Best Proximity Points for Generalized Almost $\psi$-Geraghty Type Contractions in Partially Ordered Metric Spaces

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Abstract

The notion of generalized almost $\psi$-Geraghty type contraction non-self maps in partially ordered metric spaces is introduced, and some new best proximity point theorems for this class are established. Our results are generalizations of the result of Aydi et al. (Fixed Point Theory and Applications 2014:32) and some other authors.

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1 Introduction and preliminaries

Some of among popular topics in nonlinear analysis are best approximation theory and best proximity point theory. Best approximation theory study the existence of an approximate solution, whereas best proximity point theory analyze the existence of an approximate solution that is optimal.

The best proximity point evolves as a generalization of the concept of the best approximation. The authors ([2, 5, 7, 9, 15] and reference therein) obtained best proximity point theorems for certain contractions.

Especially, Caballero et al. [3] obtained a best proximity point theorem for Geraghty contraction non-self maps in complete metric spaces. Afterward, Karapinar [8] introduced the notion of $\psi$-Geraghty contraction non-self maps and gave sufficient conditions for the existence of a unique best proximity point for $\psi$-Geraghty contractions in complete metric spaces.

Aydi et al. [1] introduced the concept of generalized almost $\psi$-Geraghty contraction non-self maps and obtained best proximity point theorems for such maps in complete metric spaces.

We recall the following notations and definitions.

Let $(X, \preceq, d)$ be a partially ordered metric space, and let $A$ and $B$ be nonempty subsets of $X$.

We use the following notations:

\begin{align*}
    d(A, B) & := \inf \{d(x, y) : x \in A \text{ and } y \in B\}, \\
    A_0 & := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\
    B_0 & := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.
\end{align*}

Note that $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$, whenever $A \cap B \neq \emptyset$.

The pair $(A, B)$ said to have p-property [12] if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

\[d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \implies d(x_1, x_2) = d(y_1, y_2).\]

Note that if $A \neq \emptyset$, then $(A, A)$ has the p-property

Recently, Zhang et al. [16] gave the following concept, which is weaker than the p-property.

The pair $(A, B)$ said to have weak p-property if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

\[d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \implies d(x_1, x_2) \leq d(y_1, y_2).\]

Let $T : A \to B$ be a map.

The map $T$ is said to be proximally nondecreasing [4] if it satisfies the condition:

\[x \preceq y, d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B) \implies u \preceq v\]

for all $x, y, u, v \in A$. 
Note that if $A = B$, then $T$ reduces to nondecreasing map, i.e. $x \leq y$ implies $Tx \leq Ty$.

A point $x \in A$ is called best proximity point of the map $T$ if

$$d(x, Tx) = d(A, B).$$

$X$ is called regular if, for any sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$,

$$x_n \preceq x \text{ for all } n \in \mathbb{N}.$$

$X$ is called $C$-regular if, for any sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.

Note that if $X$ is regular then it is $C$-regular.

Very recently, Cho [4] introduced the concept of generalized almost Geraghty type contraction non-self maps and obtained corresponding best proximity point theorems in partially ordered complete metric spaces. He obtained the following theorems.

**Theorem 1.1.** Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $(A, B)$ be a pair of nonempty closed subsets of $X$ such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following conditions are satisfied:

1. $T(A_0) \subset B_0$;
2. the pair $(A, B)$ satisfies the weak p-property;
3. $T$ is generalized almost Geraghty type contraction, i.e. $T$ satisfies
   $$d(Tx, Ty) \leq \beta(M(x, y))(\max\{d(x, y), m(x, y) - d(A, B)\}) + L(n(x, y) - d(A, B))$$
   for all $x, y \in A$ with $x \preceq y$, where $L \geq 0$ and $\beta : [0, \infty) \to [0, 1)$ is a function such that $\lim_{n \to \infty} \beta(t_n) = 1$ implies $\lim_{n \to \infty} t_n = 0$;
4. there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;
5. $T$ is proximally nondecreasing;
6. $T$ is continuous.

Then, there exists $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$. Further more, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N}$, converges to the point $x_*$. 

