On JB-Semigroups

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Abstract

In this paper, we introduce the notion of JB-semigroup. We prove that every ring determines a JB-semigroup, but the converse need not be true. We also introduce the notions of JB-field and JB-domain, and we prove that every JB-field is a JB-domain and every finite JB-domain is a JB-field. Moreover, we introduce the notion of JB-ideal of JB-semigroup, and we construct quotient JB-semigroup via JB-ideal. Furthermore, we introduce the notion of JB-homomorphism of JB-semigroups, and we provide some of its properties.

Mathematics Subject Classification: 06F35, 03G25

Keywords: JB-semigroup, JB-field, JB-domain, JB-ideal, quotient JB-semigroup, JB-homomorphism

1 Introduction

2-ary operation to BCK-algebra. In view of this process, we introduce the notion of JB-semigroup by adding the concept of semigroup to B-algebra. In this paper, we prove that every ring determines a JB-semigroup, but the converse need not be true. Furthermore, we introduce the notions of JB-field and JB-domain, and we prove that every JB-field is a JB-domain and every finite JB-domain is a JB-field. We also introduce the notion of JB-ideal of JB-semigroup, and we construct quotient JB-semigroup via JB-ideal. Furthermore, we introduce the notion of JB-homomorphism of JB-semigroups, and we provide some of its properties.

2 JB-semigroups and Rings

**Definition 2.1** [7] A B-algebra is an algebra \((X; \ast, 0)\) of type \((2, 0)\) satisfying the following axioms for all \(x, y, z \in X\):

(I) \(x \ast x = 0\),

(II) \(x \ast 0 = x\),

(III) \((x \ast y) \ast z = x \ast (z \ast (0 \ast y))\).

In [7], a B-algebra \((X; \ast, 0)\) is called *commutative* if \(a \ast (0 \ast b) = b \ast (0 \ast a)\) for all \(a, b \in X\). A nonempty subset \(N\) of a B-algebra \((X; \ast, 0)\) is called a *subalgebra* of \(X\) if \(x \ast y \in N\) for any \(x, y \in N\). It is called *normal* in \(X\) if for any \(x \ast y, a \ast b \in N\) implies \((x \ast a) \ast (y \ast b) \in N\). A normal subset of \(X\) is a subalgebra of \(X\). Moreover, in [2], the intersection of any nonempty collection of (normal) subalgebras of \(X\) is also a (normal) subalgebra of \(X\).

**Definition 2.2** A JB-semigroup is a nonempty set \(X\) together with two binary operations \(\ast\) and \(\cdot\) and a constant 0 satisfying the following:

i. \((X; \ast, 0)\) is a B-algebra;

ii. \((X, \cdot)\) is a semigroup; and

iii. The operation \(\cdot\) is left and right distributive over the operation \(\ast\).

Definition 2.2(i) implies that if \((X; \ast, \cdot, 0)\) is a JB-semigroup, then all properties pertaining to the binary operation \(\ast\) with respect to the B-algebra \((X; \ast, 0)\) also hold for the JB-semigroup \((X; \ast, \cdot)\). In particular, the following properties hold in a JB-semigroup: (P1) \(0 \ast (0 \ast x) = x [7]\), (P2) \(x \ast y = 0\) implies \(x = y [7]\), and (P3) \(x \ast y = 0 \ast (y \ast x) [8]\).

The following are examples of JB-semigroups.
Example 2.3 Let $X = \{0, a, b, c\}$ be a set with the following tables:

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Then by routine calculations, $(X; *, \cdot, 0)$ is a JB-semigroup.

Example 2.4 Let $X = \{0, a, b, c\}$ be a set with the following tables:

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Then by routine calculations, $(X; *, \cdot, 0)$ is a JB-semigroup.

Example 2.5 Let $X = \{0, a, b, c\}$ be a set with the following tables:

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Then by routine calculations, $(X; *, \cdot, 0)$ is a JB-semigroup.

Example 2.6 By routine calculations, $(\mathbb{Z}; *, \cdot, 0)$, $(\mathbb{Q}; *, \cdot, 0)$, $(\mathbb{R}; *, \cdot, 0)$, and $(\mathbb{C}; *, \cdot, 0)$ are JB-semigroups, where $x \ast y = x - y$ and $\cdot$ is the usual multiplication.

