On Partial Stability of

Retarded Functional Differential Systems

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Abstract

For nonlinear nonstationary retarded functional differential systems the problem of stability with respect to a part of the variables of a “partial” equilibrium position is considered. The conditions of stability of this type are obtained in the context of the method of Lyapunov-Razumikhin functions.

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1 Introduction

The classical definition of stability with respect to a part of variables of the zero equilibrium position of the system of ordinary differential equations [1] assumes the domain of initial perturbations to be a sufficiently small neighborhood of the zero equilibrium position. Along with this statement, the cases of arbitrary or large (belonging to an arbitrary compact set) initial perturbations for a part of variables that are non-controlled when studying stability are investigated; see [2, 3]. More general situation then initial perturbations can be large with respect to one part of non-controlled variables and arbitrary with respect to their other part is also considered [4]. Such problems are closely connected to stability problems of “partial” zero equilibrium positions of the system of ordinary differential equations [3, 4]. On the other hand, for stability problem of “partial” zero equilibrium positions also naturally assume [4] that initial perturbations of variables that do not define the given equilibrium position can be large with respect to one part of the variables and arbitrary with respect to their other part.
Contrary the assumptions that initial perturbations of this variables are are either only arbitrary or only large the combined assumption made it possible an admissible trade-off between the meaning sense for notion of stability and the respective requirements on the Lyapunov functions.

In this article the problem of stability with respect to a part of the variables of the “partial” equilibrium position is considered for nonlinear retarded systems of functional differential equations. The Razumikhin approach is used. The method of Lyapunov-Krasovskii functionals for solving of this problem was used in [5].

2 Statement of the Problem

We assume \( \tau > 0 \) is a given real number, \( R^n \) is a linear real space of \( n \)-dimensional vectors \( x \) with the norm \( |x| = \max |x_i| \) (where \( i \) is the component of the vector \( x \)), \( C \) is the Banach space of continuous functions \( \phi: [-\tau, 0] \rightarrow R^n \) with standard supremum-norm \( ||\phi|| = \sup |\phi(\theta)| \) (\( \theta \in [-\tau, 0] \)), and \( R_+ = [0, +\infty) \). If \( t_0, \beta \in R_+, \beta > t_0 \), then for a continuous function \( x: [t_0-\tau, \beta] \rightarrow R^n \) we define a function \( x_\tau \in C \) by the relation \( x_\tau = x(t + \theta) (\theta \in [-\tau, 0]) \); in what follows, \( x_\tau(t) \) denotes the right-hand derivative.

We introduce the partition \( x = (y^T, z^T)^T \) (T denotes transposition), where \( y \in R^m, z \in R^{n-m} \) (1 \( \leq m \leq n \)). According to this partition, we set \( C = C^y \times C^z \), and \( C^\tau \) is the Banach space of continuous functions \( \phi_{\tau}: [-\tau, 0] \rightarrow R^n \) with the norms \( ||\phi_{\tau}|| = \sup |\phi_{\tau}(\theta)| \) and \( ||\phi_{\tau}|| = \sup |\phi_{\tau}(\theta)| \) (\( \theta \in [-\tau, 0] \)). For \( \phi \in C \), we have \( \phi = (\phi^1, \phi^2)^T \) and \( ||\phi|| = \max(||\phi^1||, ||\phi^2||) \).

Let there be given a system of differential equations with holdover [6]

\[
\begin{align*}
x'(t) &= X(t, x_\tau), \\
y'(t) &= Y(t, y, z_\tau), \\
z'(t) &= Z(t, y, z_\tau).
\end{align*}
\]

(1)

In the space \( C \), we consider the set \( M = \{ \phi \in C: \phi_\tau = 0 \} \). If \( Y[t, \phi] \equiv 0 \) for \( \phi \in M \), then the solution \( x(t_0, \phi) \) of system (1) satisfies the condition \( |y(t_0, \phi)| \equiv 0 \). In other words, \( M = \{ x: y = 0 \} \) is a “partial” equilibrium position of system (1). In this case, system (1) does not necessarily have a zero equilibrium position \( x = 0 \).

To consider the problem with respect to a part of the variables of the “partial” equilibrium position \( y = 0 \), we assume that \( y = (y_1^T, y_2^T)^T \). Then, we also represent the component \( \phi_y \) of the vector function \( \phi \) as \( \phi_y = (\phi_y^1, \phi_y^2)^T \).

