Generic Lightlike Submanifolds of an Indefinite Trans-Sasakian Manifold of a Quasi-Constant Curvature

Dae Ho Jin

Department of Mathematics, Dongguk University,
Gyeongju 780-714, Republic of Korea

Abstract

We study generic lightlike submanifolds $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ of a quasi-constant curvature. The main result is two classification theorems of such a generic lightlike submanifold subject such that the structure vector field $\zeta$ of $\bar{M}$ is tangent to $M$.

Mathematics Subject Classification: 53C25, 53C40, 53C50

Keywords: generic lightlike submanifold, indefinite trans-Sasakian manifold, quasi-constant curvature

1 Introduction

A lightlike submanifold $M$ of an indefinite almost contact manifold $\bar{M}$, equipped with an indefinite almost contact structure tensor $J$, is called a generic lightlike submanifold [7, 9, 11] if there exists a screen distribution $S(TM)$ such that

$$J(S(TM)^\perp) \subset S(TM),$$

where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in $T\bar{M}$. The geometry of generic lightlike submanifold is an extension of the geometry of lightlike hypersurface or half lightlike submanifold of codimension 2. Much of its theory
will be immediately generalized in a formal way to general lightlike submanifolds (i.e., r-lightlike and coisotropic submanifolds).

Oubina [12] introduced the notion of a trans-Sasakian manifold of type \((\alpha, \beta)\). In case, \(\tilde{M}\) is a semi-Riemannian manifold, we say that a trans-Sasakian manifold \(\tilde{M}\) of type \((\alpha, \beta)\) is an \textit{indefinite trans-Sasakian manifold}. Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are three important kinds of indefinite trans-Sasakian manifold such that

\[
\alpha = 1(\alpha = -1), \quad \beta = 0; \quad \alpha = 0, \quad \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}
\]

In the classical theory of Riemannian geometry, Chen-Yano [2] introduced the notion of a \textit{Riemannian manifold of a quasi-constant curvature} as a Riemannian manifold \((\tilde{M}, \tilde{g})\) endowed with the curvature tensor \(\tilde{R}\) satisfying

\[
\tilde{R}(X, Y)Z = f_1\{\tilde{g}(Y, Z)X - \tilde{g}(X, Y)\} + f_2\{\theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y + \tilde{g}(Y, Z)\theta(X)\zeta - \tilde{g}(X, Z)\theta(Y)\zeta\},
\]

for any vector fields \(X, Y\) and \(Z\) on \(\tilde{M}\), where \(f_1\) and \(f_2\) are smooth functions, \(\zeta\) is a unit vector field which is called the \textit{characteristic vector field} of \(\tilde{M}\), and \(\theta\) is a 1-form associated with \(\zeta\) by \(\theta(X) = \tilde{g}(X, \zeta)\). It is well known that if the curvature tensor \(\tilde{R}\) is of the form (1.2), then the manifold is conformally flat. If \(f_2 = 0\), then the manifold is reduced to a space of constant curvature.

In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold \(\tilde{M}\) of a quasi-constant curvature in which the 1-form \(\theta\) and the vector field \(\zeta\) defined by (1.2) are identical with the 1-form \(\theta\) and the vector field \(\zeta\) of the indefinite trans-Sasakian structure \((J, \zeta, \theta, \tilde{g})\) on \(\tilde{M}\). We prove two classification theorems of such a generic lightlike submanifold subject such that the structure vector field \(\zeta\) of \(\tilde{M}\) is tangent to \(M\).

2 Preliminaries

An odd-dimensional semi-Riemannian manifold \((\tilde{M}, \tilde{g})\) is called an \textit{indefinite trans-Sasakian manifold} [12] if there exist a structure set \(\{J, \zeta, \theta, \tilde{g}\}\) and two smooth functions \((\alpha, \beta)\), where \(J\) is a \((1, 1)\)-type tensor field, \(\zeta\) is a vector field which is called the \textit{structure vector field} and \(\theta\) is a 1-form such that