Lemma 2.7 Let $(X; *, \cdot, 0)$ be a JB-semigroup and $a, b, c \in X$. Then the following properties hold:

i. $a \cdot 0 = 0 \cdot a = 0$,

ii. $a \cdot (0 \ast b) = (0 \ast a) \cdot b = 0 \ast (a \cdot b)$,

iii. $(0 \ast a) \cdot (0 \ast b) = a \cdot b$,

iv. $a \cdot (b \ast (0 \ast c)) = (a \cdot b) \ast (0 \ast (a \cdot c))$, $(b \ast (0 \ast c)) \cdot a = (b \cdot a) \ast (0 \ast (c \cdot a))$. 
Proof: Let \( a, b, c \in X \).

i. Now, \( a \cdot 0 = a \cdot (0 \ast 0) = (a \cdot 0) \ast (a \cdot 0) = 0. \) Similarly, \( 0 \cdot a = 0. \)

ii. By (i), \( a \cdot (0 \ast b) = (a \cdot 0) \ast (a \cdot b) = 0 \ast (a \cdot b) = (0 \cdot b) \ast (a \cdot b) = (0 \ast a) \cdot b. \)

iii. By (i), (ii), and (P1), \((0 \ast a) \cdot (0 \ast b) = 0 \ast ((a \cdot 0) \ast (a \cdot b)) = 0 \ast (0 \ast (a \cdot b)) = a \cdot b. \)

iv. By (ii), \( a \cdot (b \ast (0 \ast c)) = (a \cdot b) \ast (a \cdot (0 \ast c)) = (a \cdot b) \ast (0 \ast (a \cdot c)). \) Similarly, \((b \ast (0 \ast c)) \cdot a = (b \cdot a) \ast (0 \ast (c \cdot a)). \)

To discuss the relationship of JB-semigroups and rings, we recall first the relationship of B-algebras and groups. Now, every group determines a B-algebra ([1], [7]), and if the given group is commutative, then the group is commutative [1], as shown in the following theorem.

**Theorem 2.8** If \((X; \circ, e)\) is a group, then \((X; \ast, 0 = e)\) is a B-algebra, where \( x \ast y = x \circ y^{-1}. \) Moreover, if \((X; \circ, e)\) is commutative, then \((X; \ast, 0 = e)\) is commutative.

This means that every group can be transformed into a B-algebra. It is then a question of interest to determine whether or not all B-algebras can be transformed into groups. The answer is affirmative, proved by M. Kondo and Y.B. Jun [5], and if the given B-algebra is commutative, then the group is commutative [1], as shown in the following theorem.

**Theorem 2.9** If \((X; \ast, 0)\) is a B-algebra, then \((X; \circ, e = 0)\) is a group, where \( x \circ y = x \ast (0 \ast y). \) Moreover, if \((X; \ast, 0)\) is commutative, then \((X; \circ, e = 0)\) is commutative.

Combining Theorem 2.8 and Theorem 2.9, the class of B-algebras and the class of groups coincide, in some sense. With regards to JB-semigroups and rings, we prove that every ring determines a JB-semigroup.

**Theorem 2.10** If \((X; +, \cdot, e)\) is a ring, then \((X; \ast, 0 = e)\) is a JB-semigroup, where \( x \ast y = x - y. \)

**Proof:** Let \((X; +, \cdot, e)\) be a ring. By Theorem 2.8, \((X; \ast, 0 = e)\) is a B-algebra. Clearly, \((X, \cdot)\) is a semigroup since \( \cdot \) is associative. If \( x, y, z \in X, \) then \( x \cdot (y \ast z) = x \cdot (y - z) = (x \cdot y) - (x \cdot z) = (x \cdot y) \ast (x \cdot z). \) Similarly, \( (x \ast y) \cdot z = (x \cdot z) \ast (y \cdot z). \) Therefore, \((X; \ast, 0 = e)\) is a JB-semigroup. \(\square\)

