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On Some Integral Inequalities for Twice Differentiable Quasi-Convex and Convex Functions via Fractional Integrals

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Abstract

In this article, a new general identity for twice differentiable functions via Riemann-Liouville fractional integrals is established. By making use of this equality, author has obtained new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for functions whose second derivatives in absolute value at certain powers are, respectively, convex and quasi-convex functions via Riemann-Liouville fractional integrals.

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1 Introduction

Recall that a function $f : I \subseteq R \to R$ is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$, and f is said to be concave on I if this inequality holds in reversed direction.

The following double inequality is well known in the literature as Hermite-Hadamard type inequality: Let $f : I \subseteq R \to R$ be a convex function define on an interval I of real numbers, and $a, b \in I$ with $a < b$. Then the following double inequalities hold:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave.

It was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality (1) was nowhere mentioned in the mathematical literature untill 1893. In [1], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (1) was proved by Hadamard in 1893. In 1974, Mitrinovič found Hermite and Hadamard's note in Mathesis. That is why, the inequality (1) was known as Hermite-Hadamard inequality.

Recall that the following inequality is well-known in the literature as Ostrowski type inequality:

Theorem 1.1. [15] *Let $f : I \subseteq R \to R$ be a differentiable function in I^0, the interior of I, and let $a, b \in I^0$ with $a < b$. If $\mid f'(x) \mid \leq M$ for $x \in [a,b]$, then the following inequality holds:*

$$\Big|f(x) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \leq \frac{M}{b-a}\Big[\frac{(x-a)^2+(b-x)^2}{2}\Big]. \tag{2}$$

Recall that the following inequality is well-known in the literature as Simpson type inequality:

Theorem 1.2. [10] *Let $f : [a,b] \to R$ be a four times continuously differentiable function on (a,b) and $\| f^{(4)} \|_\infty = \sup_{x\in(a,b)} \mid f^{(4)}(x) \mid < \infty$. Then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{3}\Big\{\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\Big\} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \\ &\leq \frac{1}{2880}\| f^{(4)} \|_\infty (b-a)^4.\end{aligned} \tag{3}$$

For recent results and generalizations of Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality concerning convex functions, you may see [1, 3, 4, 5, 7, 9, 8, 13, 16, 17] and the references therein.

We recall that the notion of quasi-convex function generalizes the notion of convex function.

Definition 1. A function $f : [a, b] \subseteq R \to R$ is said to be quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughtout this paper.

Definition 2. [2, 5, 6, 14, 15] Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, x > a$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}\mathrm{d}t$ and $J^0_{a^+} f(x) = J^0_{b^-} f(x) = f(x)$.

In [5, 6], Imdat Işcan established the following theorem:

Theorem 1.3. *Let $f : I \subset R \to R$ be a twice differentiable function on I^0 such that $f'' \in L[a, b]$, where $a, b \in I^0$ with $a < b$. If $\mid f'' \mid^q$ is quasi–convex on $[a, b]$ for some fixed $q \geq 1$, then for $x \in [a, b], \lambda \in [0, 1]$ and $\alpha > 0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|\Big(\frac{1-\lambda(1+\alpha)}{1+\alpha}\Big)\Big[\frac{(b-x)^{\alpha+1}-(x-a)^{\alpha+1}}{b-a}\Big] f'(x) \\
&\quad + \lambda\Big[\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a}\Big] \\
&\quad + (1-\lambda)\Big[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\Big] f(x) \\
&\quad - \frac{\Gamma(\alpha+1)}{b-a}\Big[J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(b)\Big]\Big| \\
&\leq C_1(\alpha, \lambda) \\
&\quad \times \Big[\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\Big\{\max\{\mid f''(x)\mid^q, \mid f''(a)\mid^q\}\Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\Big\{\max\{\mid f''(x)\mid^q, \mid f''(b)\mid^q\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned} \tag{4}$$

where

$$C_1(\alpha, \lambda) = \begin{cases} \frac{\alpha[(\alpha+1)\lambda]^{\frac{\alpha+2}{\alpha}}+1}{\alpha+2} - \frac{(\alpha+1)\lambda}{2}, & 0 \leq \lambda \leq \frac{1}{\alpha+1} \\ \frac{(\alpha+1)(\alpha+2)\lambda-2}{2(\alpha+2)}, & \frac{1}{\alpha+1} < \lambda \leq 1. \end{cases}$$

