Independent Non-split Domination
in the Join and Corona of Graphs

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Abstract

In this paper, we explore the concept of independent non-split domination in graphs. In particular, we characterized the independent non-split dominating sets of the join and corona of graphs and obtain their independent non-split domination numbers. Also, a connected graph with a given order, independent domination number, and independent non-split domination number is constructed.

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1 Introduction and Preliminary Results

Let \( G = (V(G), E(G)) \) be a connected undirected graph. For any vertex \( v \in V(G) \), the open neighborhood of \( v \) is the set \( N(v) = \{ u \in V(G) : uv \in E(G) \} \) and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{ v \} \). For a set \( X \subseteq V(G) \), the open neighborhood of \( X \) is \( N(X) = \bigcup_{v \in X} N(v) \) and the closed neighborhood of \( X \) is \( N[X] = \bigcup_{v \in X} N[v] \).

The subgraph \( \langle C \rangle \) of \( G \) induced by \( C \) is the graph having vertex-set \( C \) and whose edge set consists of those edges of \( G \) incident with two elements of \( C \). A
graph is called connected if every two vertices are joined by a path; otherwise, it is disconnected.

A set $S \subseteq V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set $S \subseteq V(G)$ is called an independent dominating set of $G$ if for all $u, v \in S$, $uv \not\in E(G)$. The independent domination number of $G$, denoted by $i(G)$, is the smallest cardinality of an independent dominating set of $G$.

A dominating set $S \subseteq V(G)$ is a non-split dominating set of $G$ if $\langle V(G) \setminus S \rangle$ is connected. The non-split domination number of $G$, denoted by $\gamma_{ns}(G)$, is the smallest cardinality of a non-split dominating set of $G$. An independent dominating set $S \subseteq V(G)$ is an independent non-split dominating set of $G$ if $\langle V(G) \setminus S \rangle$ is connected. The independent non-split domination number of $G$, denoted by $i_{ns}(G)$, is the smallest cardinality of an independent non-split dominating set of $G$.

The concept of non-split domination was introduced by V.R. Kulli and B. Janakiram [2]. They obtained bounds on $\gamma_{ns}(G)$ and investigated relationship with other parameters. In [1], the inverse non-split domination in graphs was introduced and discussed; and in [3], the non-split dominating sets in the join and corona are characterized and the non-split and inverse non-split domination numbers of these graphs were determined. In this paper, the concept of independent non-split domination in graphs is revisited. In particular, the independent non-split dominating sets in the join and corona are characterized and their independent non-split domination numbers are obtained.

The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Let $G$ and $H$ be graphs of order $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$.

**Remark 1.1** Let $G$ be a graph and let $S$ be an independent non-split dominating set of $G$.

(i) If $v$ is a cut-vertex of $G$, then $v \not\in S$.

(ii) If $v$ is a leaf of $G$, then $v \in S$.

**Remark 1.2** Let $G$ be a graph. If $i_{ns}(G)$ exists, then $i(G) \leq i_{ns}(G)$.

**Theorem 1.3** Given positive integers $a, b, n$ with $3 \leq a \leq b$ and $n \geq a + b$, there exists a connected graph $G$ such that $i(G) = a$, $i_{ns}(G) = b$, and $|V(G)| = n$. 
Proof: Consider the path $P_a = [u_1, u_2, \ldots, u_a]$. Let $G$ be a graph obtained from $P_a$ by adding the edges $u_i x_i$ for $i = 1, 2, \ldots, a$; adding the edges $u_a v_j$ for $j = 0, 1, \ldots, b-a$; and adding the vertices $w_k$ for $k = 0, 1, \ldots, n-a-b$ and forming the complete graph $K_r$, where $V(K_r) = \{u_1, x_1, w_1, \ldots, w_{n-a-b}\}$ (see Figure 1). Then $S_1 = \{x_1, x_2, \ldots, x_{a-1}, u_a\}$ is a minimum independent dominating set of $G$ and $S_2 = \{x_1, x_2, \ldots, x_a\} \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a minimum independent non-split dominating set of $G$. Thus, $i(G) = |S_1| = a$, $i_{ns}(G) = |S_2| = b$, and $|V(G)| = n$. $\square$

2 Main Results

Theorem 2.1 Let $G$ be a graph. Then $i_{ns}(G) = 1$ if and only if $G = K_1 + H$, where $H$ is a connected graph.

Proof: Suppose $i_{ns}(G) = 1$. Let $S = \{x\}$ be an independent non-split dominating set of $G$. Then $\langle V(G) \setminus S \rangle$ is connected. Set $K_1 = \langle \{x\} \rangle$ and $H = \langle V(G) \setminus S \rangle$. Then $G = K_1 + H$.