\[
\begin{align*}
J^2X &= -X + \theta(X)\zeta, \quad \theta(\zeta) = 1, \quad \theta(X) = \epsilon\tilde{g}(X, \zeta), \\
\theta \circ J &= 0, \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y) - \epsilon\theta(X)\theta(Y), \\
\langle \nabla_X JY \rangle &= \alpha\{\tilde{g}(X, Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\tilde{g}(JX, Y)\zeta - \epsilon\theta(Y)JX\}.
\end{align*}
\]

for any vector fields \(X\) and \(Y\) on \(\tilde{M}\), where \(\epsilon = 1\) or \(-1\) according as the vector field \(\zeta\) is spacelike or timelike respectively. In this case, the set \(\{J, \zeta, \theta, \tilde{g}\}\) is called an \textit{indefinite trans-Sasakian structure of type \((\alpha, \beta)\)}.
Throughout this paper, we may assume that the structure vector field $\zeta$ is unit spacelike, i.e., $\epsilon = 1$, no loss generality. From (2.1) and (2.2), we get
\[
\bar{\nabla}_X \zeta = -\alpha JX + \beta(X - \theta(X))\zeta, \quad d\theta(X, Y) = \bar{g}(X, JY).
\] (2.3)

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. We follow Duggal and Jin [5] for notations and structure equations used in this article. The radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$ of $M$ is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of rank $r$ ($1 \leq r \leq \min\{m, n\}$). In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $\text{Rad}(TM)$ in $TM$ and $TM^\perp$ respectively, which are called the screen distribution and co-screen distribution, such that
\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),
\]
where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$. Denote by $F(M)$ the algebra of smooth functions on $M$, by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$ and by (2.3), the $i$-th equation of (2.3). We use the same notations for any others. We use the following range of indices:
\[i, j, k, \ldots \in \{1, \ldots, r\}, \quad a, b, c, \ldots \in \{r + 1, \ldots, n\}.\]

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to $TM$ in $T\bar{M}_{\mid M}$ and $TM^\perp$ in $S(TM)^\perp$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{\mid tr}$, where $U$ is a coordinate neighborhood of $M$, such that
\[
\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,
\]
where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\text{Rad}(TM)_{\mid tr}$. Then we have
\[
TM = TM \oplus tr(TM) = \{\text{Rad}(TM) \oplus tr(TM)\} \oplus_{\text{orth}} S(TM)
\]
\[
= \{\text{Rad}(TM) \oplus ltr(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).
\]

We say that a lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of $\bar{M}$ is
1. $r$-lightlike submanifold if $1 \leq r < \min\{m, n\}$;
2. co-isotropic submanifold if $1 \leq r = n < m$;
3. isotropic submanifold if $1 \leq r = m < n$;
4. totally lightlike submanifold if $1 \leq r = m = n$.

The above three classes (2)$\sim$(4) are particular cases of the class (1) as follows: $S(TM^\perp) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^\perp) = \{0\}$, respectively. The geometry of $r$-lightlike submanifolds is more general form than that of the
other three type submanifolds. For this reason, we consider only $r$-lightlike submanifolds $M$, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{E_{r+1}, \ldots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^\bot)$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

In the following, let $X, Y, Z$ and $W$ be the vector fields on $M$, unless otherwise specified. Let $M$ be an $r$-lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a Levi-Civita connection $\bar{\nabla}$ and $P$ the projection morphism of $TM$ on $S(TM)$. Then the local Gauss-Weingarten formulas of $M$ and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^\ell(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \quad (2.4)$$
$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \quad (2.5)$$
$$\bar{\nabla}_XE_a = -A_{E_a}X + \sum_{i=1}^{r} \phi_{ai}(X)N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b, \quad (2.6)$$
$$\bar{\nabla}_XPY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^s(X,PY)\xi_i, \quad (2.7)$$
$$\bar{\nabla}_X\xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \quad (2.8)$$

where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$ respectively, $h_i^\ell$ and $h_a^s$ are called the local second fundamental forms on $TM$, $h_i^\ell$ are called the local screen second fundamental forms on $S(TM)$. $A_{N_i}, A_{E_a}$ and $A^*_{\xi_i}$ are linear operators on $TM$, which are called the shape operators, and $\tau_{ij}, \rho_{ia}, \phi_{ai}$ and $\sigma_{ab}$ are 1-forms on $TM$.