Let \((X; \ast, 0)\) be a JB-semigroup. If we define \( x + y = x \ast (0 \ast y), \) then \((X; +, \cdot, e = 0)\) need not be a ring. Consider the JB-semigroup in Example 2.4, \((X; +, \cdot, e = 0)\) is not a ring since \( a + b = a \cdot (0 \ast b) = a \neq b = b \cdot (0 \ast a) = b + a. \) This means that there exists a JB-semigroup that cannot be transformed into a ring, with respect to the predefined transformation. But if we restrict the B-algebra \((X; \ast, 0)\) to be commutative, then we get the following theorem.
Theorem 2.11 If \((X; *, \cdot, 0)\) is a JB-semigroup such that \((X; *, 0)\) is commutative, then \((X; +, \cdot, e = 0)\) is a ring, where \(x + y = x * (0 * y)\).

Proof: Let \((X; *, \cdot, 0)\) be a JB-semigroup such that \((X; *, 0)\) is commutative. By Theorem 2.9, \((X; +, e = 0)\) is a commutative group. Since \((X, \cdot)\) is a semigroup, \(\cdot\) is associative. Let \(x, y, z \in X\). Then by Lemma 2.7(ii), we have

\[
x \cdot (y + z) = x \cdot (y * (0 * z)) = (x \cdot y) * (x \cdot (0 * z)) = (x \cdot y) * (0 * (x \cdot z)) = x \cdot y + x \cdot z.
\]

Similarly, \((x + y) \cdot z = x \cdot z + y \cdot z\). Therefore, \((X; +, \cdot, e = 0)\) is a ring. \(\square\)

3 JB-field and JB-domain

Throughout this section, \(X\) means a JB-semigroup \((X; *, \cdot, 0)\).

Definition 3.1 A nonempty subset \(S\) of \(X\) is called a sub JB-semigroup of \(X\) if \(x \cdot y, x \cdot y \in S\) for all \(x, y \in S\).

Clearly, 0 is an element of a sub JB-semigroup. Moreover, \(\{0\}\) and \(X\) are sub JB-semigroups of \(X\). The set \(S_1 = \{0, a\}\) is a sub JB-semigroup of the JB-semigroup in Example 2.3, while the set \(S_2 = \{0, a, b\}\) is not since \(a * b = c \notin S_2\).

Definition 3.2 If \(a \cdot b = b \cdot a\) for all \(a, b \in X\), then \(X\) is called commutative. If \(X\) is not commutative, then it is called noncommutative.

Definition 3.2 means that \(X\) is commutative if and only if the semigroup \((X, \cdot)\) is commutative. The JB-semigroup in Example 2.3 is commutative, while the JB-semigroup in Example 2.5 is noncommutative.

Definition 3.3 The subset \(C[X] = \{a \in X: a \cdot b = b \cdot a\} for all b \in X\) of \(X\) is called the center of \(X\).

Remark 3.4 \(X\) is commutative if and only if \(X = C[X]\).

Definition 3.5 An element \(y \in X\) is called a unity in \(X\) if \(x \cdot y = x = y \cdot x\) for all \(x \in X\).

The JB-semigroup in Example 2.3 has unity \(a\), while JB-semigroup in Example 2.5 has no unity. The unity of a JB-semigroup is unique (if it exists), and is denoted by 1.

Definition 3.6 Let \(1 \in X\). An element \(a \in X\) is called 1-invertible if there exists \(b \in X\) such that \(a \cdot b = 1 = b \cdot a\).
In Example 2.3, all elements of $X$ not equal to 0 are 1-invertibles, that is, elements $a$, $b$, and $c$ are 1-invertibles.

**Lemma 3.7** Let $1 \in X$. Then the following statements hold:

i. $(0 \ast 1) \cdot a = 0 \ast a = a \cdot (0 \ast 1)$ for all $a \in X$,

ii. $(0 \ast 1) \cdot (0 \ast 1) = 1$,

iii. if $a$ is 1-invertible in $X$, then $0 \ast a$ is 1-invertible in $X$.

