On the Set of Simple Hypergraph Degree

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Abstract

For a given \( m, 0 < m \leq 2^n \), let \( D_m(n) \) denote the set of all hypergraphic sequences for hypergraphs with \( n \) vertices and \( m \) hyperedges. A hypergraphic sequence in \( D_m(n) \) is upper hypergraphic if all its components are at least \( m/2 \). Let \( \tilde{D}_m(n) \) denote the set of all upper hypergraphic sequences. A structural characterization of the lowest and highest rank maximal elements of \( \tilde{D}_m(n) \) was provided in an earlier study. In the current paper we present an analogous characterization for all upper non-hypergraphic sequences. This allows determining the thresholds \( \tilde{r}_{\text{min}} \) and \( r_{\text{max}} \) such that all upper sequences of ranks lower than \( \tilde{r}_{\text{min}} \) are hypergraphic and all sequences of ranks higher than \( r_{\text{max}} \) are non-hypergraphic.

Keywords: hypergraph, degree sequence, complement

1. Introduction

A hypergraph \( H \) is a pair \((V, E)\), where \( V \) is the vertex set of \( H \), and \( E \), the set of hyperedges, is a collection of non-empty subsets of \( V \). The degree of a vertex \( v \) of \( H \), denoted by \( d(v) \), is the number of hyperedges in \( H \) containing \( v \). A hypergraph \( H \) is simple if it has no repeated hyperedges. A hypergraph \( H \) is \( r \)-uniform if all hyperedges contain \( r \)-vertices.

Let \( V = \{v_1, \ldots, v_n\} \). \( d(H) = (d(v_1), \ldots, d(v_n)) \) is the degree sequence of hypergraph \( H \). A sequence \( d = (d_1, \ldots, d_n) \) is hypergraphic if there is a simple hypergraph \( H \) with degree sequence \( d \). For a given \( m, 0 < m \leq 2^n \), let \( H_m(n) \) denote the set of all simple hypergraphs \(([n], E)\), where \([n] = \{1,2,\ldots, n\} \), and
|$E| = m$. Let $D_m(n)$ denote the set of all hypergraphic sequences of hypergraphs in $H_m(n)$. The subject of our investigation is the set $D_m(n)$, as well as its complement, the set of integer $n$-tuples which are not hypergraphic sequences for $H_m(n)$.

The problem of characterization of $D_m(n)$ remains open even for 3-uniform hypergraphs (see [4]-[13]). The problem has its interpretation in terms of multidimensional binary cubes that arises out of the discrete isoperimetric problem for $n$-dimensional binary cube [1-3]. In [6] the polytope of degree sequences of uniform hypergraphs was studied and several partial results were obtained. It was shown in [10] that any two 3-uniform hypergraphs can be transformed into each other by using a sequence of trades. Several necessary and one sufficient conditions were obtained for existence of simple 3-uniform hypergraphs in [8]. Steepest degree sequences were defined in [7] and it was shown that the whole set of degree sequences of simple uniform hypergraphs can be determined by its steepest elements. Upper and lower degree sequences were defined for $D_m(n)$ in [11] where it was proven that the whole set $D_m(n)$ can be easily determined by the set of its upper and/or lower degree sequences. Upper degree sequences of the lowest and highest ranks were characterized in our earlier study [12]. In the current paper we extend the study to the complementary area of $D_m(n)$, which can be supportive in solving the problem algorithmically.

Define the grid $\mathcal{E}^n_{m+1}$ as: $\mathcal{E}^n_{m+1} = \{(a_1, \ldots, a_n)|0 \leq a_i \leq m \text{ for all } i\}$, and place a component-wise partial order on $\mathcal{E}^n_{m+1}$: $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ if and only if $a_i \leq b_i$ for all $i$. The rank $r(a_1, \ldots, a_n)$ of an element $(a_1, \ldots, a_n)$ is defined as $(a_1 + \cdots + a_n)$. The Hasse diagram of $\mathcal{E}^n_{m+1}$ has $m \cdot n + 1$ levels according to the ranks of elements: the $i$-th level contains all elements of the rank $i$. $r(a_1, \ldots, a_n) = r(b_1, \ldots, b_n) + 1$ if $(a_1, \ldots, a_n)$ covers $(b_1, \ldots, b_n)$ (see [14] for undefined terms). In this manner, $D_m(n)$ is a subset of $\mathcal{E}^n_{m+1}$.

