queueing models with time dependent arrival and service rates are quite tedious to arrive the closed form solution (see [8]). Xiaoqian et al (see [10]), considered M/M/1-k queue with arrival rate is time dependent and service rate is constant and an asymptotic approach is used to find the probability of n customers in the system at time t, as well as the mean number. Knessal and Yang [5] have obtained an explicit expression for the probability \( P_n(t) \) considering the service rate is constant, this corresponds to \( \rho(t) = \frac{\lambda(t)}{\mu} = (b - a\mu t)^2 \) developed a non stationary Markovian queueing system starting with M(t)/M(t)/1 queue using operator analytic techniques. Asymptotic approach plays vital role for deriving performance measures for the above model (see [7]).

Queueing models with catastrophes have been received good attention and this was discussed by several authors, see for example and references therein [3, 6]. A finite time analysis of M(t)/M/1 Queue discussed and also validated the analytic results using simulation technique in [9]. Baumann and Sandmann [1] discussed a continuous time level dependent quasi birth death process extended by catastrophes. Stationary distribution of the process computed through Matrix analytic algorithm (MAA). In [4], time dependent performance measures and the busy period of M/M/1/N queueing system with catastrophes is discussed and also the steady states results are derived. One of the great interest in [11], which shows the transient behavior of the time dependent M/M/1 queue in the absence of catastrophe and the boundary probability function with numerical examples. In the present work, we extended the results of [11] to the case where the M(t)/M(t)/1 queueing system with in the presence of catastrophe. In section 3, make use of generating functions, we expressed the boundary integral equation \( P_0(t) \) which is the probability of the system being empty and which is derived by solving a pair of coupled Volterra integral equations. Modified Bessel function is involves in the kernel of Volterra integral equation. In section 4, A busy period distribution is derived using the same technique as we discussed in section 3.

## 2 Model Description

We consider a time dependent single server queue with infinite capacity and with the possibility of catastrophes. We define \( P_n(t) = P[X(t) = n | X(0) = n_0] \), considering the rate functions \( \lambda(t) \) and \( \mu(t) \) of which represent arrival and service rates respectively and there is no loss of generality in assuming that we start the process at \( t=0 \). The countably infinite set of differential- difference
equations (forward Kolmogorov equation(s)) for $P_n(t)$ are

\begin{align*}
P'_n(t) &= \lambda(t)P_{n-1}(t) - [\lambda(t) + \mu(t) + \gamma]P_n(t) + \mu(t)P_{n+1}(t), \quad n \geq 1 \quad (1) \\
P'_0(t) &= -\lambda(t)P_0(t) + \mu(t)P_1(t) + \gamma(1 - P_0(t)) \quad (2)
\end{align*}

with the initial condition $P_n(0) = \delta(n, n_0)$, where $P_n(t)$ is the probability that $n$ customers in the queue at time $t$ and $\delta$ is the Kronecker delta function.

There are different methods to approximate the solution of the above system. We use the generating function technique to derive the solution of the above system in the following section.

### 3 Transient Analysis

In this section, we use generating function to solve the system of differential-difference equations given in (1) and (2). Define the Generating function $P(z, t)$ as follows:

\begin{equation}
P(z, t) = \sum_{n=1}^{\infty} P_n(t)z^n. \quad (1)
\end{equation}

We assume $\lambda_{\text{max}} \geq \lambda(t)$ and $\mu_{\text{max}} \geq \mu(t)$. Then we can easily transform the equation (1) into the following differential equation:

\begin{equation}
P'(z, t) = P(z, t) \left[ \lambda(t)z - \lambda(t) - \mu(t) - \gamma + \frac{\mu(t)}{z} \right] + \lambda(t)zP_0(t) - \mu(t)P_1(t)
\end{equation}

with the initial conditions $P_0(0) = 1$.

The infinite series $\sum_{n=1}^{\infty} P'_n(t)z^n$ is uniformly convergent for $|z| \leq 1$ for all $t$ [see [11]].