Theorem 1.2. Let \((X, \preceq, d)\) be a partially ordered complete metric space, and let \((A, B)\) be a pair of nonempty closed subsets of \(X\) such that \(A_0 \neq \emptyset\). Let \(T : A \to B\) be a map. Suppose that the following conditions are satisfied:

1. \(T(A_0) \subset B_0\);
2. the pair \((A, B)\) satisfies the weak \(p\)-property;
3. \(T\) is generalized almost Geraghty type contraction;
4. there exist \(x_0, x_1 \in A_0\) such that \(x_0 \preceq x_1\) and \(d(x_1, Tx_0) = d(A, B)\);
5. \(T\) is proximally nondecreasing;
6. \(A\) is \(C\)-regular.

Then, there exists \(x_* \in A\) such that \(d(x_*, Tx_*) = d(A, B)\). Furthermore, the sequence \(\{x_n\}\), defined by \(d(x_{n+1}, Tx_n) = d(A, B)\) for all \(n \in \mathbb{N}\), converges to the point \(x_*\).

In this paper, we introduce the notion of generalized almost \(\psi\)-Geraghty type contraction non-self maps in partially ordered metric spaces and study corresponding best proximity point theorems, which are a generalization and extension of the result of [1] to the case of partially ordered metric spaces.

2 Best proximity points

We denote by \(\mathcal{F}\) the family of all functions \(\beta : [0, \infty) \to [0, 1)\) which satisfies the condition:

\[
\lim_{n \to \infty} \beta(t_n) = 1 \implies \lim_{n \to \infty} t_n = 0.
\]

For a metric space \((X, d)\) and a map \(T : A \to B\), where \(A\) and \(B\) are nonempty subsets of \(X\), we denote

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},
\]

\[
N(x, y) = \min\{d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\},
\]

\[
m(x, y) = \max\{d(x, Tx), d(y, Ty)\},
\]

\[
n(x, y) = \min\{d(x, Tx), d(y, Tx)\}
\]

for all \(x, y \in A\).

Note that \(M(x, y) \geq d(A, B)\), \(N(x, y) \geq d(A, B)\), \(m(x, y) \geq d(A, B)\) and \(n(x, y) \geq d(A, B)\).

Let \(\Psi\) denote the class of all functions \(\psi : [0, \infty) \to [0, \infty)\) satisfying the following conditions:
(1) $\psi$ is nondecreasing;

(2) $\psi$ is subadditive, i.e. $\psi(s + t) \leq \psi(s) + \psi(t)$;

(3) $\psi$ is continuous;

(4) $\psi(t) = 0$ if and only if $t = 0$.

Let $(X, \preceq, d)$ be a partially ordered metric space. Then, $d_\psi = \psi \circ d$ is a metric on $X$, and so $(X, \preceq, d_\psi)$ is also a partially ordered metric space.

For a metric space $(X, d)$ and a nonempty subset $Y$ of $X$, let $d_Y(x, y) = d(x, y)$ for all $x, y \in Y$.

**Lemma 2.1.** Let $(X, d)$ be a metric space, and let $A$ and $B$ be closed subsets of $X$. If $\psi \in \Psi$, then the following are satisfied:

1. $\{x_n\}$ is Cauchy in $(X, d)$ if and only if $\{x_n\}$ is Cauchy in $(X, d_\psi)$;

2. $(X, d)$ is complete if and only if $(X, d_\psi)$ is complete;

3. $Y$ is a closed subset of $(X, d)$ if and only if it is a closed subset of $(X, d_\psi)$;

4. $T : (A, d_A) \to (B, d_B)$ is continuous if and only if $T : (A, d_\psi_A) \to (B, d_\psi_B)$ is continuous.

**Lemma 2.2.** Let $(X, d)$ be a metric space, and let $A$ and $B$ be nonempty subsets of $X$. Let $\psi \in \Psi$, and let

$$A_0^{d_\psi} := \{x \in A : \psi(x, y) = \psi(A, B) \text{ for some } y \in B\},$$

$$B_0^{d_\psi} := \{y \in B : \psi(x, y) = \psi(A, B) \text{ for some } x \in A\}.$$

Then the following are satisfied:

1. $A_0 = A_0^{d_\psi}$, and $B_0 = B_0^{d_\psi}$;

2. if $(A, B)$ satisfies the $p$-weak property, then for any $x_1, x_2 \in A_0^{d_\psi}$ and $y_1, y_2 \in B_0^{d_\psi}$,

$$d_\psi(x_1, y_1) = \psi(A, B) \text{ and } d_\psi(x_2, y_2) = \psi(A, B) \text{ imply } d_\psi(x_1, x_2) \leq d_\psi(y_1, y_2);$$

3. if $A_0 \neq \emptyset$ and $T(A_0) \subset B_0$, then $A_0^{d_\psi} \neq \emptyset$ and $T(A_0^{d_\psi}) \subset B_0^{d_\psi}$.