A hypergraphic sequence $d = (d_1, \ldots, d_n) \in D_m(n)$, is called upper hypergraphic if $d_i \geq m_{\text{mid}}$ for all $i$, where $m_{\text{mid}} = (m + 1)/2$ for odd $m$ and $m_{\text{mid}} = m/2$ for even $m$. Let $\bar{D}_m(n)$ denote the set of all upper hypergraphic sequences in $D_m(n)$. According to [11], for constructing all elements of $D_m(n)$, it is sufficient to find elements of $\bar{D}_m(n)$, reducing in this manner the problem of describing the set of degree sequences from $\mathcal{E}^n_{m+1}$ to $\bar{H}$, where $\bar{H} = \{(a_1, \ldots, a_n)|m_{\text{mid}} \leq a_i \leq m \text{ for all } i\}$.

Thus $\bar{D}_m(n) = D_m(n) \cap \bar{H}$. Figure 1 illustrates Hasse diagram of $\mathcal{E}^3_5$.

Figure 2 demonstrates $\bar{D}_m(n)$ and $D_m(n)$ in $\mathcal{E}^3_5$.

$\bar{F}_m(n) = \bar{H}\setminus \bar{D}_m(n)$ is the set of all upper non-hypergraphic sequences. Recall ([11]) that $\bar{D}_m(n)$ is an ideal in $\bar{H}$ ($\bar{F}_m(n)$ is a filter in $\bar{H}$).

Figure 3 illustrates $\bar{D}_m(n)$ and $\bar{F}_m(n)$ in $\bar{H}$.
Figure 1.
Circles correspond to elements/vertices of $E^3_5$. Highlighted part (gray) composes $\bar{H}$.

Figure 2.
Whole highlighted part composes $D_m(n)$, and its lighter part is $\bar{D}_m(n)$. 
In [12] we obtained simple formulas for the lowest \( r_{\text{min}} \) and the highest \( r_{\text{max}} \) ranks of maximal elements in \( \bar{D}_m(n) \).

In this paper we present analogous results for non-hypergraphic sequences, namely we seek for the lowest and highest ranks \( \bar{r}_{\text{min}} \) and \( \bar{r}_{\text{max}} \), respectively, of minimal elements of \( \bar{F}_m(n) \). Section 2 determines a characterization of the lowest rank. We obtain a series of minimal elements of \( \bar{F}_m(n) \) and prove that these elements are the lowest rank non-hypergraphic sequences. Section 3 determines the highest rank minimal elements. We conclude that all sequences in \( \bar{H} \) of ranks lower than \( \bar{r}_{\text{min}} \) are hypergraphic and all sequences in \( \bar{H} \) of ranks higher than \( r_{\text{max}} \) are non-hypergraphic. In the last section we give some estimates on lowest and highest ranks in \( \bar{H} \) depending on the values of \( m \).

2. Lowest rank

In this section we provide a characterization of the lowest rank minimal elements of \( \bar{F}_m(n) \). We obtain a series of minimal elements of \( \bar{F}_m(n) \) and prove that these elements are the lowest rank non-hypergraphic sequences.