Theorem 1.4. *Let $f : I \subset [0,\infty) \to R$ be a differentiable function on I^0 such that $f' \in L[a,b]$, where $a, b \in I^0$ with $a < b$. If $\mid f'' \mid^q$ is s-convex in the second sense on $[a,b]$ for some fixed $q \geq 1$, then for $x \in [a,b], \lambda \in [0,1]$ and $\alpha > 0$, the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|\lambda\Big[\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}\Big] \\
&\quad + (1-\lambda)\Big[\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\Big]f(x) \\
&\quad - \frac{\Gamma(\alpha+1)}{b-a}\Big[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)\Big]\Big| \\
&\leq A_1^{1-\frac{1}{q}}(\alpha,\lambda) \\
&\quad \times \Big[\frac{(x-a)^{\alpha+1}}{b-a}\Big\{ \mid f'(x) \mid^q A_2(\alpha,\lambda,s) + \mid f'(a) \mid^q A_3(\alpha,\lambda,s)\Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\Big\{ \mid f'(x) \mid^q A_2(\alpha,\lambda,s) + \mid f'(b) \mid^q A_3(\alpha,\lambda,s)\Big\}^{\frac{1}{q}}\Big],
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
A_1(\alpha,\lambda) &= \frac{1+2\alpha\lambda^{1+\frac{1}{\alpha}}}{\alpha+1} - \lambda, \\
A_2(\alpha,\lambda,s) &= \frac{s+1+2\alpha\lambda^{1+\frac{s+1}{\alpha}}}{\alpha+1} - \frac{\lambda}{s+1}, \\
A_3(\alpha,\lambda,s) &= \lambda\Big\{\frac{1-2\big(1-\lambda^{\frac{1}{\alpha}}\big)^{s+1}}{s+1}\Big\} + \beta(\alpha+1,s+1) \\
&\quad - 2\beta(\lambda^{\frac{1}{\alpha}},\alpha+1,s+1),
\end{aligned} \tag{6}$$

where $\beta(\cdot,\cdot)$ is Euler Beta function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, x, y > 0,$$

and $\beta(\cdot,\cdot,\cdot)$ is incomplete Beta function defined by

$$\beta(a,x,y) = \int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, x, y > 0, 0 < a < 1.$$

Note that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(x,y+1) + \beta(x+1,y) = \frac{x+y}{x}\beta(x+1,y)$$

and

$$\beta(a,x,y) = \frac{1}{\beta(x,y)}\int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

In this paper, by making use of equality, the power mean inequality and Hölder inequality, author has obtained new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for functions whose q-th powers of absolute values of second derivatives are, respectively, convex and quasi-convex functions via Riemann-Liouville fractional integrals.

2 Main results

In this section, for the simplicity of the notation, let

$$\begin{aligned}
&I_f(x,\lambda,\alpha;a,b)\\
&\equiv(1-\lambda)\Big\{\frac{(b-x)^{\alpha+1}-(x-a)^{\alpha+1}}{b-a}\Big\}f'(x)\\
&\quad+(1+\alpha-\lambda)\Big\{\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\Big\}f(x)\\
&\quad+\lambda\Big\{\frac{(x-a)^{\alpha}(f(a)+(b-x)^{\alpha}f(b)}{b-a}\Big\}\\
&\quad-\frac{\Gamma(\alpha+2)}{b-a}\Big\{J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)\Big\}
\end{aligned}$$

for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$.

