Discrete Kalman Filter Algorithm for
Permanent Magnet Stepper Motor
Involving Difference Equations

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Abstract
This paper presents a discrete version of a permanent magnet stepper motor obtained from its continuous time model using difference equations. Based on the control application the control inputs are selected in terms of linearizing outputs. Verification of the controllability and observability for the linearized model and the estimation of correct state variables derived from the response of the discrete state variables to their control inputs using discrete Kalman filter algorithm is implemented with MATLAB.
1 Introduction

A permanent magnet (PM) stepper motor is an extremely precise pulse-drive motor that transforms the input pulses from a digitally controlled system to angular position in steps [1,2,8–10]. The mathematical modeling of PM stepper motor as a dynamic system was initiated in continuous time since most of the position control applications such as robotics, aerospace, machine tools, printers and scanners require the output to be controlled from their continuous states [10]. An increase in the number of state variables led to a simultaneous increase in the complexity of the system. A solution to this was defining the dynamic system in discrete time domain with difference equation instead of the usual differential equations. Another advantage of this method is the system is reduced to a simple first order system which can be easily modeled using summation elements [3, 11]. If a series of input pulses is applied to the stator winding, due to inertia the PM rotor of stepper motor will take more time to reach its final position before stabilizing [13]. To get a correct step positional output a position sensor was used. The increased pulsation in the output of the sensor led to the development of analytical methods to estimate the rotor position [4, 12]. The Kalman filter is one such recursive algorithm which can be used for reference position tracking where the data of the previous step position is used for estimating current step position [7]. The state variables of the transformed continuous time system with initial conditions at any instant of time can be estimated.

The Section II represents the generalized modeling equations to convert any non-linear continuous time system to discrete form. Section III presents the mathematical modeling of the discrete PM stepper motor. Section IV discuss the linearized system of difference equations and in addition deals with the controllability and observability of the linearized model. Section V,VI contain the discrete Kalman filter algorithm, its results and conclusion respectively.

2 Generalized Model of Discrete Time Invariant System

Consider a system with m inputs \( \{u_1(t), u_2(t) \cdots u_m(t)\} \); p outputs \( \{y_1(t), y_2(t) \cdots y_p(t)\} \) and n state state variables \( \{x_1(t), x_2(t) \cdots x_n(t)\} \). From
the knowledge of the state variables \( \{x_1(t_0), x_2(t_0) \ldots x_n(t_0)\} \) at initial time \( t = t_0 \) and input signals \( \{u_1(t), u_2(t) \ldots u_m(t)\} \) for \( t \geq t_0 \) is suffice to determine the behavior of the system for \( t \geq t_0 \). The state variables of the system can be represented as follows:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2 \ldots x_n, u_1, u_2 \ldots u_m) \\
\dot{x}_2 &= f_2(x_1, x_2 \ldots x_n, u_1, u_2 \ldots u_m) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2 \ldots x_n, u_1, u_2 \ldots u_m).
\end{align*}
\]

(1)

The above set of equations of the time invariant system can be written as \( \dot{X} = f(X(t), U(t)) \).

The equation (1) can be represented in matrix form as

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\vdots \\
\dot{x}_n(t)
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{bmatrix} +
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nm}
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
\vdots \\
u_m(t)
\end{bmatrix}
\]

(2)

where the coefficients \( a_{ij} \) and \( b_{ij} \) are constants.

The state and output equations of the system are

\[
\dot{X} = AX + BU \quad \text{and} \quad Y = CX + DU
\]

where \( [X] \) and \( [U] \) are the \( n \times 1 \) state vector and \( m \times 1 \) input vector respectively.

The vector \( [A] \) is a \( n \times n \) and \( [B] \) is an \( n \times m \) matrix, \( [Y] \) is \( p \times 1 \) output vector, \( [C] \) is \( p \times n \) output matrix, \( [D] \) is \( p \times m \) transmission matrix respectively. The set of \( n \) state variables of (2) can be transformed into discrete time system to obtain the output at any time instant \( k \) as

\[
\begin{bmatrix}
x_1(k+1) \\
x_2(k+1) \\
\vdots \\
x_n(k+1)
\end{bmatrix} = 
\begin{bmatrix}
1 + a_{11} \tau & a_{12} \tau & \cdots & a_{1n} \tau \\
a_{21} \tau & 1 + a_{22} \tau & \cdots & a_{2n} \tau \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} \tau & a_{n2} \tau & \cdots & 1 + a_{nn} \tau
\end{bmatrix}
\begin{bmatrix}
x_1(k) \\
x_2(k) \\
\vdots \\
x_n(k)
\end{bmatrix} + \tau
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nm}
\end{bmatrix}
\begin{bmatrix}
u_1(k) \\
u_2(k) \\
\vdots \\
u_m(k)
\end{bmatrix}
\]