Let \( m \) be given in the standard binary representation form:

\[
m = 2^{k_1} + \cdots + 2^{k_p} \text{ where } k_1 > \cdots > k_p > 0.
\] (1)

The lowest rank \( r_{\text{min}} \) of maximal elements of \( \bar{D}_m(n) \) is defined in [12] as follows:

\[
r_{\text{min}} = \sum_{i=1}^{p} ((n - k_i - (i - 1)) \cdot 2^{k_i} + k_i \cdot 2^{k_i-1}).
\]

Figure 3.
Highlighted part composes \( \bar{H} \). Light part in \( \bar{H} \) is \( \bar{D}_m(n) \), and dark part is \( \bar{F}_m(n) \).
The maximal element \( d_{min} \) of rank \( r_{min} \) in \( \tilde{D}_m(n) \) corresponds to a hypergraph whose edges are identified with the initial \( m \)-segment of the reverse lexicographic ordering of \( [2^n] \) and is unique (up to coordinate permutations).

\[
d_i = \left( \sum_{j=1}^{i-1} 2^{k_j-1} \right) + 2^{k_j} \quad \text{for } i = k_j + 1, \ j = 1, \ldots, p, \\
d_i = \left( \sum_{j=1}^{i-1} 2^{k_j-1} + \sum_{l=j+1}^{p} 2^{k_l} \right) \quad \text{for } k_{j+1} + 2 \leq i \leq k_j + j, \ j = 1, \ldots, p - 1, \\
d_i = \left( \sum_{l=1}^{p} 2^{k_l-1} \right) \quad \text{for } k_1 + 2 \leq i \leq n.
\]

Thus \( d_{min} \) has the following form:

\[
d_{min} = \left( \frac{n_1}{m}, \ldots, \frac{n_1}{m}, \frac{n_2}{d_{n_1+1}}, \ldots, \frac{n_2}{d_{n_1+n_2}}, \frac{n_3}{m_{mid}}, \ldots, \frac{n_3}{m_{min}} \right)
\]

where \( m > d_{n_1+1} \geq \cdots \geq d_{n_1+n_2} > m_{mid}; \ n_1 + n_2 + n_3 = n. \) Notice that \( d_{n_1+1} = 2^{k_1}. \)

Below, we establish two easily verified properties of \( d_{min} \) which will be used to prove our results.

**Property 1.** \( d_i \) is the largest possible value for fixed \( d_1, \ldots, d_i-1. \)

**Property 2.**

a) \( n_1 > 0 \) if and only if \( m \leq 2^{n-1}. \) If \( 2^{t-1} < m \leq 2^t \) for some \( t \leq n \) then \( n_1 = n - t. \)

b) \( n_2 = 0 \) if and only if \( m = 2^t \) for some \( t. \)

c) \( n_3 = k_p. \)

We will also use the notions of flatter and steeper elements defined as follows:

Let \( a_i \geq a_j + 2 \) for some \( 1 \leq i, j \leq n, \) then \( (a_3, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_n) \) is **flatter** than \( (a_1, \ldots, a_n) \) and \( (a_1, \ldots, a_n) \) is **steeper** than \( (a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_n) \) if \( (a_1, \ldots, a_n) \in D_m(n), \) then all elements flatter than \( (a_1, \ldots, a_n) \) also belong to \( D_m(n) \) (see [7]).

The following theorem determines a minimal element of the lowest rank of \( \tilde{F}_m(n). \)

**Theorem 1.** Let \( d_{min} \) be presented as in (2).

(1) If \( m \neq 2^t \) for arbitrary \( t, \) then:

a) \( \tilde{d}_{min} = \left( \frac{n_1}{n_3}, \ldots, \frac{n_1}{n_3}, \frac{n_{n-1}}{2^{k_1} + 1}, \frac{n_{n}}{m_{mid}}, \ldots, \frac{n_{n}}{m_{mid}} \right) \) is a minimal element of \( \tilde{F}_m(n), \) where \( 2^{k_1} \) is the first component in (1).

b) \( r(\tilde{d}_{min}) = \tilde{r}_{min}. \)

(2) If \( m = 2^t \) for some \( t, \) then:
a) $\tilde{d}_{\text{min}} = \left( \frac{n_1}{m, \cdots, m_{\text{mid}} + 1, m_{\text{mid}}, \cdots, m_{\text{mid}}} \right)$ is a minimal element of $\tilde{H}_m(n)$.

b) $r(\tilde{d}_{\text{min}}) = \tilde{r}_{\text{min}}$.