The converse is clear. $\square$

Corollary 2.2 $i_{ns}(K_1 + G) = 1$ if and only if $G$ is a connected graph.

Theorem 2.3 $i_{ns}(K_1 + G) = 2$ if and only if $G$ is not connected and $i(G) = 2$.

Proof: Suppose $i_{ns}(K_1 + G) = 2$. Let $K_1 = \langle \{v\} \rangle$ and $S = \{u, w\}$ be an independent non-split dominating set of $K_1 + G$. Since $G$ is not connected, $v$ is a cut-vertex of $K_1 + G$. By Remark 1.1(i), $v \notin S$ and so, $S \subseteq V(G)$. Thus, $S$ is an independent dominating set of $G$. Hence, $i(G) \leq |S| = 2$. Since $G$ is not connected, $i(G) \neq 1$. Therefore, $i(G) = 2$.

Conversely, suppose $i(G) = 2$. Let $S = \{x, y\}$ be an independent dominating set of $G$. Then $S$ is an independent dominating set of $K_1 + G$. If $K_1 = \langle \{v\} \rangle$, then $vz \in E(K_1 + G)$ for all $z \in V(G) \setminus S$. This implies that $\langle V(K_1 + G) \setminus S \rangle$ is connected. Hence, $S$ is an independent non-split dominating set of $K_1 + G$. Thus, $i_{ns}(K_1 + G) \leq |S| = 2$. Since $G$ is not connected, $i_{ns}(K_1 + G) \neq 1$ by
Corollary 2.2. Therefore, \( i_{ns}(K_1 + G) = 2 \). □

The next result characterizes the independent non-split dominating sets of \( K_1 + G \).

**Theorem 2.4** Let \( K_1 = \langle \{v\} \rangle \) and \( G \) a nonconnected graph. Then \( S \subseteq V(K_1 + G) \) is an independent non-split dominating set of \( K_1 + G \) if and only if \( S \) is an independent dominating set of \( G \).

**Proof:** Suppose \( S \) is an independent non-split dominating set of \( K_1 + G \). Since \( G \) is not connected, \( v \) is a cut-vertex of \( K_1 + G \). By Remark 1.1(i), \( v \notin S \). Thus, \( S \subseteq V(G) \) and hence, \( S \) is an independent dominating set of \( G \).

Conversely, suppose \( S \) is an independent dominating set of \( G \). Then \( S \subseteq V(K_1 + G) \). Thus, \( S \) is an independent non-split dominating set of \( K_1 + G \). Hence, \( i(G) = |S'| \geq i_{ns}(K_1 + G) \). Therefore, \( i_{ns}(K_1 + G) = i(G) \). □

**Corollary 2.5** Let \( K_1 = \langle \{v\} \rangle \) and \( G \) a nonconnected graph. Then \( i_{ns}(K_1 + G) = i(G) \).

**Proof:** Suppose \( S \) is a minimum independent non-split dominating set of \( K_1 + G \). By Theorem 2.4, \( S \) is an independent dominating set of \( G \). Thus, \( i(G) \leq |S| = i_{ns}(K_1 + G) \). On the other hand, suppose \( S' \) is a minimum independent dominating set of \( G \). By Theorem 2.4, \( S' \) is an independent non-split dominating set of \( K_1 + G \). Hence, \( i(G) = |S'| \geq i_{ns}(K_1 + G) \). Therefore, \( i_{ns}(K_1 + G) = i(G) \). □

**Theorem 2.6** Let \( G \) and \( H \) be graphs, both not isomorphic to \( K_n \). Then \( S \subseteq V(G + H) \) is an independent non-split dominating set of \( G + H \) if and only if either \( S \) is an independent dominating set of \( G \) or \( S \) is an independent dominating set of \( H \).

**Proof:** Suppose \( S \) is an independent non-split dominating set of \( G + H \). Then either \( S \subseteq V(G) \) or \( S \subseteq V(H) \). Thus, either \( S \) is an independent dominating set of \( G \) or \( S \) is an independent dominating set of \( H \).

For the converse, suppose \( S \) is an independent dominating set of \( G \). Then \( V(G) \setminus S \neq \emptyset \). Let \( x \in V(G) \setminus S \). Then \( xy \in E(G + H) \) for all \( y \in V(H) \). This implies that \( \langle V(G + H) \setminus S \rangle \) is connected. Hence, \( S \) is an independent non-split dominating set of \( G + H \). Similarly, if \( S \) is an independent dominating set of \( H \), then \( S \) is an independent non-split dominating set of \( G + H \). □

**Corollary 2.7** Let \( G \) and \( H \) be graphs, both not isomorphic to \( K_n \). Then

\[
i_{ns}(G + H) = \min\{i(G), i(H)\}.
\]
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Proof: Suppose that \(i(G) \leq i(H)\). Let \(S\) be a minimum independent dominating set of \(G\). By Theorem 2.6, \(S\) is an independent non-split dominating set of \(G + H\). Thus, \(i(G) = |S| \geq i_{ns}(G + H)\). On the other hand, let \(S'\) be a minimum independent non-split dominating set of \(G + H\). By Theorem 2.6, \(S'\) is an independent dominating set of \(G\). Hence, \(i(G) \leq |S'| = i_{ns}(G + H)\). Therefore, \(i_{ns}(G + H) = i(G)\). Consequently, \(i_{ns}(G + H) = \min\{i(G), i(H)\}\).