4. if $T$ is proximally nondecreasing, then it is proximally nondecreasing with respect to $d_\psi$. 
Proof. Obviously, (1), (2) and (3) are satisfied.

We now show that (4) is satisfied.

Suppose that \( x \leq y, d_\psi(u, Tx) = d_\psi(A, B) \) and \( d_\psi(v, Ty) = d_\psi(A, B) \) for all \( x, y, u, v \in A \).

Then, \( d_\psi(u, Tx) \leq d_\psi(A, B) \) and \( d_\psi(v, Ty) \leq d_\psi(A, B) \). Since \( \psi \) is nondecreasing, \( d(u, Tx) \leq d(A, B) \) and \( d(v, Ty) \leq d(A, B) \).

So, we have \( x \leq y, d(u, Tx) = d(A, B) \) and \( d(v, Ty) = d(A, B) \).

Since \( T \) is proximally nondecreasing, \( u \leq v \). Thus, (4) is satisfied. \( \square \)

Let \((X, \preceq, d)\) be a partially ordered metric space, and let \(A \) and \(B \) be nonempty subsets of \(X \).

A map \( T : A \to B \) is called generalized almost \( \psi \)-Geraghty type contraction if there exists \( \beta \in \mathcal{F} \) such that

\[
\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(\max\{d(x,y), m(x,y) - d(A, B)\}) + L\psi(n(x,y) - d(A, B))
\]

for all \( x,y \in A \) with \( x \preceq y \), where \( L \geq 0 \).

**Theorem 2.1.** Let \((X, \preceq, d)\) be a partially ordered complete metric space, and let \((A, B)\) be a pair of nonempty closed subsets of \(X \) such that \( A_0 \neq \emptyset \). Let \( T : A \to B \) be a map. Suppose that the following are satisfied:

1. \( T(A_0) \subset B_0 \);
2. the pair \((A, B)\) satisfies the weak p-property;
3. \( T \) is generalized almost \( \psi \)-Geraghty type contraction;
4. there exist \( x_0, x_1 \in A_0 \) such that \( x_0 \preceq x_1 \) and \( d(x_1, Tx_0) = d(A, B) \);
5. \( T \) is proximally nondecreasing;
6. \( T \) is continuous.

Then, there exists \( x_* \in A \) such that \( d(x_*, Tx_*) = d(A, B) \). Further more, the sequence \( \{x_n\} \), defined by \( d(x_{n+1}, Tx_n) = d(A, B) \) for all \( n \in \mathbb{N} \), converges to the point \( x_* \).

Proof. By hypothesis (4) and by Lemma 3.2, there exist \( x_0, x_1 \in A_0^d \) such that \( x_0 \preceq x_1 \) and \( d_\psi(x_1, Tx_0) = d_\psi(A, B) \).

By applying Theorem 1.1 with Lemma 3.1 and 3.2, there exists \( x_* \in A \) such that \( d_\psi(x_*, Tx_*) = d_\psi(A, B) \). Since \( d_\psi(x_*, Tx_*) \leq d_\psi(A, B) \) and \( \psi \) is nondecreasing, \( d(x_*, Tx_*) \leq d(A, B) \), and hence \( d(x_*, Tx_*) = d(A, B) \).

Further more, the sequence \( \{x_n\} \), defined by \( d_\psi(x_{n+1}, Tx_n) = d_\psi(A, B) \) for all \( n \in \mathbb{N} \), converges to the point \( x_* \) in \( d_\psi \), i.e. \( \lim_{n \to \infty} d_\psi(x_*, x_n) = 0 \). It is easy to see that \( d(x_{n+1}, Tx_n) = d(A, B) \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} d(x_*, x_n) = 0 \). \( \square \)
**Theorem 2.2.** Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $(A, B)$ be a pair of nonempty closed subsets of $X$ such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following are satisfied:

1. $T(A_0) \subset B_0$;
2. the pair $(A, B)$ satisfies the weak $p$-property;
3. $T$ is generalized almost $\psi$-Geraghty type contraction;
4. there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;
5. $T$ is proximally nondecreasing;
6. $A$ is $C$-regular.