**Proof.** We consider both cases separately.

1. $m \neq 2^t$. Then $n_2 > 0$ by Property 2.
   a) Here it suffices to show that $d_{\text{min}}$ is a non-hypergraphic sequence in $\tilde{H}$ and all elements of $\tilde{H}$ covered by $d_{\text{min}}$ are hypergraphic in $\tilde{H}$. $d_{\text{min}} \notin \tilde{D}_m(n)$ by Property 1. All elements of $\tilde{H}$ covered by $d_{\text{min}}$ have one of the following forms:

   $$\left( \frac{n_1}{m, \cdots, m_{\text{mid}}, 2^{k_1}, m_{\text{mid}}, \cdots, m_{\text{mid}}} \right)$$

   $$\left( \frac{n_1}{m, \cdots, m, m - 1, 2^{k_1} + 1, m_{\text{mid}}, \cdots, m_{\text{mid}}} \right)$$

   Sequence (3) is less than $d_{\text{min}}$ and hence belongs to $\tilde{D}_m(n)$. If $m = 2^{k_1} + 1$, then sequence (4) is just a permutation of (3). If $m \geq 2^{k_1} + 2$, then (4) is flatter than (3), and therefore is hypergraphic.

   b) We have to prove that all $a \in \tilde{H}$ of $r(a) < r(\tilde{d}_{\text{min}})$ belong to $\tilde{D}_m(n)$. It is sufficient to show that all $a \in \tilde{H}$ of $r(a) = r(\tilde{d}_{\text{min}}) - 1$ belong to $\tilde{D}_m(n)$.

   Consider $d' = \left( \frac{n_1}{m, \cdots, m_{\text{mid}}, 2^{k_1}, m_{\text{mid}}, \cdots, m_{\text{mid}}} \right)$ which is of rank $r(\tilde{d}_{\text{min}}) - 1$. All elements in $\tilde{H}$ of the rank $r(\tilde{d}_{\text{min}}) - 1$ can be obtained from $d'$ by combinations of the following unit operations:

   i) replace $(m, 2^{k_1})$ by $(m - 1, 2^{k_1} + 1)$;

   ii) replace $(m, m_{\text{mid}})$ by $(m - 1, m_{\text{mid}} + 1)$;

   iii) replace $(2^{k_1}, m_{\text{mid}})$ by $(2^{k_1} - 1, m_{\text{mid}} + 1)$.

   In all three cases above the resulting sequence is either flatter than $d'$ or is a permutation of $d'$, and therefore belongs to $\tilde{D}_m(n)$.

2. $m = 2^t$. Then $n_2 = 0$ by Property 2.

   a) Here it is only required to prove that $d_{\text{min}} \notin \tilde{D}_m(n)$ and all elements of $\tilde{H}$ covered by $d_{\text{min}}$ are hypergraphic sequences in $\tilde{H}$. $d_{\text{min}} \notin \tilde{D}_m(n)$ because $d_{\text{min}} > d_{\text{min}}$ and $d_{\text{min}}$ is a maximal element of $\tilde{D}_m(n)$.

   All elements of $\tilde{H}$ covered by $d_{\text{min}}$ have one of the following forms:

   $$\left( \frac{n_1}{m, \cdots, m_{\text{mid}}, m_{\text{mid}}}, \cdots, m_{\text{mid}} \right)$$

   or

   $$\left( \frac{n_1}{m, \cdots, m - 1, m_{\text{mid}} + 1}, m_{\text{mid}}, \cdots, m_{\text{mid}} \right).$$
In the former form this is just $d_{\text{min}}$ and in the latter form this is flatter than $d_{\text{min}}$, and hence belongs to $\hat{D}_m(n)$.

b) We have to prove that all elements $a$ of $r(a) = \bar{d}_{\text{min}} - 1$ in $\hat{H}$ belong to $\hat{D}_m(n)$. These are flatter than $d_{\text{min}} = \left(\frac{n_1}{m, \ldots, m, m_{\text{mid}}, \ldots, m_{\text{mid}}}\right)$, and hence belong to $\hat{D}_m(n)$.

