Optimal Homotopy Asymptotic Method
for Solving Gardner Equation

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Abstract

The Gardner equation is widely used in various areas of physics, such as plasma physics, fluid dynamics, and others. In this article, the optimal homotopy asymptotic method is applied to obtain approximate analytical solution for Gardner equation. Comparisons are made between the exact solution. This method provides us a convenient way to control the convergence of approximation series. The results reveal that our approximate solutions are very useful for problems with large domain, very effective, and easy to use. The optimal homotopy asymptotic method can be used to get fast convergent series solutions of different types of nonlinear problem with strong nonlinearity.

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1 Introduction

In nonlinear sciences, many important phenomena in various fields can be described by the nonlinear evolution equations, such as the Gardner equation. This equation is frequently used to model a variety of physical phenomena, such as the internal waves in a stratified ocean [2], ion acoustic waves in plasma
with negative ion [12], and wave propagation in a one-dimensional nonlinear lattice [13]. The Gardner equation was first derived rigorously within the asymptotic theory for long internal waves in a two-layer fluid with a density jump at the interface [10]. For some restricted initial and boundary conditions, exact analytical solutions for the Gardner equation were computed [11], [3]. However in most cases it is not possible to find such exact analytical solution and usually numerical methods are used. Various approximation methods were also employed to find approximate analytical solutions for Gardner equation, such as the homotopy analysis method [1], the discrete variational method [8], and the restrictive Taylor approximation [9].

Marinca et al. [7] introduced a new method named as optimal homotopy asymptotic method (OHAM). They applied this method for approximate solution of nonlinear equations arising in heat transfer. A considerable amount of research works have been invested recently in applying this method to a wide class of nonlinear problems [5]. The method has proved to be powerful and effective where it demonstrates fast convergence of the solution and therefore provides several significant advantages. The method was successfully applied to a large amount of applications in nonlinear sciences.

The aim of this paper is to apply optimal homotopy asymptotic method to get an approximate analytical solution of the Gardner equation. This work is divided in five sections. In the next section, the key idea of our method is described. In Section 3, the proposed method is applied to the Gardner equation. In Section 4, the numerical solution of Gardner equation is obtained and comparison is made using graph. Conclusions of this paper are presented in Section 5.

## 2 Description of the method

Let us consider a form of the nonlinear differential equation

\[ N(u(x)) = 0, \quad x \in \Omega, \]  

(1)

where \( N \) is a nonlinear operator and \( u(x) \) is an unknown function. Liao [6] in 1997 introduced such a nonzero auxiliary parameter \( h \) to construct the zeroth-order deformation equation

\[ (1 - p)L(\phi(x, p) - u_0(x)) = phN(\phi(x, p)), \]  

(2)

where \( L \) is an auxiliary linear operator, \( \phi(x, p) \) is an unknown function, \( u_0(x) \) is an initial guess, and \( p \in [0, 1] \) is an embedding parameter. In 2008, Marinca et al. [7] combined \( h \) and \( p \) in the zeroth-order deformation equation (2) as one function \( H(p) \) with \( H(0) = 0 \) and considered such a family of equations.
(1 - p)L(\phi(x, p) - u_0(x)) = H(p) N(\phi(x, p)). \quad (3)

Obviously, when \( p = 0 \) or \( p = 1 \), it holds that \( \phi(x, 0) = u_0(x) \) and \( \phi(x, 1) = u(x) \), respectively. Next, we choose the auxiliary function \( H(p) \) in the form

\[ H(p) = c_1p + c_2p^2 + c_3p^3 + \cdots, \]

where \( c_1, c_2, \cdots \) are convergence control parameters which can be determined later. When \( c_2 = c_3 = \cdots = 0 \), equation (3) becomes (2) which is used in the homotopy analysis method (HAM)[4]. Next, to get an approximate solution, we expand \( \phi(x, p, c_j) \) in Taylor series about \( p \) in the following manner

\[ \phi(x, p, c_j) = u_0(x) + \sum_{k=1}^{\infty} \phi_k(x, c_j) p^k, \quad j = 1, 2, \ldots. \quad (4) \]

By substituting (4) into (3) and then equating the coefficient of like powers of \( p \), we obtain the high-order deformation equation

\[ L(\phi_1) = c_1 N(u_0) \]

\[ L(\phi_k) = L(\phi_{k-1}) + c_k N(u_0) + \sum_{j=1}^{k-1} c_j N_j(u_0, \phi_1, \phi_2, \ldots, \phi_{k-j}) \quad (5) \]

with \( k = 2, 3, \ldots \). Convergence of the series (4) depends upon the auxiliary constant \( c_j \) \( (j = 1, 2, \ldots) \) and if it converges at \( p = 1 \), then the following form applies

\[ \phi(x, 1, c_j) = u_0(x) + \sum_{k=1}^{\infty} \phi_k(x, c_j). \quad (6) \]