**Proof:** Let $a \in X$. Then $(0 \ast 1) \cdot a = (0 \cdot a) \ast (1 \cdot a) = 0 \ast a$. Similarly, $a \cdot (0 \ast 1) = 0 \ast a$. This proves (i). By Lemma 2.7(iii), $(0 \ast 1) \cdot (0 \ast 1) = 1 \cdot 1 = 1$. This proves (ii). Suppose $a$ is 1-invertible in $X$. Then there exists $b \in X$ such that $a \cdot b = 1 = b \cdot a$. Now, $0 \ast b \in X$ with $(0 \ast b) \cdot (0 \ast a) = b \ast a = 1 = (a \cdot b) = (0 \ast a) \cdot (0 \ast b)$. Thus, $0 \ast a$ is 1-invertible in $X$. This proves (iii). □

**Theorem 3.8** Let $1 \in X$ and $T$ be the set of all 1-invertible elements of $X$. Then $T \neq \emptyset$, $0 \notin T$, and $a \cdot b \in T$ for all $a$, $b \in T$.

**Proof:** Let $T$ be the set of all 1-invertible elements of $X$. Since $1 \cdot 1 = 1 = 1 \cdot 1$, $1 \in T$. Thus, $T \neq \emptyset$. Suppose that $0 \in T$. Then there exists $b \in X$ such that $0 \cdot b = 1 = b \cdot 0$. But $0 \cdot b = 0$ and so $0 = 1$, a contradiction. Thus, $0 \notin T$. Let $a$, $b \in T$. Then there exist $c$, $d \in X$ such that $a \cdot c = 1 = c \cdot a$ and $b \cdot d = 1 = d \cdot b$. Now, $(a \cdot b) \cdot (d \cdot c) = a \cdot (b \cdot d) \cdot c = a \cdot (1) \cdot c = a \cdot c = 1$ and $(d \cdot c) \cdot (a \cdot b) = d \cdot (c \cdot a) \cdot b = d \cdot (1) \cdot b = d \cdot b = 1$. Thus, $a \cdot b \in T$. □

**Definition 3.9** Let $1 \in X$. Then $X$ is called a JB-field if the semigroup $(X, \cdot)$ is commutative and every $0 \neq a \in X$ is 1-invertible.

The JB-semigroup in Example 2.3 is a JB-field.

**Definition 3.10** An element $0 \neq a \in X$ is called 0-divisor if there exists $b \in X$ such that $b \neq 0$ and either $a \cdot b = 0$ or $b \cdot a = 0$.

**Remark 3.11** An element cannot be a 1-invertible and 0-divisor at the same time. Thus, a JB-field has no 0-divisors.

**Theorem 3.12** If $X$ has no 0-divisors, then left and right cancellation laws hold, that is, for all $a$, $b$, $c \in X$, $a \neq 0$, $a \cdot b = a \cdot c$ implies $b = c$ (left cancellation) and $b \cdot a = c \cdot a$ implies $b = c$ (right cancellation). If either left or right cancellation law holds, then $X$ has no 0-divisors.
Proof: Let \( a, b, c \in X \) such that \( a \cdot b = a \cdot c \) and \( a \neq 0 \). Then \( a \cdot (b * c) = (a \cdot b) * (a \cdot c) = 0 \). Since \( X \) has no 0-divisors and \( a \neq 0 \), we have \( b * c = 0 \). By (P2), \( b = c \). Hence, the left cancellation law holds. Similarly, the right cancellation law holds. Conversely, suppose one of the cancellation laws hold, say, the left cancellation. Let \( a \) be a nonzero element of \( X \) and \( b \in X \). Suppose \( a \cdot b = 0 \). Then \( a \cdot b = a \cdot 0 \) and so by left cancellation, \( b = 0 \). Suppose \( b \cdot a = 0 \) and \( b \neq 0 \). Then \( b \cdot a = b \cdot 0 \) and so by left cancellation, \( a = 0 \), a contradiction. Therefore, \( b = 0 \). Hence, \( X \) has no 0-divisors. Similarly, the right cancellation law implies that \( X \) has no 0-divisors. □

Definition 3.13 Let 1 \( X \). Then \( X \) is called a JB-domain if the semi-

The JB-semigroup in Example 2.3 is a JB-domain.

Remark 3.14 Every JB-field is a JB-domain.