Remark that in case of hypergraphic sequences there is a unique maximal element of rank $r_{\text{min}}$ in $\hat{D}_m(n)$, whereas in case of non-hypergraphic sequences there are a number of minimal elements of rank $\bar{r}_{\text{min}}$ in $\hat{F}_{\text{m}}(n)$.

Theorem 2 below produces a series of minimal elements of $\hat{F}_{\text{m}}(n)$.

**Theorem 2.**

(1) If $m \neq 2^t$ then

$$\left(\frac{n_1}{m, \ldots, m, m - t, 2^{k_1} + t + 1, m_{\text{mid}}, \ldots, m_{\text{mid}}}\right), \quad t = 1, \ldots, (m - 2^{k_1} - 1)/2.$$  

are minimal elements of $\bar{r}_{\text{min}}$ in $\hat{F}_{\text{m}}(n)$.

(2) If $m = 2^t$ then

$$\left(\frac{n_1}{m, \ldots, m, m - t, m_{\text{mid}} + t + 1, m_{\text{mid}}, \ldots, m_{\text{mid}}}\right), \quad t = 1, \ldots, (m - m_{\text{mid}} - 1)/2.$$  

are minimal elements of $\bar{r}_{\text{min}}$ in $\hat{F}_{\text{m}}(n)$.

The proof is obtained by an analogous reasoning as in Theorem 1.

**3. Highest rank**

In this section we determine a characterization of highest rank minimal elements of $\hat{F}_m(n)$.

Let $m$ be given in the following canonical representation form:

$$m = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} + m_1, \quad m_1 < \binom{n}{k-1} \tag{5}$$

Let the highest rank $r_{\text{max}}$ of maximal elements of $\hat{D}_m(n)$, be defined as in [12]:

$$r_{\text{max}} = \sum_{i=0}^{k} ((n - i) \cdot \binom{n}{i} + (n - k - 1) \cdot m_1.$$  

Let $D_{\text{max}}$ denote the class of maximal elements of $\hat{D}_m(n)$ of the rank $r_{\text{max}}$. $D_{\text{max}}$ defines the set of degree sequences of that class of hypergraphs which have $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$ common hyperedges (the subsets of $[n]$ of cardinalities $n, n - 1, \ldots, n - k$) and differ only in the remaining $m_1$ hyperedges (the $(n - k - 1)$-subsets of $[n]$). Thus, $|D_{\text{max}}| = \binom{m_1}{k-1}$. The components of all $d_{\text{max}} \in D_{\text{max}}$
are calculated as follows: $d_i = \sum_{i=0}^{k} C_{n-i}^{n-1} + s_i$, where $(s_1, \cdots, s_n)$ defines the set of hypergraphic sequences for $(n-k-1)$-uniform hypergraphs with $m_1$ edges.

**Theorem 3.**
Let $m = C_n^n + C_n^{n-1} + \cdots + C_n^{n-k} + m_1$, $m_1 < C_n^{n-k-1}$.

a) If $m_1 \geq 1 + \frac{k}{n} - k - 1$, then $\bar{r}_{\text{max}} = r_{\text{max}} + 1$.

b) If $m_1 < 1 + \frac{k}{n} - k - 1$, then $\bar{r}_{\text{max}} \leq r_{\text{max}}$.

**Proof.**
a) It is easy to check that there is a sequence $d' = (d_1, \cdots, d_n)$ in $D_{\text{max}}$ such that $d_1 \geq \cdots \geq d_{n-1} > d_n$ (we can always choose $m_1$ hyperedges such that $s_1 \geq \cdots \geq s_{n-1} > s_n$). Consider $d'' = (d_1, \cdots, d_{n-1}, d_n + 1)$, which belongs to $\hat{F}_m(n)$. All elements covered by $d''$ are either flatter than $d'$ or permutations of $d''$ and, thus, belong to $\hat{D}_m(n)$. Therefore $d''$ is a minimal element of the rank $r_{\text{max}} + 1$ in $\hat{F}_m(n)$. Since all elements of the rank $r_{\text{max}} + 1$ in $\hat{H}$ belong to $\hat{F}_m(n)$, there is no minimal element of $\hat{F}_m(n)$ of rank higher than $r_{\text{max}} + 1$. Thus $\bar{r}_{\text{max}} = r_{\text{max}} + 1$.

b) This case is obvious.