We assume that the operator \( X: R_+ \times C \rightarrow R^n \) determining the right-hand side of system (1) is completely continuous in the domain

\[
G = R_+ \times C^1 \times C^2 \times C^\tau = \{ t \geq 0, \, ||\phi_{y_1}|| < h, \, ||\phi_{y_2}|| + ||\phi_{\tau}|| < \infty \},
\]

(2)

We also assume that the Cauchy-Lipschitz condition is satisfied on each compact subset \( K \) in the domain (2). Then [6], for each point \( t_0, \phi \) in the domain (2) there is
a unique solution \(x(t_0, \varphi)\) of system (1) which can be continued to the boundary of the domain \(C^1 \times C^2 \times C^2\) and continuously depends on \(t_0, \varphi\), while the “partial” equilibrium position \(y = 0\) is an invariant set of this system.

Following [6], we let \(x(t) = x(t, t_0, \varphi)\) denote the value of \(x(t_0, \varphi)\) at time \(t\). Just as in [6], we can prove the following assertion [5]: if for each bounded closed subset \(S\) in \(R^n \times C^1 \times R^n\), the operator \(Y(t, \varphi, y_1, \varphi y_2, \varphi z)\) maps the set \(S \times C^2 \times C^2\) into a bounded set (in \(R^n\)), then the inequality \(|y_1(t, t_0, \varphi)| \leq h_1 < h\) means that the \(y_1\)-components of the corresponding solutions of system (1) are determined for all \(t \geq t_0\). In this case, the \((y_2, z)\)-components of the solutions can be determined only on a finite time interval \(t \in [t_0 - \tau, \beta), \beta < +\infty\), and \(|y_2(t, t_0, \varphi)| + |z(t, t_0, \varphi)| \rightarrow 0\) as \(t \rightarrow \beta\). As a result, when studying the stability with respect to a part of the variables (with respect to \(y_1\)) of the “partial” equilibrium position of system (1), we additionally assume that the solutions are \((y_2, z)\)-continuable [3], namely, the solutions of the system are determined for \(t \geq t_0\) such that \(|y_1(t, t_0, \varphi)| < h\). In this case, if \(|y_1(t, t_0, \varphi)| \leq h_1 < h\) for all \(t \geq t_0\), then the corresponding functions \(x(t, t_0, \varphi)\) are determined for all \(t \geq t_0\).

We represent the component \(\varphi_z\) of the vector function \(\varphi\) in the form \(\varphi_z = [\varphi z_1^T, \varphi z_2^T]\) and let \(D\delta\) denote the domain \(\varphi\) such that \(||\varphi_z\| < \delta, ||\varphi z_1|| \leq L, ||\varphi z_2|| < \infty\).

**Definition** [5]. A “partial” equilibrium position \(y = 0\) of system (1) is \(y_1\)-stable for a large values of \(\varphi z_1\) and on the whole with respect to \(\varphi z_2\), if for any \(\varepsilon > 0, t_0 \geq 0\) and for any given number \(L > 0\) there is \(\delta(\varepsilon, t_0, L) > 0\) such that from \(\varphi \in D\delta\) it follows that \(|y_1(t, t_0, \varphi)| < \varepsilon\) for all \(t \geq t_0\).

**Remark 1.** In the case \(y_1 = y, ||\varphi y|| < \delta, ||\varphi z|| < \infty\) [7], we have the definition of the stability of a “partial” equilibrium position \(y = 0\) of system (1). If \(z \equiv 0 (\varphi z \equiv 0),\) then we obtain the definition of stability with respect to all of the variables (if \(y_1 = y\)) [6] or with respect to a part of the variables (if \(y_1 \neq y\)) [2] of the zero equilibrium position. (The zero equilibrium position \(x = 0\) of the system (1) is \(y_1\)-stable [2], if for any \(\varepsilon > 0, t_0 \geq 0\) there is \(\delta(\varepsilon, t_0, 0) > 0\) such that from \(||\varphi|| < \delta\) it follows that \(|y_1(t, t_0, \varphi)| < \varepsilon\) for all \(t \geq t_0\).

**3 Main Results**

We consider single-valued scalar continuously differentiable functions \(V = V(t, x), V(t, 0) = 0\), defined in the domain

\[E = \{t \geq 0, |y_1| < h, |y_2| + |z| < \infty\}.\] (3)

The derivatives \(V'\) of the \(V\)-functions along the solutions of system (1) is functionals are understood as (we denote by symbol \(<\cdot x>\) the scalar product)

\[V'(t, \varphi) = \partial V(t, \varphi(0))/\partial t + <\partial V(t, \varphi(0))/\partial x, X(t, \varphi)>.