In order to prove our main results, we need the following identity:

Lemma 1. *Let $f:I\subseteq R\to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. Then, for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$ we have*

$$\begin{aligned}
&I_f(x,\lambda,\alpha;a,b)\\
&=\frac{(x-a)^{\alpha+2}}{b-a}\int_0^1 t(\lambda-t^{\alpha})f''\big(tx+(1-t)a\big)\mathrm{d}t\\
&\quad+\frac{(b-x)^{\alpha+2}}{b-a}\int_0^1 t(\lambda-t^{\alpha})f''\big(tx+(1-t)b\big)\mathrm{d}t.
\end{aligned}$$

Proof. From Definition 2, by letting $tx+(1-t)a=u$ and $tx+(1-t)b=u$, respectively, we have

$$\begin{aligned}
\int_0^1 t^{\alpha-1}f\big(tx+(1-t)a\big)\mathrm{d}t&=\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}J^{\alpha}_{x^-}f(a),\\
\int_0^1 t^{\alpha-1}f\big(tx+(1-t)b\big)\mathrm{d}t&=\frac{\Gamma(\alpha)}{(b-x)^{\alpha}}J^{\alpha}_{x^+}f(b).
\end{aligned}\tag{7}$$

By integration by parts, using the properties (7) and changing the variables, for $x \neq a$ we can state

$$\begin{aligned}(i) &\int_0^1 t(\lambda - t^\alpha) f''\big(tx + (1-t)a\big)\mathrm{d}t \\ &= (\lambda - 1)\frac{f'(x)}{x-a} + (1+\alpha-\lambda)\frac{f(x)}{(x-a)^2} \\ &\quad + \lambda\frac{f(a)}{(x-a)^2} - \frac{\Gamma(\alpha+2)}{(x-a)^{\alpha+2}} J^\alpha_{x^-} f(a), \end{aligned} \tag{8}$$

and, similarly, for $x \neq b$ we get

$$\begin{aligned}(ii) &\int_0^1 t(\lambda - t^\alpha) f''\big(tx + (1-t)b\big)\mathrm{d}t \\ &= (1 - \lambda)\frac{f'(x)}{b-x} + (1+\alpha-\lambda)\frac{f(x)}{(b-x)^2} \\ &\quad + \lambda\frac{f(b)}{(b-x)^2} - \frac{\Gamma(\alpha+2)}{(b-x)^{\alpha+2}} J^\alpha_{x^+} f(b). \end{aligned} \tag{9}$$

Multiplying both sides of (8) and (9) by $\frac{(x-a)^{\alpha+2}}{b-a}$ and $\frac{(b-x)^{\alpha+2}}{b-a}$, respectively, and adding the resulting identities we obtain the desired result.

Theorem 2.1. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a,b]$, where $a, b \in I^0$ with $a < b$. If $\mid f'' \mid^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\Big|I_f(x,\lambda,\alpha;a,b)\Big| \\ &\leq A_1^{1-\frac{1}{q}}(\alpha,\lambda)\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big\{A_2(\alpha,\lambda)\big|f''(x)\big|^q + A_3(\alpha,\lambda)\big|f''(a)\big|^q\Big\}^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big\{A_2(\alpha,\lambda)\big|f''(x)\big|^q + A_3(\alpha,\lambda)\big|f''(b)\big|^q\Big\}^{\frac{1}{q}}\Big] \end{aligned} \tag{10}$$

for any $x = ta + (1-t)b$, $t \in [0,1]$, $\lambda \in [0,1]$, and $\alpha > 0$, where

$$\begin{aligned} A_1(\alpha,\lambda) &= \frac{\alpha\lambda^{1+\frac{2}{\alpha}} + 1}{\alpha+2} - \frac{\lambda}{2}, \\ A_2(\alpha,\lambda) &= \frac{3 - (\alpha+3)\lambda + 2\alpha\lambda^{1+\frac{3}{\alpha}}}{3(\alpha+3)}, \\ A_3(\alpha,\lambda) &= \frac{1}{(\alpha+2)(\alpha+3)} - \frac{\lambda}{6} + \big(\frac{\alpha}{\alpha+2}\big)\lambda^{1+\frac{2}{\alpha}}\big(1 - \frac{2}{3}\lambda^{\frac{1}{\alpha}}\big). \end{aligned}$$