**4. Concluding remarks**

The last section gives estimates on the lowest and highest ranks in $\hat{H}$ depending on parameter $m$ and brings several concluding remarks.

As it was stated above all sequences in $m$ with ranks lower than $\bar{r}_{\text{min}}$ are hypergraphic and all sequences in $\hat{H}$ with ranks higher than $r_{\text{max}}$ are non-hypergraphic. Hence all maximal elements of $\hat{D}_m(n)$ and all minimal elements of $\hat{F}_m(n)$ have ranks ranging between $\bar{r}_{\text{min}}$ and $r_{\text{max}} + 1$, and, thus, are located between $\bar{r}_{\text{min}}$ and $r_{\text{max}} + 1$ levels of $\hat{H}$. An illustration of the upper hypergraphic and non-hypergraphic sequences in $\hat{H}$ for $n = 3$ and $m = 4$ is given in Figure 3.

$\hat{D}_m(n) = \{(3,3,3), (4,2,2), (3,3,2), (3,2,3), (2,4,2), (2,3,3), (2,2,4), (3,2,2), (2,3,2), (2,2,3), (2,2,2)\}$.

Maximal elements of $\hat{D}_m(n)$ are: $(3,3,3), (4,2,2), (2,4,2), (2,2,4)$.

$\hat{F}_m(n) = \{(4,4,4), (4,4,3), (4,3,4), (3,4,4), (4,4,2), (4,3,3), (2,4,4), (3,4,2), (3,4,3), (3,3,4), (2,4,4), (4,3,2), (4,2,3), (3,4,2), (3,2,4), (2,4,3), (2,3,4)\}$.

Maximal elements of $\hat{F}_m(n)$ are: $(4,3,2), (4,2,3), (3,4,2), (3,2,4), (2,4,3), (2,3,4)$.

And thus $r_{\text{min}} = 8$, $r_{\text{max}} = 9$, $\bar{r}_{\text{min}} = 9$ and $\bar{r}_{\text{max}} = 9$. 
We consider the lowest, highest and middle levels in $\tilde{H}$: the lowest level consists of the lowest element $(m_{\text{mid}}, \ldots, m_{\text{mid}})$, the highest level consists of the highest element $(m, \ldots, m)$ and the middle level consists of all elements of the rank $n \cdot (m + m_{\text{mid}})/2$, particularly it contains the element $((m + m_{\text{mid}})/2, \ldots, (m + m_{\text{mid}})/2)$.

Next we shall examine the distance of $\tilde{r}_{\text{min}}$ and $r_{\text{max}}$ ranks/levels from the lowest, middle and highest levels of $\tilde{H}$.

First distinguish the following cases for $\tilde{r}_{\text{min}}$:

a) If $2^{t-1} < m < 2^t$ for some $t \leq n$, then:

$$\tilde{d}_{\text{min}} = \left( \frac{m-t}{m, \ldots, m}, 2^{t-1} + 1, \frac{t-1}{m_{\text{mid}}, \ldots, m_{\text{mid}}} \right)$$

$$\tilde{r}_{\text{min}} = (n-t) \cdot m + 2^{t-1} + (t-1) \cdot m_{\text{mid}} + 1$$

The distance from the lowest level in $\tilde{H}$ will be:

$$\tilde{r}_{\text{min}} - r(m_{\text{mid}}, \ldots, m_{\text{mid}}) \approx (n-t) \cdot \frac{m}{2} + \left( 2^{t-1} - \frac{m}{2} + 1 \right) = \frac{m}{2} \cdot (n-t-1) + 2^{t-1} + 1$$