The converse of Remark 3.14 need not be true. The JB-semigroup \((Z; *, \cdot, 0)\) in Example 2.6 is a JB-domain, but not a JB-field.

Theorem 3.15 A finite commutative JB-semigroup \( X \) with more than one

Proof: Let \( a_1, a_2, \ldots, a_n \) be the distinct elements of \( X \). Let \( a \in X \) with \( a \neq 0 \).

Corollary 3.16 Every finite JB-domain is a JB-field.

4 JB-ideals and JB-homomorphisms

Definition 4.1 A nonempty subset \( I \) of \( X \) is called a JB-ideal of \( X \) if the following hold:

i. \((x * a) * (y * b) \in I \) for any \( x * y, a * b \in I \),
ii. \( a \cdot x, x \cdot a \in I \) for any \( a \in I, x \in X \).

Definition 4.1(i) means that \( I \) is a normal subalgebra of the B-algebra \((X; *, 0)\). Obviously, the subsets \( \{0\} \) and \( X \) are JB-ideals of \( X \). These JB-ideals are called trivial JB-ideals. All other JB-ideals are called nontrivial JB-ideals. Consider the JB-semigroup \( X \) in Example 2.4, the set \( I = \{0, b\} \) is a JB-ideal of \( X \), while the set \( J = \{0, a\} \) is not since \( c \cdot a = c \notin J \).

**Remark 4.2** Let \( I \) be a JB-ideal of \( X \). Then \( I \) is a sub JB-semigroup of \( X \) and \( I \) is a JB-ideal for every sub JB-semigroup of \( X \) containing \( I \).

**Theorem 4.3** Let \( \{I_\alpha : \alpha \in \mathcal{A}\} \) be a nonempty collection of JB-ideals of \( X \). Then \( \bigcap_{\alpha \in \mathcal{A}} I_\alpha \) is a JB-ideal of \( X \).

Let \( \emptyset \neq A, B \subseteq X \). Define \( A \ast B \) as the set \( \{a \ast (0 \ast b) : a \in A, b \in B\} \).

In [2], if \( A \) is a subalgebra of a B-algebra \((X; *, 0)\), then \( A \ast A = A \). Moreover, if \( A \) and \( B \) are normal subalgebras of \( X \), then \( A \ast B = B \ast A \) is a normal subalgebra of \( X \).

**Theorem 4.4** Let \( A, B, \) and \( C \) be JB-ideals of \( X \). Then the following properties hold:

i. \( A \ast B = B \ast A \) is a JB-ideal of \( X \),

ii. \( A \ast A = A \),

iii. \( (A \ast B) \ast C = A \ast (B \ast C) \).

**Proof:**  
i. Let \( y \in A \ast B \) and \( x \in X \). Then \( y = a \ast (0 \ast b) \) for some \( a \in A, b \in B \). Since \( A \) and \( B \) are JB-ideals, \( a \cdot x \in A \) and \( b \cdot x \in B \). Hence, by Lemma 2.7(ii), we have \( y \cdot x = (a \ast (0 \ast b)) \cdot x = (a \cdot x) \ast ((0 \ast b) \cdot x) = (a \cdot x) \ast (0 \ast (b \cdot x)) \in A \ast B \). Similarly, \( x \cdot y \in A \ast B \). Therefore, \( A \ast B = B \ast A \) is a JB-ideal of \( X \).

iii. By (III), (P1), and (P3), we have \( x \in (A \ast B) \ast C \iff x = (a \ast (0 \ast b)) \ast (0 \ast c) \iff x = a \ast ((0 \ast c) \ast (0 \ast b)) \iff x = a \ast ((0 \ast c) \ast b) \iff x = a \ast [0 \ast (b \ast (0 \ast c))] \iff x \in A \ast (B \ast C) \). Hence, \( (A \ast B) \ast C = A \ast (B \ast C) \). \( \square \)