Since $\bar{\nabla}$ is torsion-free, $\nabla$ is also torsion-free, and both $h_i^\ell$ and $h_a^s$ are symmetric. From the fact that $h_i^\ell(X,Y) = \bar{g}(\nabla_X Y, \xi_i)$, we know that each $h_i^\ell$ is independent of the choice of $S(TM)$. The above three local second fundamental forms are related to their shape operators by

$$g(A^*_{\xi_i}X,Y) = h_i^\ell(X,Y) + \sum_{j=1}^{r} h_j^\ell(X,\xi_i)\eta_j(Y), \quad \bar{g}(A^*_{\xi_i}X,N_j) = 0, \quad (2.9)$$
$$g(A_{E_a}X,Y) = \epsilon_a h_a^s(X,Y) + \sum_{i=1}^{r} \phi_{ai}(X)\eta_i(Y), \quad (2.10)$$
$$\bar{g}(A_{E_a}X,N_i) = \epsilon_a \rho_{ia}(X), \quad \epsilon_b \sigma_{ab} = -\epsilon_a \sigma_{ba},$$

$$g(A_{N_i}X,PY) = h_i^s(X,PY), \quad \eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \quad (2.11)$$

$$g(A_{N_i}X,Y) = h_i^\ell(X,Y), \quad \eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \quad (2.12)$$
$$\bar{g}(A_{N_i}X,N_j) = 0, \quad (2.13)$$
$$g(A_{E_a}X,N_i) = \epsilon_a \rho_{ia}(X), \quad \epsilon_b \sigma_{ab} = -\epsilon_a \sigma_{ba},$$

$$g(A_{E_a}X,N_j) = \epsilon_a \rho_{ia}(X), \quad \epsilon_b \sigma_{ab} = -\epsilon_a \sigma_{ba},$$

$$g(A_{N_i}X,N_j) = 0, \quad (2.14)$$
$$\bar{g}(A_{N_i}X,N_j) = 0, \quad (2.15)$$

$$g(A_{N_i}X,N_j) = \epsilon_a \rho_{ia}(X), \quad \epsilon_b \sigma_{ab} = -\epsilon_a \sigma_{ba},$$
where $\eta_i$ are the 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i).$$

Replacing $Y$ by $\xi_j$ to (2.9)1, we have

$$h^\ell_i(X, \xi_j) + h^\ell_j(X, \xi_i) = 0, \quad h^\ell_i(X, \xi_i) = 0, \quad h^\ell_i(\xi_j, \xi_k) = 0. \quad (2.12)$$

For any $r$-lightlike submanifold, replacing $Y$ by $\xi_i$ to (2.10), we have

$$h^s_a(X, \xi_i) = -\epsilon_a \phi_{ai}(X). \quad (2.13)$$

Denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of the connection $\bar{\nabla}$, $\nabla$ and $\nabla^*$ on $\bar{M}$, $M$ and $S(TM)$ respectively. Using the Gauss-Weingarten equations, we obtain the Gauss equations for $M$ and $S(TM)$ such that

$$\bar{R}(X, Y)Z = R(X, Y)Z$$

$$+ \sum_{i=1}^{r} \{ h^\ell_i(X, Z)A_{N_i} Y - h^\ell_j(Y, Z)A_{N_j} X \} \quad (2.14)$$

$$+ \sum_{a=r+1}^{n} \{ h^s_a(X, Z)A_{E_a} Y - h^s_b(Y, Z)A_{E_b} X \}$$

$$+ \sum_{i=1}^{r} \{ (\nabla_X h^\ell_i)(Y, Z) - (\nabla_Y h^\ell_i)(X, Z) \} \quad \text{if } R = 0, \text{ we say that } M \text{ is flat.} \quad (2.15)$$

In the case $R = 0$, we say that $M$ is flat.
3 Indefinite trans-Sasakian manifolds

We shall assume that $\zeta$ is tangent to $M$. Călin [1] proved that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assume this paper. For a generic lightlike submanifold $M$, from (1.1) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. There exists a non-degenerate almost complex distribution $H_\circ$ with respect to $J$, i.e., $J(H_\circ) = H_\circ$, such that

$$S(TM) = \{ J(Rad(TM)) \oplus J(ltr(TM)) \} \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H_\circ.$$ 