The distance from the middle level in $\tilde{H}$ will be:

$$\tilde{r}_{\text{min}} - r((m + m_{\text{mid}})/2, \ldots, (m + m_{\text{mid}})/2) \approx (n-t) \cdot \frac{m}{4} - (t-1) \cdot \frac{m}{4} + 2^{t-1} - 3m/4 = \frac{m}{4} (n-t) - (2t) + 2^{t-1} + 1.$$  

b) If $m = 2^t$ for some $t$, then:

$$\tilde{d}_{\text{min}} = \left( \frac{m-t}{m, \ldots, m}, m_{\text{mid}}, \ldots, m_{\text{mid}} + 1, \frac{t-1}{m_{\text{mid}}, \ldots, m_{\text{mid}}} \right)$$

$$\tilde{r}_{\text{min}} = (n-t) \cdot m + t \cdot m_{\text{mid}} + 1$$

The distance from the lowest level will be:

$$\tilde{r}_{\text{min}} - r(m_{\text{mid}}, \ldots, m_{\text{mid}}) \approx \frac{m}{2} \cdot (n-t) + 1$$

The distance from the middle level will be:

$$\tilde{r}_{\text{min}} - r((m + m_{\text{mid}})/2, \ldots, (m + m_{\text{mid}})/2) \approx (n-t) \cdot \frac{m}{4} - t \cdot \frac{m}{4} + 1 = \frac{m}{4} \cdot (n-2t) + 1.$$  

As we see in both cases $\tilde{r}_{\text{min}}$ goes up from the lowest level in $\tilde{H}$ with decrease of $m$. 

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Now we estimate the case of $r_{\text{max}}$. We have $r_{\text{max}} = n \cdot \sum_{i=0}^{k} C_{n-1}^{n-i} + m_1 \cdot (n - k - 1)$.

a) $m_1 = 0$. $m = \sum_{i=0}^{k} C_{n-1}^{n-i} = 2 \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i} + C_{n-1}^{n-k-1}$, and thus $m_{\text{mid}} = \sum_{i=0}^{k-1} C_{n-1}^{n-i} + C_{n-1}^{n-k-1}/2$.

The distance from the lowest level in $\tilde{H}$ element is:
$$d_i = \sum_{i=0}^{k} C_{n-1}^{n-i}.$$ 

The distance from the lowest level in $\tilde{H}$ element is:
$$r_{\text{max}} - r(m_{\text{mid}}, \ldots, m_{\text{mid}}) = C_{n-1}^{n-k-1} \cdot n/2.$$ 

The distance from the highest level is:
$$r(m, \ldots, m) - r_{\text{max}} = n \cdot \left(2 \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i} + C_{n-1}^{n-k-1}\right) -$$
$$n \cdot \sum_{i=0}^{k} C_{n-1}^{n-i} = n \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i-1}.$$ 

b) $m_1 > 0$. $m = \sum_{i=0}^{k} C_{n-1}^{n-i} + m_1 = 2 \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i} + C_{n-1}^{n-k-1} + m_1$, and thus $m_{\text{mid}} = \sum_{i=0}^{k-1} C_{n-1}^{n-i} + (C_{n-1}^{n-k-1} + m_1)/2$.

The distance from the lowest level is:
$$r_{\text{max}} - r(m_{\text{mid}}, \ldots, m_{\text{mid}}) = C_{n-1}^{n-k-1} \cdot n/2 + m_1 \cdot (n - k - 1) - n \cdot m_1/2 =$$
$$C_{n-1}^{n-k-1} \cdot n/2 + m_1 (n/2 - k - 1).$$ 

The distance from the highest level is:
$$n \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i} - m_1 \cdot (n - k - 1).$$

Thus we have determined the layouts of $r_{\text{max}}$ and $\tilde{r}_{\text{min}}$ in $\tilde{H}$ depending on parameter $m$. The obtained formulas show when $r_{\text{max}}$ and $\tilde{r}_{\text{min}}$ are close, or when $\tilde{r}_{\text{min}}$ is greater than the middle rank depending on $m$, etc.

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