Corollary 2.8 Let \(G\) be a connected graph and \(n\) a positive integer integer greater than or equal to 2. Then \(i_{ns}(G + K_n) = \min\{i(G), n\}\).

The next result characterizes the independent non-split dominating set of \(G \circ H\).

Theorem 2.9 Let \(G\) be a connected graph and let \(H\) be any graph. Then \(C \subseteq V(G \circ H)\) is an independent non-split dominating set of \(G \circ H\) if and only if \(C = \bigcup_{v \in V(G)} S^v\), where \(S^v\) is an independent dominating set of \(H^v\) for all \(v \in V(G)\).

Proof: Suppose \(C \subseteq V(G \circ H)\) is an independent non-split dominating set of \(G \circ H\). Let \(v \in V(G)\). Suppose \(v \notin C\). Then \(v \notin V(G \circ H)\). Since \(v\) is a cut-vertex of \(G \circ H\), it follows \(C = V(G \circ H)\). This contradicts the assumption that \(C\) is an independent non-split dominating set of \(G \circ H\). Hence, \(v \notin C\) for all \(v \in V(G)\). Thus, \(C \cap V(H^v) \neq \emptyset\) and \(C \cap V(H^v)\) is a dominating set of \(H^v\) for all \(v \in V(G)\). Set \(S^v = C \cap V(H^v)\) for all \(v \in V(G)\). Then \(C = \bigcup_{v \in V(G)} S^v\).

Conversely, suppose \(C = \bigcup_{v \in V(G)} S^v\), where \(S^v\) is an independent dominating set of \(H^v\) for all \(v \in V(G)\). Clearly \(C\) is an independent dominating set of \(G \circ H\). Let \(x, y \in V(G \circ H)\). If \(xy \notin E(G \circ H)\), then we are done. Suppose \(xy \notin E(G \circ H)\). Consider the following cases:

Case 1. \(x, y \in V(G)\).

Since \(G\) is connected, there exists an \(x - y\) path in \(V(G \circ H)\). Thus, there exists an \(x - y\) path in \(V(G \circ H)\). By Case 1, there exists an \(x - y\) path in \(V(G \circ H)\).

Case 2. \(x \in V(G)\) and \(y \in V(H^v)\) for some \(v \in V(G)\).

Then \(vy \in E(G \circ H)\). By Case 1, there is a \(u - v\) path in \(V(G \circ H)\). Hence, there is an \(x - y\) path in \(V(G \circ H)\).

Case 3. \(x \in V(H^v)\) and \(y \in V(H^v)\) for some \(u, v \in V(G)\).

Then \(xu, vy \in E(G \circ H)\). By Case 1, there is a \(u - v\) path in \(V(G \circ H)\). Therefore, \(V(G \circ H)\) is connected. Accordingly, \(C\) is is an independent non-split dominating set of \(G \circ H\).
Corollary 2.10 Let $G$ be a connected graph and let $H$ be any graph. Then $i_{ns}(G \circ H) = i(H)|V(G)|$.

Proof: Suppose $C$ is a minimum independent non-split dominating set of $G \circ H$. By Theorem 2.9, $C = \bigcup_{v \in V(G)} S^v$, where $S^v$ is an independent dominating set of $H^v$ for all $v \in V(G)$. Thus, $i_{ns}(G \circ H) = |C| = |S^v||V(G)| \geq i(H)|V(G)|$. On the other hand, let $D$ be a minimum independent dominating set of $H$. For each $v \in V(G)$, let $S^v \subseteq V(H^v)$ such that $\langle S^v \rangle \cong \langle D \rangle$. Then $C = \bigcup_{v \in V(G)} S^v$ is an independent non-split dominating set of $G \circ H$ by Theorem 2.9. Hence, $i_{ns}(G \circ H) \leq |C| = |S^v||V(G)| = i(H)|V(G)|$. Therefore, $i_{ns}(G \circ H) = i(H)|V(G)|$. $\square$

Corollary 2.11 Let $G$ be a connected graph of order $m$. Then

(i) $i_{ns}(G \circ K_n) = m$.

(ii) $i_{ns}(G \circ \overline{K_n}) = mn$.

References


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