Denote by $H$ the almost complex distribution with respect to $J$ such that

$$H = Rad(TM) \oplus_{\text{orth}} J(Rad(TM)) \oplus_{\text{orth}} H_\circ.$$ 

Therefore the general decomposition form of $TM$ in Section 2 is reduced to

$$TM = H \oplus J(ltr(TM)) \oplus_{\text{orth}} J(S(TM^\perp)). \quad (3.1)$$

Consider local null vector fields $U_i$ and $V_i$ for each $i$, local non-null unit vector fields $W_a$ for each $a$, and their 1-forms $u_i$, $v_i$ and $w_a$ defined by

$$U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_a = -JE_a, \quad (3.2)$$

$$u_i(X) = g(X,V_i), \quad v_i(X) = g(X,U_i), \quad w_a(X) = \epsilon_a g(X,W_a). \quad (3.3)$$

Denote by $S$ the projection morphism of $TM$ on $H$. Then, for any vector field $X$ on $M$, $JX$ is expressed as follow:

$$JX = FX + \sum_{i=1}^r u_i(X)N_i + \sum_{a=r+1}^n w_a(X)E_a, \quad (3.4)$$

where $F$ is a tensor field of type $(1,1)$ globally defined on $M$ by $F = J \circ S$.

Applying $\nabla_X$ to (3.2)~(3.4) by turns and using (2.2), (2.4)~(2.6), (2.8)~(2.11), (2.13) and (3.2)~(3.4), we have

$$h^j\ell(X,U_i) = h^i\ell(X,V_j), \quad \epsilon_a h^i\ell(X,W_a) = h^a\ell(X,U_i), \quad (3.5)$$

$$h^j\ell(X,V_i) = h^i\ell(X,V_j), \quad \epsilon_a h^i\ell(X,W_a) = h^a\ell(X,V_i),$$

$$\epsilon_b h^a\ell(X,W_b) = \epsilon_a h^a\ell(X,W_b),$$

$$\nabla_X U_i = F(A_{N_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a \quad (3.6)$$

$$- \{ \alpha_{i\ell}(X) + \beta v_i(X) \} \zeta,$$

$$\nabla_X V_i = F(A_{\xi_i}^*X) - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h^\ell\ell(X,\xi_i)U_j \quad (3.7)$$

$$- \sum_{a=r+1}^n \epsilon_a \phi_{ai}(X)W_a - \beta u_i(X) \zeta,$$
\[ \nabla_X W_a = F(A_{Ea} X) + \sum_{i=1}^{r} \phi_{ai}(X) U_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) W_b \] (3.8)
\[ - \epsilon_a \beta w_a(X) \zeta, \]
\[ (\nabla_X u_i)(Y) = - \sum_{j=1}^{r} u_j(Y) \tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y) \phi_{ai}(X) \] (3.9)
\[ - \beta \theta(Y) u_i(X) - h^i_\ell(X, FY), \]
\[ (\nabla_X v_i)(Y) = \sum_{j=1}^{r} v_j(Y) \tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y) \rho_{ia}(X) \] (3.10)
\[ - \sum_{j=r+1}^{n} u_j(Y) \eta_j(A_{N_i} X) - g(A_{N_i} X, FY) \]
\[ - \theta(Y) \{ \alpha \eta_h(X) + \beta v_i(X) \}, \]
\[ (\nabla_X F)(Y) = \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{Ea} X \] (3.11)
\[ - \sum_{i=1}^{r} h^i_\ell(X, Y) U_i - \sum_{a=r+1}^{n} h^a_\ell(X, Y) W_a \]
\[ + \alpha \{ g(X, Y) \zeta - \theta(Y) X \} + \beta \{ \bar{g}(JX, Y) \zeta - \theta(Y) FX \}. \]

Applying \( \bar{\nabla}_X \) to \( \bar{g}(\zeta, \zeta_i) = 0, \bar{g}(\zeta, E_a) = 0 \) and \( \bar{g}(\zeta, N_i) = 0 \) by turns and using (2.1), (2.3), (2.4) \sim (2.11), (3.2) and (3.3), we have
\[ h^i_\ell(X, \zeta) = -\alpha u_i(X), \quad h^a_\ell(X, \zeta) = -\alpha w_a(X), \]
\[ h^i_*(X, \zeta) = -\alpha v_i(X) + \beta \eta_h(X). \] (3.12)

Substituting (3.4) into (2.3) and using (2.4), we have
\[ \nabla_X \zeta = -\alpha FX + \beta (X - \theta(X) \zeta). \] (3.13)

We quote the following result for generic lightlike submanifold of an indefinite trans-Sasakian manifold \( M \) [10].