To obtain the partial stability conditions, we also consider: 1) auxiliary scalar continuous in the domain (3) functions \(V'(t, y, z_1)\) and auxiliary generally vector functions \(\mu(t, x)\), which are continuously differentiable in the domain (3); 2 conti-
uous monotonically increasing for \( r \in R_+ \) scalar function \( a(r), a(0) = 0 \).

We associate functions \( \mu(t, x) \) in the space \( E \) with functions \( \mu(t, \phi) \) in the space \( G \) and let us define \( \| \mu(t, \phi) \| = \sup \| \mu(t, \phi(\theta)) \|, \theta \in [-\tau, 0], \ t \in R_+ \).

Let us assume that it is possible to represent \( V \)-function in the form

\[
V(t, x) = V^{**}(t, y_1, \mu(t, x), y_2, z),
\]

where \( V^{**} \) is continuously differentiable function in the domain (3).

We also denote

\[
\Omega(V) = \{ \phi \in C^n : V^{**}(t + \theta, \phi_{y1}(\theta), \mu(t + \theta, \phi(\theta)), \phi_{y2}(0), \phi_{z}(0)) \leq V^{**}(t, \phi_{y1}(0), \mu(t, \phi(0)), \phi_{y2}(0), \phi_{z}(0)), \ \theta \in [-\tau, 0], \ t \in R_+ \},
\]

\[G^* = ||\phi_{y1}|| + ||\mu(t, \phi)|| < h_1 < h, \ ||\phi_{y2}|| + ||\phi_{z}|| < \infty.\]

**Theorem 1.** Suppose that for system (1) along with a \( V \)-function, it is possible to find a vector function \( \mu(t, x), \mu(t, 0) = 0 \), such that:

i) it is possible to represent \( V \)-function in the form (4);

ii) in domain

\[t \geq 0, \ |y_1| + |\mu(t, x)| < h_1 < h, \ |y_2| + |z| < \infty\]

the conditions

\[V(t, x) \geq a(|y_1| + |\mu(t, x)|),\]

\[V(t, x) \leq V'(t, y, z_1), \ V'(t, 0, z_1) \equiv 0\]

are satisfied;

iii) \( V'(t, \phi) \leq 0 \) for all \( t \in R_+ \), \( \phi \in \Omega(V) \).

Then, the “partial” equilibrium position \( y = 0 \) of system (1) is \( y_1 \)-stable for a large values of \( \phi_{y1} \) and on the whole with respect to \( \phi_{y2} \).

**Proof.** We introduce the functional

\[W(t, \phi) = \sup V^{**}(t + \theta, \phi_{y1}(\theta), \mu(t + \theta, \phi(\theta)), \phi_{y2}(0), \phi_{z}(0)), \ \theta \in [-\tau, 0]\]

for all \( t \in R_+, \ \phi \in \Omega(V) \). For any \( \varepsilon > 0, t_0 \geq 0 \) and for any given number \( L > 0 \), it follows from the continuity of the function \( V \) (functional \( W \)), the condition \( V(t, \phi) \equiv 0 \) (the condition \( W(t, 0) \equiv 0 \)), and the conditions (7) (the conditions \( W(t, \phi) \leq V'(t, \phi_{y1}, \phi_{y2}) \)) that there is \( \delta(\varepsilon, t_0, L) > 0 \) such that from \( \phi \in D_\delta \) it follows that \( W(t_0, \phi) < a(\varepsilon) \).

Let us show that for the arbitrary solution \( x(t_0, \phi), \phi \in D_\delta \) of the system (1) the function \( W_0(t) = W(t, x(t_0, \phi)) \) is a non-increasing in \( t \). We assume the contrary. Then, for a some \( t > t_0 \) the relation \( W_0(t_0) > W_0(t) \) holds for \( t \in (t_0, t_0-r) \), where \( r > 0 \) is sufficiently small. It is possible if the relations \( W_0(t_r) = V(t_r, x(t_r)) \), \( V(t_r, x(t_r)) > V(t, x(t)) \) holds for \( t \in (t_0, t_0-r) \). From the first the relation by the condition iii it follows inequality \( V'(t_r) \leq 0 \) which contradicts the second relation.

As a result, taking into account the condition (6) for the arbitrary solution \( x(t_0, \phi), \phi \in D_\delta \) of the system (1) we have the inequalities
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\[ a(|\phi_2(0)| + |\mu(t, \phi(0))|) \leq W(t, x(t_0, \phi)) \leq W(t, \phi) \leq W(t_0, \phi) < a(\varepsilon). \]

Consequently, we obtain \( |y_1(t; t_0, \phi)| < \varepsilon \) for all \( t \geq t_0 \), if \( \phi \in D\delta \).