Proof. From Lemma 1, using the property of the modulus and the power mean inequality, we have

$$\begin{aligned}
&\Big|I_f(x,\lambda,\alpha;a,b)\Big| \\
&\le \frac{(x-a)^{\alpha+2}}{b-a}\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)a\big)\big|\mathrm{d}t \\
&\quad + \frac{(b-x)^{\alpha+2}}{b-a}\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)b\big)\big|\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\le A_1^{1-\frac{1}{q}}(\alpha,\lambda)\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big(\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)a\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big(\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)b\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]. \qquad (11)
\end{aligned}$$

Since $\mid f' \mid^q$ is convex on $[a,b]$, we have

$$\begin{aligned}
&(i)\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)a\big)\big|^q\mathrm{d}t \\
&\le \int_0^1 \big|t(\lambda-t^\alpha)\big|\big\{t\big|f''\big(x\big)\big|^q+(1-t)\big|f''\big(a\big)\big|^q\big\}\mathrm{d}t \\
&= A_2(\alpha,\lambda)\big|f''\big(x\big)\big|^q + A_3(\alpha,\lambda)\big|f''\big(a\big)\big|^q \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
&(ii)\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)b\big)\big|^q\mathrm{d}t \\
&\le A_2(\alpha,\lambda)\big|f''\big(x\big)\big|^q + A_3(\alpha,\lambda)\big|f''\big(b\big)\big|^q. \qquad (13)
\end{aligned}$$

By substituting (12) and (13) in (11), we get the desired result.

Corollary 2.1. *In Theorem 2.1, if we take $q=1$, then the inequality (10) reduces to the following inequality:*

$$\begin{aligned}
&\Big|I_f(x,\lambda,\alpha;a,b)\Big| \\
&\le \frac{(x-a)^{\alpha+2}}{b-a}\Big\{A_2(\alpha,\lambda)\big|f''\big(x\big)\big| + A_3(\alpha,\lambda)\big|f''\big(a\big)\big|\Big\} \\
&\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big\{A_2(\alpha,\lambda)\big|f''\big(x\big)\big| + A_3(\alpha,\lambda)\big|f''\big(b\big)\big|\Big\}.
\end{aligned}$$

Corollary 2.2. *In Theorem 2.1, if we choose* $x = \frac{a+b}{2}$ *and* $\lambda = \frac{1}{3}$*, then the inequality (10) reduces to the following inequality:*

$$\begin{aligned}
&\Big|I_f(\frac{a+b}{2}, \frac{1}{3}, \alpha; a, b)\Big| \\
&\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+2}} A_1^{1-\frac{1}{q}}(\alpha, \frac{1}{3}) \\
&\quad \times \Big[\Big\{A_2(\alpha, \frac{1}{3})\big|f''(\frac{a+b}{2})\big|^q + A_3(\alpha, \frac{1}{3})\big|f''(a)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{A_2(\alpha, \frac{1}{3})\big|f''(\frac{a+b}{2})\big|^q + A_3(\alpha, \frac{1}{3})\big|f''(b)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Corollary 2.3. *In Theorem 2.1, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = 0$*, then the inequality (10) reduces to the following inequality:*