**Theorem 3.1.** Let \( M \) be a generic lightlike submanifold of an indefinite trans-Sasakian manifold \( \bar{M} \). Then the functions \( \alpha \) and \( \beta \) are satisfied
\[ \beta(\alpha - 1) = 0. \]

**4 Manifolds of quasi-constant curvatures**

**Theorem 4.1.** Let \( M \) be a generic lightlike submanifold of an indefinite trans-Sasakian manifold \( \bar{M} \) of a quasi-constant curvature. Then \( \bar{M} \) is an indefinite Sasakian manifold of a constant curvature such that
\[ \alpha \text{ is a constant, } \beta = 0, \quad f_1 = \alpha^2, \quad f_2 = 0. \]
Proof. Comparing the tangential, lightlike transversal and co-screen components of the two equations (1.2) and (2.14), and using (3.4), we get

\[ R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} \]
\[ + f_2\{\theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y \]
\[ + \bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta \]
\[ + \sum_{i=1}^{r}\{h_i^\ell(Y,Z)A_{N_i}X - h_i^\ell(X,Z)A_{N_i}Y\} \]
\[ + \sum_{a=r+1}^{n}\{h_a^s(Y,Z)A_{E_a}X - h_a^s(X,Z)A_{E_a}Y\}, \]

(\nabla_X h_i^\ell)(Y, Z) - (\nabla_Y h_i^\ell)(X, Z) \quad (4.2)
\[ + \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y, Z) - \tau_{ji}(Y)h_j^\ell(X, Z)\} \]
\[ + \sum_{a=r+1}^{n}\{\phi_{ai}(X)h_a^s(Y, Z) - \phi_{ai}(Y)h_a^s(X, Z)\} = 0, \]

(\nabla_X h_a^s)(Y, Z) - (\nabla_Y h_a^s)(X, Z) \quad (4.3)
\[ + \sum_{i=1}^{r}\{\rho_{ia}(X)h_i^r(Y, Z) - \rho_{ia}(Y)h_i^r(X, Z)\} \]
\[ + \sum_{b=r+1}^{n}\{\sigma_{ba}(X)h_b^r(Y, Z) - \sigma_{ba}(Y)h_b^r(X, Z)\} = 0. \]

Taking the scalar product with \( N \) to (2.15), we have
\[ g(R(X,Y)PZ, N_i) = (\nabla_X h_i^s)(Y, PZ) - (\nabla_Y h_i^s)(X, PZ) \]
\[ + \sum_{j=1}^{r}\{\tau_{ij}(Y)h_j^s(X, PZ) - \tau_{ij}(X)h_j^s(Y, PZ)\}. \]

Substituting (4.1) into the last equation and using (2.7)_2, we obtain

(\nabla_X h_i^\ell)(Y, PZ) - (\nabla_Y h_i^\ell)(X, PZ) \quad (4.4)
\[ + \sum_{j=1}^{r}\{\tau_{ij}(Y)h_j^s(X, PZ) - \tau_{ij}(X)h_j^s(Y, PZ)\} \]
\[ + \sum_{a=r+1}^{n}\{\rho_{ia}(X)h_a^r(X, PZ) - \rho_{ia}(Y)h_a^r(Y, PZ)\} \]
\[ + \sum_{j=1}^{r}\{h_j^\ell(X, PZ)\eta_j(A_{N_j} Y) - h_j^\ell(Y, PZ)\eta_j(A_{N_j} X)\} \]
\[ = f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \]
\[ + f_2\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ). \]
Applying $\nabla_Y$ to (3.5)\(_1\) and using (2.9), (2.11), (3.4) and (3.5), we obtain

\[
(\nabla_X h^k_i(Y, U_i)) = (\nabla_X h^k_i(Y, V_j)) + g(A_{N_j} Y, \nabla_X V_j) - g(A_{N_j} Y, \nabla_X U_i)
\]

\[
+ \sum_{k=1}^r h^k_i(X, U_k) h^k_k(Y, \xi_j).
\]