The solution of the linear differential equation (2) can be expressed in terms of the boundary probability function $P_0(t)$ and $P_1(t)$ through the following convolution integral.

\begin{equation}
P(z, t) = \int_0^t \left[ z\lambda(\eta)P_0(\eta) - \mu(\eta)P_1(\eta) \right] \phi_z(t, \eta) d\eta, \quad (3)
\end{equation}

where

\begin{equation}
\phi_z(t, \eta) = \exp \left[ \int_\eta^t \left[ \lambda(\tau)z + \mu(\tau)z^{-1} - (\lambda(\tau) + \mu(\tau) + \gamma) \right] d\tau \right]. \quad (4)
\end{equation}

Since $P_n(t)$ is conservative for each $n$ and $t$ and $\sum_{n=0}^{\infty} P_n(t) = 1$.

Equation (3) yields,

\begin{equation}
P(z, t) = \sum_{n=-\infty}^{\infty} \int_0^t \left[ \lambda(\eta)P_0(\eta)r_{n-1}(t, \eta) - \mu(\eta)P_1(\eta)r_n(t, \eta) \right] z^n d\eta, \quad (5)
\end{equation}
where \( r_n(t, \eta) \) can also be written as

\[
r_n(t, \eta) = \frac{1}{2\pi i} \oint z^{n-1} \exp \left[ \int_{\eta}^{t} \lambda(\tau) z + \mu(\tau) z^{-1} - \lambda(\tau) - \mu(\tau) \right] dz.
\]

From (1), the expression for \( P(z,t) \) is analytic for \(|z| \leq 1\) in the complex plane. From Cauchy’s theorem, this requirement on \( P(z,t) \) implies the following property of \( P(z,t) \).

**Theorem 1** [11] The generating function of the probability distribution \( P(z,t) \) satisfies

\[
\frac{1}{2\pi i} \oint P(z,t) dz = 0,
\]

where \( \frac{1}{2\pi i} \oint (.) dz \) is the counter clockwise complex integration along the unit circle \(|z|=1\).

By applying the above theorem we can obtain the following integral equation with the unknown functions \( P_0(t) \) and \( P_1(t) \). From (5), we get

\[
\int_{0}^{t} \left[ \lambda(t, \tau) P_0(\tau) r_{-2}(t, \tau) - \mu(t, \tau) P_1(\tau) r_{-1}(t, \tau) \right] d\tau = 0 \tag{6}
\]

which gives

\[
\int_{0}^{t} \left[ \lambda(\eta) P_0(\eta) R_1(t, \eta) - \mu(\eta) P_1(\eta) R_2(t, \eta) \right] d\eta = 0 \tag{7}
\]

with

\[
R_1(t, \eta) = \frac{e^{-a(t,\eta)-b(t,\eta)}}{a(t,\eta)^2} \sum_{n=1}^{\infty} \frac{(a(t,\eta)b(t,\eta))^n}{n!(n-2)!}, \quad R_2(t, \eta) = \frac{e^{-a(t,\eta)-b(t,\eta)}}{a(t,\eta)} \sum_{n=1}^{\infty} \frac{(a(t,\eta)b(t,\eta))^n}{n!(n-1)!},
\]

where

\[
a(t, \eta) = \int_{\eta}^{t} \lambda(\tau) d\tau, \quad b(t, \eta) = \int_{\eta}^{t} \mu(\tau) d\tau
\]

The following theorem relates the expression for \( R_1(t, \eta) \) and \( R_2(t, \eta) \) to modified Bessel functions.

**Theorem 2** [11] \( R_1(t, \eta) \) and \( R_2(t, \eta) \) can be written as

\[
R_1(t, \eta) = \frac{b(t, \eta)}{a(t, \eta)} e^{-a(t,\eta)-b(t,\eta)} I_2 \left( 2\sqrt{a(t, \eta)b(t, \eta)} \right),
\]

\[
R_2(t, \eta) = \sqrt{\frac{b(t, \eta)}{a(t, \eta)}} e^{-a(t,\eta)-b(t,\eta)} I_1 \left( 2\sqrt{a(t, \eta)b(t, \eta)} \right),
\]

where \( I_1 \) and \( I_2 \) are modified Bessel functions of the first kind. \( I_1(x) \) and \( I_2(x) \) are given by