$$\begin{aligned}
&\Big|I_f(\frac{a+b}{2}, 0, \alpha; a, b)\Big| \\
&\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+2}} \Big(\frac{1}{\alpha+2}\Big)^{1-\frac{1}{q}} \\
&\quad \times \Big[\Big\{\frac{1}{\alpha+3}\big|f''(\frac{a+b}{2})\big|^q + \frac{1}{(\alpha+2)(\alpha+3)}\big|f''(a)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\frac{1}{\alpha+3}\big|f''(\frac{a+b}{2})\big|^q + \frac{1}{(\alpha+2)(\alpha+3)}\big|f''(b)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Corollary 2.4. *In Theorem 2.1, if we choose* $\lambda = 1$*, then the inequality (10) reduces to the following inequality:*

$$\begin{aligned}
&\Big|I_f(x, 1, \alpha; a, b)\Big| \\
&\leq A_1^{1-\frac{1}{q}}(\alpha, 1) \\
&\quad \times \Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big\{A_2(\alpha, 1)\big|f''(x)\big|^q + A_3(\alpha, 1)\big|f''(a)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big\{A_2(\alpha, 1)\big|f''(x)\big|^q + A_3(\alpha, 1)\big|f''(b)\big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Theorem 2.2. *Let* $f : I \subset [0, \infty) \to R$ *be a twice differentiable function on the interior* I^0 *of an interval* I *such that* $f'' \in L_1[a, b]$*, where* $a, b \in I^0$ *with* $a < b$*. If* $\mid f'' \mid^q$ *is convex on* $[a, b]$ *for some fixed* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, then the following inequality for fractional integrals*

$$\begin{aligned}
&\Big|I_f(x, \lambda, \alpha; a, b)\Big| \\
&\leq B^{\frac{1}{p}}(\alpha, \lambda, p)\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big\{\frac{\big|f''(x)\big|^q + \big|f''(a)\big|^q}{2}\Big\}^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big\{\frac{\big|f''(x)\big|^q + \big|f''(b)\big|^q}{2}\Big\}^{\frac{1}{q}}\Big], \qquad (14)
\end{aligned}$$

holds, for any $x \in [a,b], \lambda \in [0,1]$ *and* $\alpha > 0$, *where* ${}_2F_1$ *is hypergeometric function defined by*

$$ {}_2F_1(a,b,c,z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t $$

for $0 < b < c$ *and* $\mid z \mid < 1$, *and*

$$ \begin{aligned} & B(\alpha,\lambda,p) \\ &= \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\Big\{\Gamma(1+p)\Gamma(\frac{1+p+\alpha}{\alpha})\ \ {}_2F_1(1,1+p,2+p+\frac{1+p}{\alpha},1) \\ &\quad + \beta(1+p,-\frac{1+p+\alpha p}{\alpha} - \beta(\lambda,1+p,-\frac{1+p+\alpha p}{\alpha})\Big\}. \end{aligned} $$

Proof. From Lemma 1, using the power mean inequality, we have

$$ \begin{aligned} & \Big|I_f(x,\lambda,\alpha;a,b)\Big| \\ &\le \frac{(x-a)^{\alpha+2}}{b-a}\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)a\big)\big|\mathrm{d}t \\ &\quad + \frac{(b-x)^{\alpha+2}}{b-a}\int_0^1 \big|t(\lambda-t^\alpha)\big|\big|f''\big(tx+(1-t)b\big)\big|\mathrm{d}t \\ &\le \frac{(x-a)^{\alpha+2}}{b-a}\Big(\int_0^1 \big|t(\lambda-t^\alpha)\big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \big|f''\big(tx+(1-t)a\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big(\int_0^1 \big|t(\lambda-t^\alpha)\big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \big|f''\big(tx+(1-t)b\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\alpha,\lambda,p)\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big(\int_0^1 \big|f''\big(tx+(1-t)a\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^{\alpha+2}}{b-a}\Big(\int_0^1 \big|f''\big(tx+(1-t)b\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]. \end{aligned} \tag{15} $$