(3)

where the sampling time \( \tau \) is a constant.
3 Mathematical Modeling of a Discrete PM Stepper Motor

In [5, 6, 8, 9] the continuous linear time-invariant model of the permanent magnet stepper motor is

\[
\begin{align*}
\dot{i}_d &= \frac{v_d - Ri_d + N_r L \omega_i}{L} \\
\dot{i}_q &= \frac{v_q - Ri_q + N_r L \omega_i - K_m \omega}{L} \\
\dot{\omega} &= \frac{K_m i_q - B \omega}{L} \\
\dot{\theta} &= \omega.
\end{align*}
\] (4)

From (3) and (4), we have

\[
\begin{align*}
i_d(k+1) &= \left(1 - \frac{R \tau}{L}\right) i_d(k) + N_r \tau i_q(k) \omega(k) \frac{v_d(k) \tau}{L} \\
i_q(k+1) &= \left(1 - \frac{R \tau}{L}\right) i_q(k) + N_r \tau i_d(k) \omega(k) \frac{K_m \tau}{L} \omega(k) + \frac{v_q(k) \tau}{L} \\
\omega(k+1) &= \frac{K_m \tau}{J} i_q(k) + \left(1 - \frac{B \tau}{J}\right) \omega(k) \\
\theta(k+1) &= \tau \omega(k) + \theta(k).
\end{align*}
\] (5)

4 Linearized System of Difference Equations

The PM stepper motor makes a single step angular displacement for every input pulse fed to its stator winding. It is desired that the rotor position tracks the given constant reference position. This can be done only if the rotor current tracks the reference current along \(d\)-axis. Linearizing the output in terms of the control inputs as

\[
y_1(k) = i_d(k) \quad y_2(k) = \theta(k).
\] (6)

Applying the control input \(i_d(k)\) in the system of difference equation (5), to obtain the linearized form of state equations as follows:

\[
\begin{align*}
i_d(k+1) &= i_d(k) + \tau v_1(k) \\
i_q(k+1) &= \left(1 + \frac{B \tau}{J}\right) i_q(k) - \frac{B^2 \tau}{JK_m} \omega(k) + \frac{\tau J}{K_m} v_2(k) \\
\omega(k+1) &= \frac{K_m \tau}{J} i_q(k) + \left(1 - \frac{B \tau}{J}\right) \omega(k) \\
\theta(k+1) &= \tau \omega(k) + \theta(k).
\end{align*}
\] (7)
where \( v_1(k) = \frac{i_d(k+1) - i_d(k)}{\tau} \)

and \( v_2(k) = \frac{\theta(k+3) - 3\theta(k+2) + 3\theta(k+1) - \theta(k)}{\tau^3} \) \( (8) \)

are the auxiliary inputs. Hence the linearized form of the above discrete time invariant system is obtained from the equations (6) and (7) as

\[
X(k+1) = A_\tau X(k) + G_\tau V(k) \quad ; \quad Y(k) = CX(k) \quad (9)
\]

where

\[
A_\tau = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 + \frac{B_\tau}{J} & -\frac{B^2_\tau}{\tau J K_m} & 0 \\
0 & \frac{K_m \tau}{J} & 1 - \frac{B_\tau}{\tau J} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad G_\tau = \begin{bmatrix}
\tau & 0 \\
0 & \frac{\tau J}{K_m} \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}. \quad (10)
\]

### 4.1 Controllability and Observability for the Permanent Magnet Stepper Motor

Controllability can be defined in two ways. The input transfers any state to zero state which is called Controllability to the origin. The input transfers zero state to any state. This is referred as controllability from the origin or reachability. The discrete time invariant model of the PM stepper motor satisfies the Kalman’s test for controllability as

\[
Q_{c_\tau} = \begin{bmatrix}
G_\tau & A_\tau G_\tau & A^2_\tau G_\tau & A^3_\tau G_\tau
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.1000 & 0 & 0 & 0.1000 \\
0 & 0.0001 & 0 & 0.1000 \\
0 & 0 & 0 & 0.1000 \\
0 & 0 & 0 & 0.1000 \\
0 & 0 & 0 & 0.1000 \end{bmatrix}
\]

which implies that the rank of \( Q_{c_\tau} \) is equal to number of state variables, where \( A_\tau \) and \( G_\tau \) are given in (10) in the system of the difference equations. Hence the system (9) is controllable.