Substituting (3.6) and (3.7) into this and using (3.2)\~(3.5) and (3.12), we have

\[
(\nabla_X h^k_i(Y, U_i)) = (\nabla_X h^k_i(Y, V_j))
\]

\[- \sum_{k=1}^r \{\tau_{k j}(X) h^k_i(Y, U_i) + \tau_{i k}(X) h^k_k(Y, V_j)\}\]

\[- \sum_{a=r+1}^n \{\phi_{a j}(X) h^a_a(Y, U_i) + \epsilon_a \rho_{ia}(X) h^a_a(Y, V_j)\}\]

\[- g(A_{\xi_j}^* X, F(A_{N_j} Y)) - g(A_{\xi_j}^* Y, F(A_{N_j} X))\]

\[+ \sum_{k=1}^r \{h^k_i(Y, U_k) h^k_k(X, \xi_j) + h^k_i(Y, U_k) h^k_k(Y, \xi_j)\}\]

\[- \sum_{k=1}^r h^k_i(X, V_k) \eta_k(A_{N_j} Y) - \alpha^2 u_j(Y) \eta_i(X)\]

\[= (\alpha^2 - \beta^2) \{u_j(X) \eta_i(Y) - u_j(Y) \eta_i(X)\}\]

\[+ 2\alpha \beta \{u_j(X) v_i(Y) - u_j(Y) v_i(X)\} = 0.
\]

Comparing this and (4.4) with $PZ = V_j$ and using the facts that $h^i_k(X, V_j)$ are symmetric and $\eta_i(A_{N_j} X)$ are skew-symmetric with respect to $i$ and $j$ due to (3.5)\(_3\) and (2.11)\(_2\), we get

\[
\{f_1 - (\alpha^2 - \beta^2)\} [u_j(Y) \eta_i(X) - u_j(X) \eta_i(Y)]
\]

\[= 2\alpha \beta \{u_j(X) v_i(Y) - u_j(Y) v_i(X)\}.
\]

Taking $X = V_j$ and $Y = U_i$, and $X = \xi_j$ and $Y = U_j$ by turns, we have $\alpha \beta = 0$ and $f_1 = \alpha^2 - \beta^2$. As $\alpha \beta = 0$, by Theorem 3.1, we have $\beta = 0$. Thus

\[f_1 = \alpha^2; \quad \beta = 0.
\]
Applying $\nabla_X$ to $h_i^*(Y, \zeta) = -\alpha v_i(Y)$ and using (3.10) and (3.13), we have

$$(\nabla_X h_i^*)(Y, \zeta) = -(X\alpha)v_i(Y) + \alpha^2\theta(Y)\eta_i(X)$$

$$+ \alpha\{g(A_{Ni}X, FY) + g(A_{Ni}X, FX) - \sum_{j=1}^r v_j(Y)\tau_{ij}(X)$$

$$- \sum_{a=r+1}^n \epsilon_au_a(Y)\rho_{ia}(X) - \sum_{j=1}^r u_j(Y)\eta_i(A_{Ni}X)\}.$$ 

Substituting this and (3.12) into (4.4) with $PZ = \zeta$ and using $f_1 = \alpha^2$, we get

$$f_2\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\} = (X\alpha)v_i(Y) - (Y\alpha)v_i(X).$$

Taking $X = \zeta$ and $Y = \xi_i$, and taking $X = U_k$ and $Y = V_i$ by turns, we get

$$f_2 = 0, \quad U_i\alpha = 0, \quad \forall i.$$ 

Applying $\nabla_X$ to $h_i^*(Y, \zeta) = -\alpha u_i(Y)$ and using (3.9) and (3.13), we get

$$(\nabla_X h_i^*)(Y, \zeta) = -(X\alpha)u_i(Y) + \alpha\{\sum_{j=1}^r u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^n w_a(Y)\phi_{ai}(X)$$

$$+ h_i^*(X, FY) + h_i^*(Y, FX)\}.$$ 

Substituting this and (3.12) into (4.2) such that $Z = \zeta$, we obtain

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Replacing $Y$ by $U_i$ to this, we obtain $X\alpha = 0$. Thus $\alpha$ is a constant.