Substituting equation (6) into equation (1) yields the following residual

\[ R(x, c_j) = N(\phi(x, 1, c_j)). \]

If \( R(x, c_j) = 0 \), then \( \phi(x, 1, c_j) \) will be the exact solution. The least squares method can be used to determine \( c_j \) \( (j = 1, 2, \ldots) \). We consider the functional

\[ J(c_j) = \int_{\alpha}^{\beta} (R(x, c_j))^2 dx, \]

where \( \alpha \) and \( \beta \) are the endpoints of the given problem. The unknown convergence control parameters \( c_j \) \( (j = 1, 2, \ldots, m) \) can be calculated from the system of equations.
\[
\frac{\partial J}{\partial c_j} = 0, \ j = 1, 2, \cdots, m.
\]

It should be noted that we included the auxiliary function \(H(p)\) which provides us an easy way to set an optimal control the convergent area of the solution series.

## 3 Application of method

In this section, we apply OHAM to find approximate analytical solutions for the Gardner equation. Let us consider the Gardner equation

\[
\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} + \gamma u^2 \frac{\partial u}{\partial x} + \delta \frac{\partial^3 u}{\partial x^3} = 0 \quad (7)
\]

which is written in dimensionless form. In equation (7), \(u = u(x, t)\) is a function of the two independent variables \(x\) and \(t\), which are the direction of wave propagation and time, respectively. In most applications \(u = u(x, t)\) represents the amplitude of the relevant wave mode (e.g., it may represent the vertical displacement of the pycnocline). The coefficients of the nonlinear terms \(\mu\) and \(\gamma\) and the dispersive term \(\delta\) are determined by the steady oceanic background density and flow stratification through the linear eigenmode (vertical structure function) of the internal waves. Making a transformation \(u(x, t) = \eta(\zeta)\), with \(\zeta = x - ct, c > 0\), equation (7) can be reduced to the following ordinary differential equation

\[
-cd\eta d\zeta + \mu\eta d\eta d\zeta + \gamma\eta^2 d\eta d\zeta + \delta d^3\eta d\zeta^3 = 0, \quad (8)
\]

where \(c\) is wave velocity which moves along the direction of \(x\) axis. Exact soliton solution of the equation (8) was given by equation

\[
\eta(\zeta) = \frac{2A}{-B + \sqrt{B^2 - 4AE \cosh(\sqrt{A}\zeta)}}, \quad (9)
\]

where

\[
A = \frac{c}{\delta}, \quad B = \frac{\mu}{3\delta}, \quad E = \frac{\gamma}{6\delta}.
\]

Suppose

\[
\eta(\zeta) \sim D \exp(-\lambda\zeta), \ \text{as} \ \zeta \to \infty, \quad (10)
\]

where \(\lambda > 0\) and \(D\) is a constant. Substituting equation (10) into equation (8) and balancing the main terms, we have \(\lambda = \sqrt{A}\). Define \(\xi = \lambda\zeta\), equation (8)
Optimal homotopy asymptotic method

becomes

\[-c \frac{d\eta}{d\xi} + \mu \eta \frac{d\eta}{d\xi} + \gamma \eta^2 \frac{d\eta}{d\xi} + c \frac{d^3\eta}{d\xi^3} = 0.\] (11)

Assuming the nondimensional wave elevation \( \eta \) arrives its maximum at the origin, we have the boundary condition as follows

\[\eta(0) = 1, \text{ and } \eta(\infty) = 0.\] (12)

Next, the solution of equation (11) can be expressed by the base functions

\[\{\exp(-n\xi) \mid n = 1, 2, \ldots\}.\]

In the form

\[\eta(\xi) = \sum_{n=1}^{\infty} b_n \exp(-n\xi),\]

where \( b_n \) is a coefficient to be determined. From equation (11), we define the nonlinear operator \( N \) as

\[N(\phi) = -c \frac{d\phi}{d\xi} + \mu \phi \frac{d\phi}{d\xi} + \gamma \phi^2 \frac{d\phi}{d\xi} + c \frac{d^3\phi}{d\xi^3}\] (13)

and the linear operator \( L \) can be defined as below

\[L(\phi) = \frac{d^3\phi}{d\xi^3} - \frac{d\phi}{d\xi}\] (14)

which possesses the property \( L(A_1 \exp(-\xi) + A_2 \exp(\xi) + A_3) = 0 \), where \( A_1, A_2 \) and \( A_3 \) are integration coefficients. Using the above definition, the solution of equation (5) is

\[\phi_m(\xi) = A_1 e^{-\xi} + A_2 e^{\xi} + A_3 + \widetilde{\phi}_m(\xi),\] (15)

where \( \widetilde{\phi}_m(\xi) \) is a special solution of equation (5). The integral coefficients of equation (15)

\[A_2 = A_3 = 0, \quad A_1 = -\widetilde{\phi}_m(0)\] (16)

are determined by boundary conditions (12) and equation (10).