\[
I_1(x) = \frac{1}{\pi} \int_{0}^{\pi} e^{x \cos(\theta)} \sin(\theta) d\theta
\]

and

\[
I_2(x) = \frac{1}{\pi} \int_{0}^{\pi} e^{x \cos(\theta)} \cos(\theta) d\theta
\]
where $I_k(x)$ is the modified Bessel function of $k$th order given by

$$I_k(x) = \left(\frac{x}{2}\right)^k \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^n}{n!(n+k)!}$$

with

$$R_1(t, t) = R_2(t, t) = 0$$

and

$$\left[ \frac{\partial}{\partial \eta} R_2(t, \eta) \right]_{\eta=t} = -\mu(t), \; P_0(0) = 1$$

The proof of the above theorem is discussed in [11].

Using (2) in (6) and after considerable simplifications, we obtain

$$\int_0^t \bar{k}(t, \eta) P_0(\eta) d\eta = \gamma \alpha(t) - \bar{g}(t)$$

(8)

with

$$\bar{k}(t, \eta) = \lambda(\eta) [R_1(t, \eta) - R_2(t, \eta)] + \frac{\partial}{\partial \eta} R_2(t, \eta) - \gamma R_2(t, \eta).$$

and

$$\bar{g}(t) = R_2(t, 0).$$

also assume

$$\int_0^t R_2(t, \eta) d\eta = -\alpha(t).$$

Differentiate (8), we derive

$$P_0(t) = g(t) + f(t) + \int_0^t P_0(\eta) k(t, \eta) d\eta,$$

(9)

where

$$k(t, \eta) = \frac{1}{\mu(t)} \frac{\partial}{\partial t} \left[ \bar{k}(t, \eta) \right], \; g(t) = \frac{1}{\mu(t)} \frac{d}{dt} \bar{g}(t).$$

and also

$$f(t) = \frac{1}{\mu(t)} R_2(t, \eta).$$

We can show $P(z, t)$ in terms of $P_0(t)$ only, and once $P_0(t)$ is found we express any $P_n(t)$ as a function of $P(z, t)$. Therefore

$$P_n(t) = \frac{1}{2\pi i} \int_{|Z|=1} Z^{-(n+1)} P(z, t) dz, \; n \geq 1$$

(10)
4 Busy period Analysis

In the case of a single-server system, the busy period is defined to be the time from when the server first becomes busy until the server first goes idle. Let $M(t)$ denote number of customers in the system at time $t$, where $0 < t < T$, $M(t) \neq 0$ and the server never fails in this period. Let

$$q_n(t) = P[M(t) = n], \ 0 \leq t \leq T, \ n \geq 1.$$  

and $q_n(0) = \delta_{1,n}$.

Governing equations are

$$q'_0(t) = \mu(t)q_1(t) + \gamma(1 - q_0(t)) \quad (1)$$

$$q'_1(t) = - (\lambda(t) + \mu(t) + \gamma)q_1(t) + \mu(t)q_2(t) \quad (2)$$

$$q'_n(t) = \lambda(t)q_{n-1}(t) - (\lambda(t) + \mu(t) + \gamma)q_n(t) + \mu(t)q_{n+1}(t), \ n = 2, 3, 4, \ldots \quad (3)$$

Define the generating function

$$Q(z, t) = \sum_{n=2}^{\infty} q_n(t)z^n.$$  

Equation (3) follows

$$Q'(z, t) = \left[z\lambda(t) - (\lambda(t) + \mu(t) + \gamma) + \frac{\mu(t)}{z}\right]Q(z, t) - \left[z^2\lambda(t)q_1(t) + \mu(t)q_1(t) + \mu(t)zq_2(t)\right]$$  

Solving the above equation, we get

$$Q(z, t) = -\int_0^t \left[z^2\lambda(\eta)q_1(\eta) + \mu(\eta)q_1(\eta) + \mu(\eta)zq_2(\eta)\right]\phi_z(t, \eta)d\eta, \quad (5)$$

where

$$\phi_z(t, \eta) = \exp\int_{\eta}^{t} [z\lambda(\tau) - (\lambda(\tau) + \mu(\tau) + \gamma) + \frac{\mu(\tau)}{z}]d\tau. \quad (6)$$