Similarly from the observability test condition the state variables of the
respective system is measurable for

\[ Q_{o_r} = \begin{bmatrix} C^T & A_r^T C^T & (A_r^T)^2 C^T & (A_r^T)^3 C^T \end{bmatrix} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1000 & 1.0000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 10.0512 & 0.1803 & 1.0000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 30.1536 & 0.2410 & 1.0000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

From the above matrix it is observed that the rank of \( Q_{o_r} \) is 4 which is equal to the number of state variables.
Hence the system (9) is linearized and completely controllable and observable.

5 Discrete Kalman Filter Algorithm

The Kalman filter algorithm estimates the angular displacement of a PM stepper motor process by using a form of feedback measurement from the rotor position. The filter estimates of the system, the process state at some time and then obtains feedback in the form of (noisy) measurements. As such, the equations for the Kalman filter fall into two groups: time update equations and measurement update equations. The time update (or) Predictor equations are responsible for projecting forward (in time) the current state and error covariance estimates to obtain the a priori estimates for the next time step. By incorporating a new measurement into the prior estimate an improved posterior estimate is obtained. Hence the measurement update equations are called as corrector equations. The resulting estimation algorithm resembles that of a predictor-corrector algorithm for solving numerical problems of time and measurement updates.

The time updates is as given in (9) and the measurement updates of the PM stepper motor equation is

\[ P_k = A_r P_{k-1} A_r^T + Q_{k-1} \]

(11)
Discrete Kalman filter algorithm for PM stepper motor

where $P_{k-1}$ is the covariance estimates forward from time step $k - 1$ to step $k$ and $Q$ is the process noise covariance matrices respectively. Also, the process noise covariance matrices $P_k$ and measurement noise covariance matrices $Q_{k-1}$ are Gaussian and might change with each time step or measurement. A correct estimate of the state variables is dependent on the initial value of the co-variance matrix that is chosen. The initial values taken for the covariance matrices are given below.

$$P_0 = \begin{bmatrix} 10^{-4} & 0 & 0 & 0 \\ 0 & 10^{-4} & 0 & 0 \\ 0 & 0 & 10^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_0 = \begin{bmatrix} 10^{-4} & 0 & 0 & 0 \\ 0 & 10^{-4} & 0 & 0 \\ 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-6} \end{bmatrix}$$

The discrete Kalman filter measurement update equations of PM stepper motor are calculated as follows:

$$K_k = P_k H^T (HP_k H^T + R_{k-1})^{-1}$$  \hspace{1cm} (12)

$$x_{k+1} = x_k + K_k (y_{k+1} - H x_k)$$  \hspace{1cm} (13)

$$P_{k+1} = (I - K_k H) P_k$$  \hspace{1cm} (14)

where the white gaussian noise of the initial covariance matrix $R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. After each time and measurement update pair, the process is repeated with the previous a posterior estimates used to project or predict the new a prior estimates of PM stepper motor. The discrete Kalman filter instead recursively conditions the current estimate on all of the past measurements of PM stepper motor.

6 Results and Conclusion

The discretized model of the PM stepper motor is linear and controllable. The matrices $A, G, C$ of the system of equations (9) define the linearized model of the above system which is used for the Kalman filter algorithm. The Kalman filter algorithm yields the following steady-state error co-variance matrices.

$$P_{k+1/k} = \begin{bmatrix} 0.0032 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0099 & 0.0208 \\ 0 & 0 & 0.0208 & 0.0653 \end{bmatrix}.$$
\[ P_{k+1/k+1} = \begin{bmatrix}
0.0032 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0.0095 & 0.0195 \\
0 & 0 & 0.0195 & 0.0613 \\
\end{bmatrix} \]

and the kalman gain matrix \( K_{k+1} = \begin{bmatrix}
0.0032 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0.0195 & 0 & 0 \\
0 & 0.0613 & 0 & 0 \\
\end{bmatrix} \).

\[ E = [0.9968, 0.9689, 0.9839 + 0.0270i, 0.9839 - 0.0270i] \] are distinct and complex conjugate eigen values of the discrete PM stepper motor and is placed within the unity circle. The programming is carried out in MATLAB for the following data \( R = 0.37 \, \text{ohms}, \, L = 0.9 \, \text{mH}, \, K_m = 0.157 \, \text{Nm/A}, \, J = 15.62 \times 10^{-5} \, \text{Kgm}^2, \, B = 0.00307 \, \text{Nm/Rd/sec}, \, N_r = 50 \) and \( \tau = 0.1 \) to implement the discrete kalman filter algorithm for the system states brought to zero from any initial condition in finite time. It can be observed that if the system is subjected to a disturbance it will oscillate about the equilibrium point, hence the need to design a controller to suppress the same. The discrete Kalman filter algorithm can also be used for validation of gates and numerical rounding off issues other than measurement.

References


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