In [7], if \( N \) is a normal subalgebra of a B-algebra \((X; *, 0)\), then we have a B-algebra \((X/N; *, [0]_N)\), where \( X/N = \{[x]_N : x \in X\} \) and \( * \) is defined by \([x]_N \ast [y]_N = [x \ast y]_N \). For \( x \in X \), \([x]_N \) is the equivalence class containing \( x \), that is, \([x]_N = \{y \in X : x \sim_N y\} \), where \( x \sim_N y \) if and only if \( x \ast y \in N \) for any \( x, y \in X \). The algebra \( X/N \) is called the quotient B-algebra of \( X \) by \( N \).
Let $I$ be a JB-ideal of $X$. Then $I$ is a normal subalgebra of the B-algebra $(X;\ast,0)$ and $(X/I;\ast,[0]_I)$ is a B-algebra. Now, define $\cdot$ on $X/I$ by $[x]_I \cdot [y]_I = [x \cdot y]_I$. The binary operation $\cdot$ on $X/I$ is well-defined. To see this, let $[x]_I = [x']_I$ and $[y]_I = [y']_I$. Then $x \cdot x' \in I$ and $y \cdot y' \in I$. Since $I$ is a JB-ideal, $(x \cdot y) \ast (x' \cdot y') = x \cdot (y \ast y') \in I$ and so $x \cdot y \sim_I x \cdot y'$. Also, $(x \cdot y') \ast (x' \cdot y') = (x \ast x') \cdot y' \in I$ and so $x \cdot y' \sim_I x' \cdot y'$. Thus, $x \cdot y \sim_I x' \cdot y'$. Hence, $[x]_I \cdot [y]_I = [x \cdot y]_I = [x']_I \cdot [y']_I$.

**Theorem 4.5** Let $I$ be a JB-ideal of $X$. Then $(X/I;\ast,[0]_I)$ is a JB-semigroup, where $\ast$ and $\cdot$ defined as above. If $X$ is commutative or has a unity, then the same is true of $X/I$.

The JB-semigroup $X/I$ in Theorem 4.5 is called *quotient JB-semigroup* of $X$ by $I$.

Let $(X;\ast,\cdot,0_X)$ and $(Y;\ast,\cdot,0_Y)$ be JB-semigroups. A map $\varphi : X \rightarrow Y$ is called a **JB-homomorphism** from $X$ into $Y$ if $\varphi(x \ast y) = \varphi(x) \ast \varphi(y)$ and $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ for any $x, y \in X$. A JB-homomorphism $\varphi$ is called a **JB-monomorphism**, **JB-epimorphism**, or **JB-isomorphism** if $\varphi$ is one-to-one, onto, or a bijection, respectively. A JB-homomorphism $\varphi : X \rightarrow X$ is called a **JB-endomorphism** and a JB-isomorphism $\varphi : X \rightarrow X$ is called a **JB-automorphism**.

**Remark 4.6** Let $\varphi : X \rightarrow Y$ be a JB-homomorphism. Then $\varphi$ maps $0_X$ to $0_Y$, that is, $\varphi(0_X) = 0_Y$, and for all $x \in X$, $\varphi(0_X \ast x) = 0_Y \ast \varphi(x)$.

**Lemma 4.7** If $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ are JB-homomorphisms, then $\psi \circ \varphi : X \rightarrow Z$ is also a JB-homomorphism.

Assuming compatibility of functions so that composition is defined, the following corollary easily follows.

**Corollary 4.8** The composition of JB-monomorphisms is a JB-monomorphism, the composition of JB-epimorphisms is a JB-epimorphism, the composition of JB-isomorphisms is a JB-isomorphism, and the composition of JB-automorphisms is a JB-automorphism.

Let $\varphi : X \rightarrow Y$ be a B-homomorphism, that is, $\varphi(x \ast y) = \varphi(x) \ast \varphi(y)$. In [2], if $J$ is a normal subalgebra of $Y$, then $\varphi^{-1}(J)$ is a normal subalgebra of $X$. If $I$ is a normal subalgebra of $X$ and $\varphi$ is onto, then $\varphi(I)$ is a normal subalgebra of $Y$.