From (5) we get

$$Q(z, t) = -\sum_{n=-\infty}^{\infty} \phi_z(t, \eta)z^n \sum_{n=-\infty}^{\infty} r_n(t, \eta)z^n d\eta,$$

$$Q(z, t) = -\sum_{n=-\infty}^{\infty} \int_0^t [\lambda(\eta)q_{n-1}(\eta)r_{n-2}(t, \eta) + \mu(\eta)q_1(\eta)r_n(t, \eta) + \mu(\eta)q_2(\eta)r_{n-1}(t, \eta)]z^n d\eta.$$
Since $Q(z,t)$ is analytic for $|z| \leq 1$ is that of coefficient of $z^{-1}$ in the above series is zero. Therefore,

$$\int_0^t [\lambda(t,\eta)q_1(\eta)r_{-3}(t,\eta) + \mu(t,\eta)q_1(\eta)r_{-2}(t,\eta)] d\eta = 0$$

$$\int_0^t [\lambda(\eta)q_1(\eta)R_1(t,\eta) + \mu(\eta)q_1(\eta)R_3(t,\eta) + \mu(\eta)q_2(\eta)R_2(t,\eta)] d\eta = 0 \quad (7)$$

where

$$R_1(t,\eta) = e^{-a(t,\eta)-b(t,\eta)} \sum_{n=3}^{\infty} \frac{(a(t,\eta)b(t,\eta))^n}{n!(n-3)!} = r_{-3}(t,\eta)$$

$$R_2(t,\eta) = e^{-a(t,\eta)-b(t,\eta)} \sum_{n=2}^{\infty} \frac{(a(t,\eta)b(t,\eta))^n}{n!(n-2)!} = r_{-2}(t,\eta)$$

$$R_3(t,\eta) = e^{-a(t,\eta)-b(t,\eta)} \sum_{n=1}^{\infty} \frac{(a(t,\eta)b(t,\eta))^n}{n!(n-1)!} = r_{-1}(t,\eta)$$

and also $R_1(t,t) = R_2(t,t) = R_3(t,t) = 0$.

Using (2) in (7) and after some simple manipulations, we deduce

$$\int_0^t [\lambda(\eta)q_1(\eta) [R_1(t,\eta) + R_2(t,\eta)] + \mu(\eta)q_1(\eta) [R_2(t,\eta) + R_3(t,\eta)] + \gamma q_1(\eta)R_2(t,\eta)] d\eta + \bar{g}(t) - \int_0^t \frac{\partial}{\partial \eta} [R_2(t,\eta)] q_1(\eta) d\eta = 0$$

After some simple calculations,

$$\int_0^t \tilde{k}(t,\eta)q_1(\eta)d\eta = -\bar{g}(t),$$

where

$$\bar{g}(t) = R_2(t,0), \quad \tilde{k}(t,\eta) = \lambda(\eta) [R_1(t,\eta) + R_2(t,\eta)] + \mu(\eta) [R_2(t,\eta) + R_3(t,\eta)] - \frac{\partial}{\partial \eta} [R_2(t,\eta)]$$

Using Leibnitz rule, we deduce

$$q_1(t) = g(t) + \int_0^t q_1(\eta)K(t,\eta)d\eta, \quad (8)$$

where

$$K(t,\eta) = \frac{1}{\mu(t)} \frac{\partial}{\partial \eta} [\tilde{k}(t,\eta)], \quad g(t) = \frac{1}{\mu(t)} \frac{d}{dt} [\bar{g}(t)].$$
Equation (1) yields,

\[ q_0(t) = e^{-\gamma t} \int_0^t e^{\gamma t} \mu(t)q_1(t) dt + 1 - e^{-\gamma t} \]  

(9)

Thus the busy period distribution \( b(t) \) is given by

\[ b(t) = \mu(t)q_1(t) + \gamma(1-q_0(t)). \]  

(10)

Substitute (8) and (9) in (10) we get the busy period distribution.

References


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