**Theorem 2.** Let us assume system (1) have the zero equilibrium position \( x = 0 \). Suppose that for system (1) along with a \( V \)-function, it is possible to find a vector function \( \mu(t, x) \), \( \mu(t, 0) = 0 \), such that:

i) it is possible to represent \( V \)-function in the form (4);

ii) in domain (5) the conditions (6) is satisfied;

iii) \( V'(t, \phi) \leq 0 \) for all \( t \in R_+ \), \( \phi \in \Omega_2(V) \).

Then, the zero equilibrium position \( x = 0 \) of system (1) is \( y_1 \)-stable.

**Remark 2.** Theorems 1 and 2 are generalizations of the Razumikhin [8] and Rumyantsev [1] theorems and of the corresponding results obtained in [2,4,9].

For comparison, the stability problem with respect to all of the variables of the zero equilibrium \( x = 0 \) of system (1) was considered in [8]. The stability with respect to a part of the variables (\( y_1 \)-stability) of the zero equilibrium \( x = 0 \) of the system (1) was considered in [2] under the assumptions \( V(t, x) = V^*(t, y_1, \mu(t, x)) \), \( ||\phi|| < \delta \) and was also considered in [9] under the assumptions \( \mu = 0 \), \( V(t + \theta, \phi(\theta)) \leq V(t, \phi(0)) \), and \( ||\phi|| < \delta \). The aftereffect was not taken into account when these problems were analyzed in [1, 4].

**Remark 3.** The proposed approach allows us to prove \( y_1 \)-stability by using functions which cannot be sign-alternating either with respect to \( y_1 \) in the Rumyantsev sense [1] (i.e., the condition \( V(t, x) \geq a(|y_1|) \) cannot be satisfied in the domain

\[ t \geq 0, \quad |y_1| < h_1 < h, \quad |y_2| + |z| < \infty, \]

or in the Lyapunov sense (with respect to all of the variables).

Moreover, the derivative of the \( V \)-function in Theorems 1 and 2 are generally sign-alternating in the domain (8).

**Example.** We assume that system (1) has the form

\[ y_1'(t) = -(t + 4)y_1(t) + y_1(t - \tau) + y_2^2(t - \tau)z_1(t - \tau), \]

\[ y_2'(t) = y_2(t)[1 + e^t y_1(t) + y_2^2(t)z_1(t)], \]

\[ z_1'(t) = 2[1 + e^t y_1(t)]z_1(t), \quad z_2'(t) = e^t y_1(t)z_2(t). \]  

We consider an auxiliary functions \( V \) and \( \mu_1 \) of the form

\[ V(x) = \frac{1}{2}(y_1^2 + y_2^2z_1^2), \quad \mu_1 = y_2^2z_1, \]

satisfying the conditions (i) and (ii) of theorem 1. On the set

\[ \Omega_2(V) = \{ \phi \in G^*: |\phi_1(0)| = \max |\phi_1(\theta)|, \big| \mu_1(\phi(0)) \big| = \max \big| \mu_1(\phi(\theta)) \big|, \theta \in [-\tau, 0] \}, \]

\[ G^* = \{ ||\phi_2|| + ||\mu_1(t, \phi)|| < h_1 < h, ||\phi_2|| + ||\phi_2|| < \infty \} \]
the derivative $V'$ due to system (9) can be estimated as
\[
V'(t, \phi) = -(t + 2)\varphi_{\Phi t}^2(0) + \varphi_{\Psi 1}(0)\varphi_{\Psi 1}(-\tau) + \varphi_{\Psi 1}(0)\mu_1(-\tau) - \mu_1^2(0) + 2\mu_1^3(0)
\]
\[
\leq -(t + 2)\varphi_{\Phi t}^2(0) + |\varphi_{\Psi 1}(0)||\varphi_{\Psi 1}(-\tau)| + |\varphi_{\Psi 1}(0)||\mu_1(-\tau)| - \mu_1^2(0) + 2\mu_1^3(0)
\]
\[
\leq -\gamma(\varphi_{\Phi t}^2(0) + \mu_1^2(0)) \leq 0, \quad \gamma = \text{const} > 0,
\]
and condition (iii) of Theorem 1 also is satisfied.

It follows from Theorem 1 that the “partial” equilibrium position $y_1= y_2 = 0$ of the system (9) is $y_1$-stable for a large values of $\varphi_{\Psi 1}$ and on the whole with respect to $\varphi_{\Psi 2}$. We also note that, in the domain (8), the derivative of the chosen $V$-function due to the system (9) is sign-alternating and the conditions of Theorem 1 are not satisfied for $\mu \neq 0$.

References


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