Since $\mid f' \mid^q$ is convex on $[a,b]$, we have

$$ \begin{aligned} &(i) \int_0^1 \big|f''\big(tx+(1-t)a\big)\big|^q\mathrm{d}t \le \frac{\big|f''(x)\big|^q + \big|f''(a)\big|^q}{2}, \\ &(ii) \int_0^1 \big|f''\big(tx+(1-t)b\big)\big|^q\mathrm{d}t \le \frac{\big|f''(x)\big|^q + \big|f''(b)\big|^q}{2}. \end{aligned} \tag{16} $$

By letting $\lambda - t^\alpha = u$ and $t^\alpha = u$, respectively, we have

$$ \begin{aligned} &(i) \int_0^{\lambda^{\frac{1}{\alpha}}} \big\{t(\lambda-t^\alpha)\big\}^p\mathrm{d}t = \frac{1}{\alpha}\int_0^\lambda u^p(\lambda-u)^{\frac{1+p-\alpha}{\alpha}}\mathrm{d}u \\ &\quad = \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\Gamma(1+p)\Gamma(\frac{1+p+\alpha}{\alpha})\ \ {}_2F_1(1,1+p,2+p+\frac{1+p}{\alpha},1), \end{aligned} \tag{17} $$

and

$$(ii)\int_{\lambda^{\frac{1}{\alpha}}}^{1}\big\{t(t^{\alpha}-\lambda)\big\}^{p}\mathrm{d}t=\frac{1}{\alpha}\int_{\lambda}^{1}u^{\frac{1+p-\alpha}{\alpha}}(u-\lambda)^{p}\mathrm{d}u$$
$$=\frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\Big\{\beta(1+p,-\frac{1+p+\alpha p}{\alpha})-\beta(\lambda,1+p,-\frac{1+p+\alpha p}{\alpha})\Big\}. \quad (18)$$

By (17) and (18), since

$$\int_{0}^{1}\big|t(\lambda-t^{\alpha})\big|^{p}\mathrm{d}t=\int_{0}^{\lambda^{\frac{1}{\alpha}}}\big\{t(\lambda-t^{\alpha})\big\}^{p}\mathrm{d}t+\int_{\lambda^{\frac{1}{\alpha}}}^{1}\big\{t(t^{\alpha}-\lambda)\big\}^{p}\mathrm{d}t, \quad (19)$$

we have

$$\int_{0}^{1}\big|t(\lambda-t^{\alpha})\big|^{p}\mathrm{d}t=B(\alpha,\lambda,p).$$

By substituting (16) and (19) in (15), we get the desired result.

Corollary 2.5. *In Theorem 2.2, if we take $x=\frac{a+b}{2}$, then the inequality (18) reduces to the following inequality:*

$$\Big|I_f(\frac{a+b}{2},\lambda,\alpha;a,b)\Big|$$
$$\leq\frac{(b-a)^{\alpha+1}}{2^{\alpha+2}}B^{\frac{1}{p}}(\alpha,\lambda,p)\Big[\Big\{\frac{\big|f''\big(\frac{a+b}{2}\big)\big|^{q}+\big|f''\big(a\big)\big|^{q}}{2}\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{\big|f''\big(\frac{a+b}{2}\big)\big|^{q}+\big|f''\big(b\big)\big|^{q}}{2}\Big\}^{\frac{1}{q}}\Big].$$

Here note that $B(\alpha,0,p)=\frac{1}{(\alpha+1)p+1}$ and $B(\alpha,1,p)=\frac{\alpha}{2(\alpha+2)}$.