Denote by $H'$ the distribution on $S(TM)$ such that

$$H' = J(ltr(TM)) \oplus_{orth} J(S(TM^{-1})), \quad TM = H \oplus H'.$$

**Theorem 4.2.** Let $M$ be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ of a quasi-constant curvature. If one of the three kind objects $F$, $U_i$ and $V_i$ is parallel with respect to $\nabla$, then $\bar{M}$ is a flat manifold with an indefinite cosymplectic structure. In case $F$ is parallel with respect to $\nabla$, $H$ and $H'$ are parallel and $M$ is locally a product manifold $M_1 \times M_2$, where $M_1$ and $M_2$ are leaves of $H'$ and $H$ respectively.

**Proof.** (1) Taking the scalar product with $U_j$ to (3.11), we get

$$\sum_{i=1}^r u_i(Y)g(A_{Ni}X, U_j) + \sum_{a=r+1}^n w_a(Y)g(A_{fa}X, U_j) - \alpha\theta(Y)v_j(X) = 0,$$

due to $\beta = 0$. Replacing $Y$ by $\zeta$ to this equation, we have $\alpha = 0$. Therefore, $f_1 = 0$ and $M$ is a flat manifold with an indefinite cosymplectic structure.
Taking $Y = U_k$ and $Y = W_b$ to the last equation by turns, we obtain
\[ \bar{g}(A_{N_i} X, U_j) = 0, \quad \bar{g}(A_{E_a} X, U_j) = 0, \] (4.5)
as $\alpha = 0$. Replacing $Y$ by $\xi_j$ to (3.11) such that $\alpha = \beta = 0$, we get
\[ \sum_{i=1}^{r} h_i^l(X, \xi_j) U_i + \sum_{a=r+1}^{n} h_a^s(X, \xi_j) W_a = 0. \]
From this equation and (2.13), we obtain
\[ h_i^l(X, \xi_j) = 0, \quad \phi_{ai}(X) = h_a^s(X, \xi_i) = 0. \] (4.6)
Taking the scalar product with $N_j$ to (3.11) such that $\alpha = \beta = 0$, we have
\[ \sum_{i=1}^{r} u_i(Y) \bar{g}(A_{N_i} X, N_j) + \sum_{a=r+1}^{n} w_a(Y) \bar{g}(A_{E_a} X, N_j) = 0. \]
From this equation and (2.10), we obtain
\[ \bar{g}(A_{N_i} X, N_j) = 0, \quad \rho_{ia}(X) = \bar{g}(A_{E_a} X, N_i) = 0. \] (4.7)
Taking $Y \in \Gamma(H)$ and then, taking $Y = V_j$ to (3.11) by turns, we have
\[ \sum_{i=1}^{r} h_i^l(X, Y) U_i + \sum_{a=r+1}^{n} h_a^s(X, Y) W_a = 0, \]
\[ \sum_{i=1}^{r} h_i^l(X, V_j) U_i + \sum_{a=r+1}^{n} h_a^s(X, V_j) W_a = 0, \]
respectively. Taking the scalar product with $U_j$ and $W_b$ to these two equations by turns, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$, we have
\[ h_i^l(X, Y) = 0, \quad h_a^s(X, Y) = 0, \quad h_i^l(X, V_j) = 0, \quad h_a^s(X, V_j) = 0, \] (4.8)
respectively. Taking the scalar product with $Z \in \Gamma(H_o)$ to (3.11), we get
\[ \sum_{i=1}^{r} u_i(Y) h_i^s(X, Z) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y) h_a^s(X, Z) = 0. \]
Taking $Y = U_k$ to this equation, we have
\[ h_i^s(X, Y) = 0, \quad \forall X \in \Gamma(TM), \ Y \in \Gamma(H_o). \] (4.9)
By straightforward calculations from the decomposition (3.1) and by using (2.4), (3.5), (3.7), (3.8), (4.6), (4.7), (4.8) and $\phi_{ai} = \rho_{ia} = 0$, we derive
\[ g(\nabla_X \xi_i, V_j) = g(\nabla_X V_i, V_j) = g(\nabla_X Y, V_i) = 0, \]
\[ g(\nabla_X \xi_i, W_a) = g(\nabla_X V_i, W_a) = g(\nabla_X Y, W_a) = 0, \]
for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H_0)$, or equivalently, we get
\[ \nabla_X Y \in \Gamma(H), \quad \forall \, X \in \Gamma(TM), \quad \forall \, Y \in \Gamma(H). \]