4 Result and Discussion

The sign of the coefficient \( \mu \) of the quadratic nonlinear term dictates the polarity of such solitons. Suppose given the following data: \( \mu = 1, \gamma = -1, \delta = 1, \)
and \( c = 1 \). According to the equation (10) and the boundary condition (12), the initial guess is chosen as

\[
u_0(\xi) = \phi_0(\xi) = \exp(-\xi).
\]

(17)

Now, we use equation (5) and equation (13) to obtain a series problems. The first-order deformation is given as

\[
L(\phi_1) = c_1 N(\phi_0)
\]

and has the solution

\[
\phi_1(\xi, c_1) = -0.1250 \ e^{-\xi} \ c_1 + 0.1667 \ e^{-2\xi} \ c_1 - 0.04167 \ e^{-3\xi} \ c_1.
\]

The second-order deformation is given as

\[
L(\phi_2) = L(\phi_1) + c_2 N(\phi_0) + c_1 [-c \frac{d\phi_1}{d\xi} + \mu \frac{d}{d\xi} (\phi_0 \phi_1) \\
\gamma \left( \frac{\phi_0^2 \frac{d\phi_1}{d\xi}}{d\xi} + 2\phi_0 \phi_1 \frac{d\phi_0}{d\xi} \right) + c \frac{d^3 \phi_1}{d\xi^3}\]

and has the solution

\[
\phi_2(\xi, c_1, c_2) = (0.1250 \ c_1 - 0.1250 \ c_2 - 0.1076 \ c_1^2) \ e^{-\xi} \\
+ (0.1667 \ c_2 + 0.1250 \ c_2^2 + 0.1667 \ c_1) \ e^{-2\xi} \\
+ (-0.04167 \ c_1 - 0.04167 \ c_2 - 0.00521 \ c_1^2) \ e^{-3\xi} \\
- 0.013889 \ c_1^2 \ e^{-4\xi} + 0.00173 \ c_1^2 \ e^{-5\xi}.
\]

The second-order approximate solution by optimal homotopy asymptotic method is

\[
\phi(\xi, c_1, c_2) = \phi_0(\xi) + \phi_1(\xi, c_1) + \phi_2(\xi, c_1, c_2).
\]

By using the proposed method of Section 2 on \([0, \infty)\), we use the residual error

\[
R(\xi, c_1, c_2) = N(\phi(\xi, c_1, c_2)) = -c \frac{d\phi}{d\xi} + \mu \frac{d\phi}{d\xi} + \gamma \phi^2 \frac{d\phi}{d\xi} + c \frac{d^3 \phi}{d\xi^3}.
\]

(18)

The less square error can be formed as

\[
J(c_1, c_2) = \int_0^\infty (R(\xi, c_1, c_2))^2 \ d\xi,
\]

\[
\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = 0.
\]
Thus, the following optimal values of $c_j$‘s are obtained

\[ c_1 = 1.025956055, \quad c_2 = -4.102004686. \]

This fact is obvious from the use of the auxiliary function $H(p)$. In OHAM, it is important to solve a set of nonlinear algebraic equations with $m$ unknown convergence control parameters $c_j$ and this makes it time consuming, especially for large $m$. In this case, our approximate solution is

\[
\eta(\xi) = 1.1429624028 e^{-\xi} - 0.2101088673 e^{-2\xi} + 0.07993830614 e^{-3\xi} - 0.01461924760 e^{-4\xi} + 0.001827405950 e^{-5\xi}.
\]

(19)

In the first-term OHAM, we get $c_1 = -0.9634566897$.

Figure 1. Comparison between the approximate solution (OHAM and HAM) and the exact solution (9). The OHAM has an accelerated convergence compared to the HAM. From Figure 1, we observe that the results agree very well with the exact solution; as we increase the order of the problem the accuracy increases and the residual error will decrease as shown in Figure 2 and Figure 3. The residual error in equation (18) for second-term OHAM and first-term OHM is plotted in Figure 2 and Figure
3, respectively. We noted that the absolute maximum error via second-order OHAM is $6.56 \times 10^{-4}$.

![Graph](image)

Figure 2. Residual error $R$ given by (18) using the second-term OHAM approximate solution.

![Graph](image)

Figure 3. Residual error $R$ given by (18) using the first-term OHAM approximate solution.
5 Conclusions

In this paper, optimal homotopy analysis method (OHAM) was applied to the Gardner equation easily. The OHAM provides us a simple way to optimally control and adjust the convergence solution series and it gives a good approximation in few terms which is converged to the exact solution and proved the efficiency and reliability of the method. The method is a powerful one since the approximations computed consist of fewer terms than the previous solutions. This method proved to be an accurate and efficient technique for finding approximate solutions for the Gardner equation. The OHAM has an accelerated convergence compared to the HAM, fact proved by the included applications.

References


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