**Lemma 4.9** Let $\varphi : X \rightarrow Y$ be a JB-homomorphism.
i. If $I$ is a sub JB-semigroup of $X$, then $\varphi(I)$ is a sub JB-semigroup of $Y$. Moreover, if $I$ is commutative, then $\varphi(I)$ is commutative.

ii. If $J$ is a sub JB-semigroup of $Y$, then $\varphi^{-1}(J)$ is a sub JB-semigroup of $X$ containing $\ker \varphi$.

iii. If $I$ is a JB-ideal of $X$ and $\varphi$ is onto, then $\varphi(I)$ is a JB-ideal of $Y$.

iv. If $J$ is a JB-ideal of $Y$, then $\varphi^{-1}(J)$ is a JB-ideal of $X$.

Proof: Let $\varphi: X \to Y$ be a JB-homomorphism.

iii. Let $a \in \varphi(I)$ and $y \in Y$. Since $\varphi$ is onto, there exists $b \in I$ such that $\varphi(b) = a$. Since $\varphi$ is onto, there exists $x \in X$ such that $\varphi(x) = y$. Since $I$ is a JB-ideal of $X$, we have $b \cdot x, x \cdot b \in I$. Thus, $a \cdot y = \varphi(b) \cdot \varphi(x) = \varphi(b \cdot x) \in \varphi(I)$ and $y \cdot a = \varphi(x) \cdot \varphi(b) = \varphi(x \cdot b) \in \varphi(I)$. Therefore, $\varphi(I)$ is a JB-ideal of $Y$.

iv. Let $a \in \varphi^{-1}(J)$ and $x \in X$. Then $\varphi(a) \in J$ and $\varphi(x) \in Y$. Since $J$ is a JB-ideal of $Y$, $\varphi(a \cdot x) = \varphi(a) \cdot \varphi(x) \in J$ and $\varphi(x \cdot a) = \varphi(x) \cdot \varphi(a) \in J$. Therefore, $a \cdot x, x \cdot a \in \varphi^{-1}(J)$ and so $\varphi^{-1}(J)$ is a JB-ideal of $X$. \hfill \Box

Definition 4.10 Let $\varphi: X \to Y$ be a JB-homomorphism from $(X; *, \cdot, 0_X)$ into $(Y; *, \cdot, 0_Y)$. We define $\ker \varphi$ to be the set $\ker \varphi = \{ x \in X : \varphi(x) = 0_Y \}$.

In [6], if $\varphi : X \to Y$ is a B-homomorphism, then $\ker \varphi$ is a normal subalgebra of $X$, and $\varphi$ is one-to-one if and only if $\ker \varphi = \{ 0_X \}$.

Lemma 4.11 Let $\varphi : X \to Y$ be a JB-homomorphism. Then

i. $\varphi$ is one-to-one if and only if $\ker \varphi = \{ 0_X \}$.

ii. $\ker \varphi$ is a JB-ideal of $X$.

Proof: ii. Let $a \in \ker \varphi$ and $x \in X$. Then $\varphi(a) = 0_Y$. Hence, by Lemma 2.7(i), $\varphi(a \cdot x) = \varphi(a) \cdot \varphi(x) = 0_Y \cdot \varphi(x) = 0_Y$ and $\varphi(x \cdot a) = \varphi(x) \cdot \varphi(a) = \varphi(x) \cdot 0_Y = 0_Y$. Therefore, $a \cdot x, x \cdot a \in \ker \varphi$ and so $\ker \varphi$ is a JB-ideal of $X$. \hfill \Box

Proposition 4.12 Let $I$ be a JB-ideal of $X$. Then the map $\gamma: X \to X/I$, given by $\gamma(x) = [x]_I$, is a JB-epimorphism, and $\ker \gamma = I$.

Proof: For all $x, y \in X$, $\gamma(x \ast y) = [x \ast y]_I = [x]_I \ast [y]_I = \gamma(x) \ast \gamma(y)$ and $\gamma(x \ast y) = [x \ast y]_I = [x]_I \cdot [y]_I = \gamma(x) \cdot \gamma(y)$. Hence, $\gamma$ is a JB-homomorphism. Since $\gamma$ is obviously onto, $\gamma$ is a JB-epimorphism. Furthermore, $\ker \gamma = \{ x \in X : [x]_I = I \} = \{ x \in X : x \in I \} = I$. \hfill \Box

The mapping $\gamma$ in Proposition 4.12 is called the natural (or canonical) JB-homomorphism of $X$ onto $X/I$.

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References


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