Then, there exists $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$. Further more, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N}$, converges to the point $x_*$. 

**Proof.** Since $A$ is $C$-regular, and noticing that $\lim_{n \to \infty} d(x, x_n) = 0$ if and only if $\lim_{n \to \infty} d_\psi(x, x_n) = 0$, we have that, for any sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} d_\psi(x, x_n) = 0$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.

From (4) $x_0, x_1 \in A_0^{d_\psi}$ such that $x_0 \preceq x_1$ and $d_\psi(x_1, Tx_0) = d_\psi(A, B)$.

By applying Theorem 1.2 with Lemma 3.1 and 3.2, there exists $x_* \in A$ such that $d_\psi(x_*, Tx_*) = d_\psi(A, B)$. As in the proof of Theorem 2.1, we have $d(x_*, Tx_*) = d(A, B)$ and the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N}$, converges to the point $x_*$. \hfill \Box

Let $T : A \to B$ be a map.

We denotes by $\text{Bes}(T, A, B)$ the set of all best proximity points of $T$.

For the uniqueness of a best proximity point of a generalized almost $\psi$-Geraghty type contraction, we consider the following hypothesis which is found in [4].

(C) For all $x, y \in \text{Bes}(T, A, B)$, there exists $z \in A$ such that

either $x \preceq z$ and $z \preceq y$ or $y \preceq z$ and $z \preceq x$.

**Theorem 2.3.** Adding condition (C) to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), we obtain that $x_*$ is the unique best proximity point of $T$. 
Proof. From Theorem 2.1 (resp. Theorem 2.2), we have a best proximity point, namely \( x_* \in A \). Let \( y_* \in X \) be another fixed point of \( T \) such that \( x_* \neq y_* \).

Then, \( d(x_*, y_*) > 0 \) and \( d(x_*, Tx_*) = d(A, B) = d(y_*, Ty_*) \).

Since \((A, B)\) satisfies weak \( p\)-property,
\[
d(x_*, y_*) \leq d(Tx_*, Ty_*).
\]

Then, by assumption, there exists \( z \in A \) such that
\[
x_* \leq z \text{ and } z \leq y_*.
\]
or
\[
y_* \leq z \text{ and } z \leq x_*.
\]

Assume that \( x_* \leq z \) and \( z \leq y_* \).
Then \( x_* \leq y_* \), and since \( T \) is generalized almost \( \psi \)-Geraghty type contraction, we obtain
\[
\begin{align*}
\psi(d(x_*, y_*)) & \leq \psi(d(Tx_*, Ty_*)) \\
& \leq \beta(\psi(M(x_*, y_*))) \psi(\max\{d(x_*, y_*), m(x_*, y_*) - d(A, B)\}) \\
& \quad + L \psi(n(x_*, y_*) - d(A, B)).
\end{align*}
\]

We deduce
\[
\begin{align*}
\max\{d(x_*, y_*), m(x_*, y_*) - d(A, B)\} & = \max\{d(x_*, y_*), \max\{d(x_*, Tx_*), d(y_*, Ty_*)\} - d(A, B)\} \\
& = \max\{d(x_*, y_*), 0\} \\
& = d(x_*, y_*) \\
\text{and} \\
0 & \leq n(x_*, y_*) - d(A, B) \\
& = \min\{d(x_*, Tx_*), d(y_*, Ty_*)\} - d(A, B) \\
& \leq \min\{d(A, B), d(x_*, y_*) + d(A, B)\} - d(A, B) \\
& = 0.
\end{align*}
\]

Thus, \( n(x_*, y_*) - d(A, B) = 0 \).
From (2.1) we obtain
\[
\begin{align*}
\psi(d(x_*, y_*)) & \leq \beta(\psi(M(x_*, y_*))) \psi(d(x_*, y_*)) \\
& < \psi(d(x_*, y_*)),
\end{align*}
\]
which is a contradiction. Therefore, \( x_* = y_* \). \( \square \)
Remark 2.1. We consider the following conditions:

1. there exists $\beta \in F$ such that, for all $x, y \in A$ with $x \preceq y$,
   $$\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(d(x, y));$$

2. there exists $\beta \in F$ such that, for all $x, y \in A$ with $x \preceq y$,
   $$\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(\max\{d(x, y), m(x, y) - d(A, B)\});$$

3. $T$ is generalized almost $\psi$-Geraghty type contraction;

4. there exists $\beta \in F$ such that, for all $x, y \in A$ with $x \preceq y$,
   $$\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y) - d(A, B));$$

5. there exists $\beta \in F$ such that, for all $x, y \in A$ with $x \preceq y$,
   $$\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y) - d(A, B)) + L\psi(N(x, y) - d(A, B)).$$

6. there exists $\beta \in F$ such that, for all $x, y \in A$ with $x \preceq y$,
   $$\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y) - d(A, B)) + L\psi(n(x, y) - d(A, B)).$$

Then, we have the following implications:

$(1) \Rightarrow (2) \Rightarrow (3)$, and $(4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (3)$.

Remark 2.2. If the condition (3) of Theorem 2.1 and Theorem 2.2 be replaced by the condition (5) of Remark 3.1, then we obtain an extension of Theorem 2.1 of [1] to the case of partially ordered metric spaces.

Corollary 2.4. Let $(X, \preceq, d)$ be a partially ordered complete metric space, and let $(A, B)$ be a pair of nonempty closed subsets of $X$ such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a map. Suppose that the following are satisfied:

1. $T(A_0) \subseteq B_0$;

2. the pair $(A, B)$ satisfies the weak $p$-property;

3. there exists $\beta \in F$ such that, for all $x, y \in A$ with $x \preceq y$,
   $$\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(d(x, y));$$

4. there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;

5. $T$ is proximally nondecreasing;
Corollary 2.5. Let \((X, \preceq, d)\) be a partially ordered complete metric space, and let \((A, B)\) be a pair of nonempty closed subsets of \(X\) such that \(A_0 \neq \emptyset\). Let \(T : A \rightarrow B\) be a map. Suppose that the following are satisfied:

1. \(T(A_0) \subset B_0\);
2. the pair \((A, B)\) satsfies the weak \(p\)-property;
3. there exists \(\beta \in F\) such that
   \[
   \psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))
   \]
   for all \(x, y \in X\) with \(x \preceq y\);
4. there exist \(x_0, x_1 \in A_0\) such that \(x_0 \preceq x_1\) and \(d(x_1, Tx_0) = d(A, B)\);
5. \(T\) is proximally nondecreasing;
6. either \(T\) is continuous or \(A\) is \(C\)-regular.

Then, there exists \(x_* \in A\) such that \(d(x_*, Tx_*) = d(A, B)\). Further more, the sequence \(\{x_n\}\), defined by \(d(x_{n+1}, Tx_n) = d(A, B)\) for all \(n \in \mathbb{N}\), converges to the point \(x_*\). Further if the condition \((C)\) is satisfied, then \(x_*\) is unique best proximity point of \(T\).

Remark 2.3. (1) Corollary 2.5 is a generalization of Theorem 2.1 of [8] to the case of partially ordered metric spaces.

(2) If \(A = B = X\) in Theorem 2.1, Theorem 2.2, then \(T\) has a fixed pont \(x_* \in X\), and \(\{Tx_0\}\) cnoverges to \(x_*\). Further if, for any \(x, y \in F(T)\), there exists \(z \in X\) such that \(x \preceq z\) and \(z \preceq y\) (or, \(y \preceq z\) and \(z \preceq x\)), then \(x_*\) is unique fixed point.

In Corollary 2.5, if we have \(A = B = X\), then we obtain the following corollary.

Corollary 2.6. Let \((X, \preceq, d)\) be a partially ordered complete metric space. Suppose that \(T : X \rightarrow X\) is a map. Assume that the following conditions are satisfied:
(1) there exists $\beta \in F$ such that
\[
\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))
\]
for all $x, y \in X$ with $x \preceq y$;

(2) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(3) $T$ is nondecreasing;

(4) either $T$ is continuous or $X$ is $C$-regular.

Then, $T$ has a fixed point $x_* \in X$, and $\{T^n x_0\}$ converges to $x_*$.

References


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