This result implies that $H$ is a parallel distribution on $M$.

By using (3.4), (3.6), (4.5), (4.7), (4.8), (4.9) and $\rho_{ia} = 0$, we derive
\[
\begin{split}
g(\nabla_X U_j, N_j) &= g(\nabla_X U_i, U_j) = g(\nabla_X U_i, Y) = 0, \\
g(\nabla_X W_a, N_j) &= g(\nabla_X W_a, U_j) = g(\nabla_X W_a, Y) = 0,
\end{split}
\]
for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H_0)$, or equivalently, we get
\[ \nabla_X Z \in \Gamma(H'), \quad \forall \, X \in \Gamma(TM), \quad Z \in \Gamma(H'). \]

Thus $H'$ is also a parallel distribution of $M$.

As $TM = H \oplus H'$, and $H$ and $H'$ are parallel distributions, by the decomposition theorem of de Rham [3], $M$ is locally a product manifold $M_1 \times M_2$, where $M_1$ and $M_2$ are leaves of the distributions $H'$ and $H$ respectively.

(2) If $U_i$ is parallel with respect to $\nabla$ for some $i$, then we have
\[
\begin{split}
J(A_{\eta_i}, X) &= -\sum_{j=1}^r h^*_i(X, V_j)N_j - \sum_{a=r+1}^n h^*_i(X, W_a)E_a + \sum_{j=1}^r \tau_{ij}(X)U_j \\
&\quad + \sum_{a=r+1}^n \rho_{ia}(X)W_a - \alpha \eta_i(X)\zeta = 0.
\end{split}
\]

Taking the scalar product with $\zeta$, we have $\alpha \eta_i(X) = 0$. Thus $\alpha = 0$ and $\tilde{M}$ is a flat manifold with an indefinite cosymplectic structure.

(3) If $V_i$ is parallel with respect $\nabla$ for some $i$, then we have
\[
\begin{split}
J(A^*_i, X) &= -\sum_{j=1}^r u_j(A^*_i, X)N_j - \sum_{a=r+1}^n w_a(A^*_i, X)E_a - \sum_{j=1}^r \tau_{ji}(X)V_j \\
&\quad + \sum_{j=1}^r h^c_j(X, \xi_i)U_j - \sum_{a=r+1}^n \phi_{ai}(X)W_a = 0.
\end{split}
\]

Taking the scalar product with $U_k$, $V_k$ and $W_b$ by turns, we get $\tau_{ji} = 0$, $h^c_j(X, \xi_i) = 0$ and $\phi_{ai} = 0$, respectively. Applying $J$ to the last equation and using (2.1) and (3.12), we have
\[
A^*_i, X = -\alpha u_i(X)\zeta + \sum_{j=1}^r h^c_i(X, V_j)U_j + \sum_{a=r+1}^n h^c_i(X, W_a)W_a. \tag{4.11}
\]

Taking the scalar product with $U_k$, we get $h^c_i(X, U_j) = 0$. Taking $X = U_i$ to (3.12) and using the last equation, we get $-\alpha = -\alpha u_i(U_i) = h^c_i(U_i, \zeta) = 0$. Thus $\tilde{M}$ is a flat manifold with an indefinite cosymplectic structure.
References


[10] D. H. Jin, Generic lightlike submanifolds of an indefinite trans-Sasakian manifold, submitted in Mathematical Problems in Engineering,


Received: March 9, 2015; Published: April 12, 2015