Synergetic Effects in Multiserver Queuing Systems with Alternating Input Flow

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Abstract

In this paper an aggregation of oneserver queueing systems with ON-OFF input flows into multiserver queueing system is considered. A synergetic effect of a queue disappearance in the aggregated system is analyzed using a convergence of an aggregated input flow to partial Brownian motion widely used in modern queueing theory applications.

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1 Multiserver queuing system with input flow defined by partial Brownian motion

Consider a scheme of series in which characteristics of a multiserver queuing system are defined by the parameter \( T \to \infty \), which characterizes a convergence to the infinity of an input flow intensity and a number of servers \( n = n(T) \). Denote \( e(t) = e_T(t) \), \( e(0) = 0 \), a number of customers arrived in the system at the moment \( t \) inclusive and \( Ee(t) = nat, \ t \geq 0 \), for some
a > 0. Assume that \( q(t) = q_T(t) \) is a number of busy servers in the system at the moment \( t \), \( q(0) = 0 \). Put \( \tau_j \) a service time of \( j \)-th customer of the input flow, \( \tau_j, j \geq 1 \), is a sequence of independent and identically distributed random variables (i.i.d.r.v’s) with the common distribution function \( F(t) (F = 1 - \bar{F}) \), which has continuous and bounded by \( f > 0 \) density.

**Theorem 1.** Assume that the following conditions are true.

1) There is the function \( B(n) \to \infty, T \to \infty \), such that the sequence of random processes \( x_T(t) = \frac{e(t) - E(e(t))}{B(n)}, T = 1,2,..., C \) - converges (see \( C \) - convergence definition in [1, chapter I, §2]) on \([0,t_0]\) to the partial Brownian motion \( \xi_H(t), 1/2 < H < 1 \), multiplied by \( \sigma \neq 0 \). 2) The inequality \( \rho = aE\tau_j < 1 \) is true. 3) The number of servers \( n = n(T) \) satisfies the conditions \( B(n)/A(n) \to B \geq 0, \sqrt{n}/A(n) \to K \geq 0, T \to \infty \), where \( A(n) = \max(B(n), \sqrt{n}) \). Then for any \( t_0 > 0 \) the relation \( P\left( \sup_{0 \leq t \leq t_0} q(t) = n \right) \to 0, T \to \infty \), is true.

Proof. From [1, ch. II, § 1, Theorem 1] the process \( z(t) = (q(t) - nQ(t))/A(n) \) with \( Q(t) = \int_0^t \bar{F}(u)du \) on any segment \([0,t_0]\) \( C \)- converges to the process \( \zeta(t) = \sigma \int_0^t \bar{F}(t - u)d\xi_H(u) + K\Theta(t), T \to \infty \). Here \( \Theta(t) \) is the centered gaussian process independent from \( \xi_H(t) \) with the covariation function \( R(t,t+u) = M\Theta(t)\Theta(t+u) = \int_0^t \bar{F}(v+u)F(v)adv \). For \( 0 \leq t \leq t+u \leq t_0, C = K^2a(2t_0^2+1)+\sigma^2\bar{F}^{-1}(f_{t_0}+2H) \) from [2, Lemma 1], [3, Formula (1.1)] we have \( \varepsilon^2(t, t+u) = E(\zeta(t) - \zeta(t+u))^2 = K^2[R(t,t) + R(t+u, t+u) - 2R(t, t+u)] + \sigma^2E \int_0^t \bar{F}(t - v)d\xi_H(u) - \int_0^{t+u} \bar{F}(t + u - v)d\xi_H(u) \leq uC \).

Consequently the minimal number \( N(r) \) of balls with a radius \( r \) in the metric space \([0,t_0], \varepsilon \) covering the segment \([0,t_0] \) (here \( \varepsilon(t, t + u) \) is the half metric on \([0,t_0] \) satisfies the inequality \( N(r) \leq t_0Cr^{-2} \) and so Dadly integral \( \Psi(z) = \int_0^r (ln N(r))^{1/2}dr \) built from the relative entropy \( ln N(r) \) satisfies the condition: \( \Psi(t_0) < \infty \). So from [4] we have that \( P(\sup_{0 \leq t \leq t_0} \zeta(t) > u) \to 0, u \to \infty \). As \( n/\sqrt{n} \to \infty \) for \( n \to \infty \) so \( P\left( \sup_{0 \leq t \leq t_0} \zeta(t) \geq \frac{(1 - Q(T))n}{\max(\sigma, 1)\sqrt{n}} \right) \to 0. \) From \( C \)- convergence of the random process \( z_n(t) \) to the random process \( \zeta(t) \) for \( n \to \infty \) it is not difficult to obtain that \( P\left( \sup_{0 \leq t \leq t_0} q_n(t) = n \right) = P\left( \sup_{0 \leq t \leq t_0} q_n(t) \geq n \right) \leq P\left( \sup_{0 \leq t \leq t_0} z_n(t) \geq \frac{(1 - Q(T))n}{\max(\sigma, 1)\sqrt{n}} \right) \to 0. \) Theorem 1.1 is proved.
2 Multiserver queuing system with alternating input flow

Assume that an input flow is defined by ON and OFF periods [5, 6]: nonnegative i.i.d.r.v's \( X_0, X_1, X_2, \ldots \) are lengths of ON-periods, nonnegative i.i.d.r.v's \( Y_0, Y_1, Y_2, \ldots \) are lengths of OFF-periods and these random sequences are independent. Denote \( F_1(t) = P(X_1 < t), F_2(t) = P(Y_1 < t), t \geq 0, \) and put \( F_1(t) = t^{-\alpha_1}L_1(t), F_2(t) = t^{-\alpha_2}L_2(t), 1 < \alpha_1 < \alpha_2 < 2, \) where \( L_1(t), L_2(t) \) are slowly varying for \( t \to \infty \) functions.