Corollary 2.6. *In Theorem 2.1, if we choose $x=\frac{a+b}{2}$ and $\lambda=\frac{1}{3}$, then the inequality (10) reduces to the following inequality:*

$$\Big|I_f(\frac{a+b}{2},\frac{1}{3},\alpha;a,b)\Big|$$
$$\leq\frac{(b-a)^{\alpha+1}}{2^{\alpha+2}}B^{\frac{1}{p}}(\alpha,\frac{1}{3},p)\Big[\Big\{\frac{\big|f''\big(\frac{a+b}{2}\big)\big|^{q}+\big|f''\big(a\big)\big|^{q}}{2}\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{\big|f''\big(\frac{a+b}{2}\big)\big|^{q}+\big|f''\big(b\big)\big|^{q}}{2}\Big\}^{\frac{1}{q}}\Big].$$

Corollary 2.7. *In Theorem 2.1, if we take $x=\frac{a+b}{2}$ and $\lambda=0$, then the inequality (10) reduces to the following inequality:*

$$\Big|I_f(\frac{a+b}{2},0,\alpha;a,b)\Big|$$
$$\leq\frac{(b-a)^{\alpha+1}}{2^{\alpha+2}}\Big(\frac{1}{(\alpha+1)p+1}\Big)^{\frac{1}{p}}\Big[\Big\{\frac{\big|f''\big(\frac{a+b}{2}\big)\big|^{q}+\big|f''\big(a\big)\big|^{q}}{2}\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{\big|f''\big(\frac{a+b}{2}\big)\big|^{q}+\big|f''\big(b\big)\big|^{q}}{2}\Big\}^{\frac{1}{q}}\Big].$$

Corollary 2.8. *In Theorem 2.1, if we choose $\lambda = 1$, then the inequality (10) reduces to the following inequality:*

$$\begin{aligned}&\Big|I_f(x,1,\alpha;a,b)\Big|\\&\leq \frac{\beta(\frac{p+1}{\alpha},p+1)}{\alpha}\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big\{\frac{\big|f''(x)\big|^q+\big|f''(a)\big|^q}{2}\Big\}^{\frac{1}{q}}\\&\quad+\frac{(b-x)^{\alpha+2}}{b-a}\Big\{\frac{\big|f''(x)\big|^q+\big|f''(b)\big|^q}{2}\Big\}^{\frac{1}{q}}\Big].\end{aligned}$$

By the definition of quasi-convex function also we have the following theorem:

Theorem 2.3. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a,b]$, where $a,b \in I^0$ with $a < b$. If $|\ f''\ |^q$ is quasi-convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals*

$$\begin{aligned}&\Big|I_f(x,\lambda,\alpha;a,b)\Big|\\&\leq A_1(\alpha,\lambda)\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big(\max\big\{\big|f''(x)\big|^q,\big|f''(a)\big|^q\big\}\Big)^{\frac{1}{q}}\\&\quad+\frac{(b-x)^{\alpha+2}}{b-a}\Big(\max\big\{\big|f''(x)\big|^q,\big|f''(b)\big|^q\big\}\Big)^{\frac{1}{q}}\Big]\end{aligned}$$

holds, for $x \in [a,b], \lambda \in [0,1]$ and $\alpha > 0$, where A_1 is defined as in Theorem 2.1.

Theorem 2.4. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a,b]$, where $a,b \in I^0$ with $a < b$. If $|\ f''\ |^q$ is quasi-convex on $[a,b]$ for some fixed $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality for fractional integrals*

$$\begin{aligned}&\Big|I_f(x,\lambda,\alpha;a,b)\Big|\\&\leq B^{\frac{1}{p}}(\alpha,\lambda,p)\Big[\frac{(x-a)^{\alpha+2}}{b-a}\Big(\max\big\{\big|f''(x)\big|^q,\big|f''(a)\big|^q\big\}\Big)^{\frac{1}{q}}\\&\quad+\frac{(b-x)^{\alpha+2}}{b-a}\Big(\max\big\{\big|f''(x)\big|^q,\big|f''(b)\big|^q\big\}\Big)^{\frac{1}{q}}\Big],\end{aligned}$$

holds, for $x \in [a,b], \lambda \in [0,1]$ and $\alpha > 0$, where the functions ${}_2F_1$ and B are defined as in Theorem 2.2.

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