Introduce independent r.v's \( B, X, Y \) which are independent from \( X_n, Y_n, n \geq 1, \) and \( Y_0 \) with distributions \( P(B=1) = \frac{\mu_1}{\mu}, P(B=0) = \frac{\mu_2}{\mu}, \mu = \mu_1 + \mu_2, \mu_1 = EX_1, \mu_2 = EY_1, P(X \leq x) = \frac{1}{\mu_1} \int_0^x F_1(s)ds, P(Y \leq x) = \frac{1}{\mu_2} \int_0^x F_2(s)ds. \)

Then the random sequence \( \{T_n\} : T_0 = B(X + Y_0) + (1 - B)Y, T_0 = T_0 + \sum_{i=1}^{n}(X_i + Y_i), n \geq 1, \) creates ON-OFF process \( (I_A(x) \) is the indicator function of the set \( A) \)

\[ W(t) = BI_{[0, X)}(t) + \sum_{n=0}^{\infty} I_{[T_n, T_{n+1}, X_{n+1}]}(t), t \geq 0. \]

The process \( W(t) \) is binary: \( W(t) = 1, \) if \( t \) contains in ON-period, \( W(t) = 0, \) if \( t \) contains in OFF-period and stationary as \( EW(t) = P(W(t) = 1) = \mu_1/\mu = \alpha. \)

Assume further that r.v's \( X_0, X_1, \ldots \) have only integer meanings (but r.v. \( X \) is not integer). Confront to the random ON-OFF process \( W(t) \) the sampled random flow of customers arrival moments: for \( B = 1 - \{1, ..., [X], T_0 + 1, ..., T_0 + X_1, T_1 + 1, ..., T_1 + X_2, ... \} \) and for \( B = 0 - \{T_0 + 1, ..., T_0 + X_1, T_1 + 1, ..., T_1 + X_2, ... \}. \) Here \([c] \) is the integer part of the real number \( c. \) Denote by \( A^*(t) \) the number of customers in the sample flow arrived to the moment \( t \geq 0, A_1^*(t), \ldots, A_M^*(t) \) are independent copies of the random process \( A^*(t). \)

Consider now the aggregated queuing system with \( n(T) = TM \) servers where \( M(T) = [T^\gamma], \gamma > 0, \) and with \( A_M^*(T) \) customers arrived to the moment \( t \) and with \( q_T(t) \) busy servers at the moment \( t. \)

**Theorem 2.1** Assume that \( 3 - \alpha_1 > \gamma > \alpha_1 - 1, \rho = aE\tau_j < 1. \) Then for \( T \to \infty \) and for any \( t_0 > 0 \) the relation \( P(q_T(t) = n(T)) \to 0 \) is true.

**Proof.** Without a restriction of a generality put that \( L_1(T) \to l > 0, T \to \infty. \) Assume that the sequence \( W_m(t), m = 1, ..., M, \) of the process \( W(t) \) independent copies by which the processes \( A_1^*(t), \ldots, A_M^*(t) \) are constructed. Denote

\[ A_M(t) = \sum_{m=1}^{M} \int_0^t W_m(s)ds, d_T = [T^{3-\alpha_1}L_1(T)M]^{1/2}, \sigma^2 = \frac{2\mu_2^2\Gamma(2 - \alpha_1)}{(\alpha_1 - 1)\mu^3\Gamma(4 - \alpha_1)}, \]

\[ A_M(t) = \sum_{m=1}^{M} \int_0^t W_m(s)ds, d_T = [T^{3-\alpha_1}L_1(T)M]^{1/2}, \sigma^2 = \frac{2\mu_2^2\Gamma(2 - \alpha_1)}{(\alpha_1 - 1)\mu^3\Gamma(4 - \alpha_1)}, \]
b(t) is the function converse to the function $1/F_1(t)$. Simple geometric considerations give the inequalities

$$0 \leq \int_0^t W(s)ds - A(t) \leq 2$$

and consequently

$$0 \leq A_M(Tt) - \sum_{m=1}^M A_m^*(Tt) \leq 2M, \ t \geq 0. \quad (1)$$

It is obvious that $d_T \sim l^{1/2}T^{(3+\gamma-\alpha_1)/2}, \ b(T) \sim (IT)^{1/\alpha_1}, \ T \to \infty$, and consequently $b(MT)/T \to \infty, \ M = o(d_T), \ T \to \infty$. So [6, Theorem 4] and Formula (1) lead to the C - convergence of the random process $(A_M^*(Tt) - EA_M^*(Tt))/d_T$ to the partial Brownian motion $\sigma \xi_H(t), \ H = (3 - \alpha_1)/2, \ T \to \infty$. So the condition 1 of Theorem 2.1 is true.

Put $B(MT) = d_T, \ A(MT) = \max(B(MT), \sqrt{MT})$. As for $T \to \infty \sqrt{MT}/B(MT) \to 0$ so $B(MT)/A(MT) \to 1, \ \sqrt{MT}/A(MT) \to 0$. Then the conditions 2, 3 of Theorem 1.1 are true and so Theorem 2.1 is proved.

References


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