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Local Low Speed Preconditioning in Rotating Reference Frame

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Abstract

The aim of this paper is to present local preconditioning for the compressible fluid flow at low speeds in the rotating reference frame. The formulation of the preconditioner is derived and the basic properties are shown. Moreover, relation to some of the preconditioners available in the literature is described. The proposed formulation is implemented into the CFD code and few illustrative examples are presented.

Mathematics Subject Classification: 65N08, 76M12

Keywords: low speed preconditioning, compressible fluid flow

1 Introduction

It is a well known fact that solving low speed flow problems with numerical methods designed for compressible flow presents difficulty, which is related to the disparity between the flow speed and the speed of sound. Since we want to avoid using dedicated numerical methods depending on the velocity of the flow (let us note that in some cases the speed may vary significantly within one computational domain) the preconditioning is a simple way to remedy undesired behaviour of the compressible flow solver.

The aim of this paper is to present local preconditioner for the compressible flow in the rotating reference frame. The presented method can be seen as an extension of the one studied in [8] to the rotating reference frame. It is also a generalization of [7], although the governing flow equations are considered in a slightly different form.

2 Governing equations

The flow of the compressible fluid is described by the system of Euler (or Navier–Stokes) equations. Since the preconditioning is based on the inviscid part, we shall restrict our attention to the system of compressible Euler equations in three dimensions, which can be written with the aid of the Einstein summation convention in the following differential and quasi-linear form:

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} = \mathbf{S}, \quad \text{or} \quad \frac{\partial \mathbf{W}}{\partial t} + A_i \frac{\partial \mathbf{W}}{\partial x_i} = \mathbf{S}, \quad \text{where} \quad A_i = \frac{D\mathbf{F}_i}{D\mathbf{W}} \tag{1}$$

is the Jacobian of the convective flux vector $\mathbf{F}_i$ with respect to the conservative variables. The formulation used throughout this paper coincides with that of the CFD solver Edge [3, 2]. For the system (1) in the reference frame rotating around an axis given by a vector $\mathbf{\Omega}$ and a point $\mathbf{x_0}$ with angular velocity $|\mathbf{\Omega}|$ we define conservative variables, inviscid flux and the source term by

$$\mathbf{W} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E_r \end{pmatrix}, \qquad \mathbf{F}_i = \begin{pmatrix} \rho w_i \\ \rho u_1 w_i + \delta_{i1} p \\ \rho u_2 w_i + \delta_{i2} p \\ \rho u_3 w_i + \delta_{i3} p \\ (\rho E_r + p) w_i \end{pmatrix}, \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} 0 \\ \rho \mathbf{\Omega} \times \mathbf{u} \\ 0 \end{pmatrix}, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ are the absolute velocities, $\mathbf{w} = (w_1, w_2, w_3)^T$ denotes the velocity vector relative to the rotating reference frame, related by

$$\mathbf{w} = \mathbf{u} - \mathbf{\Omega} \times \mathbf{r}, \tag{3}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x_0}$ denotes the position vector from the rotation axis. Assuming the ideal gas, we have the following relations between density ρ, pressure p, temperature T, energy E and the speed of sound a:

$$p = \rho R T, \qquad E = \frac{p}{\rho(\gamma - 1)} + \frac{|\mathbf{u}|^2}{2}, \qquad \text{and} \qquad a = \sqrt{\frac{\gamma p}{\rho}}, \tag{4}$$

where R denotes the specific gas constant and γ the heat capacity ratio. In the definition (2) the energy equation is modified to contain contribution from the rotation,

$$E_r = E - \mathbf{u} \cdot (\mathbf{\Omega} \times \mathbf{r}). \tag{5}$$

For the definition of preconditioning it is useful to re-write the governing equations (1) in terms of other sets of variables, e. g., primitive variables $\mathbf{Q} =$

$(p, u, v, w, T)^T$ or entropy variables $\mathbf{W}^0 = (p, u, v, w, S)^T$, where S is defined by $dS = dp - a^2 d\rho$. Denoting $M = \frac{D\mathbf{W}}{D\mathbf{Q}}$ and $N = \frac{D\mathbf{Q}}{D\mathbf{W}^0}$ the transformation matrices between variables, for details we refer to the Appendix, we can rewrite (1) as

$$\frac{\partial \mathbf{Q}}{\partial t} + A_i^Q \frac{\partial \mathbf{Q}}{\partial x_i} = M^{-1}\mathbf{S}, \quad \text{where} \quad A_i^Q = M^{-1}A_i M = M^{-1}\frac{D\mathbf{F}_i}{D\mathbf{Q}}, \tag{6}$$

or

$$\frac{\partial \mathbf{W}^0}{\partial t} + A_i^0 \frac{\partial \mathbf{W}^0}{\partial x_i} = N^{-1}M^{-1}\mathbf{S}, \qquad A_i^0 = N^{-1}A_i^Q N. \tag{7}$$

It is important to note that the change of variables does not change the eigenvalues of the system matrix (A_i, A_i^Q or A_i^0). The form of the matrix is, however, simpler for primitive or entropy variables, e. g.,

$$A_1^Q = \begin{bmatrix} w_1 & \rho a^2 & 0 & 0 & 0 \\ \rho^{-1} & w_1 & 0 & 0 & 0 \\ 0 & 0 & w_1 & 0 & 0 \\ 0 & 0 & 0 & w_1 & 0 \\ 0 & \gamma_1 T & 0 & 0 & w_1 \end{bmatrix}, \quad A_1^0 = \begin{bmatrix} w_1 & \rho a^2 & 0 & 0 & 0 \\ \rho^{-1} & w_1 & 0 & 0 & 0 \\ 0 & 0 & w_1 & 0 & 0 \\ 0 & 0 & 0 & w_1 & 0 \\ 0 & 0 & 0 & 0 & w_1 \end{bmatrix}, \tag{8}$$

where $\gamma_1 = \gamma - 1$. From this expression we conclude that:

Lemma 2.1. *The matrix A_i^0 (and also A_i or A_i^Q) have the eigenvalues $\lambda_{1,2,3} = w_i$ and $\lambda_{4,5} = w_i \pm a$.*

3 Preconditioning

Preconditioning is introduced by multiplying the time derivative by the matrix Γ_0 defined later, instead of (7) we write

$$\Gamma_0 \frac{\partial \mathbf{W}^0}{\partial t} + A_i^0 \frac{\partial \mathbf{W}^0}{\partial x_i} = N^{-1}M^{-1}\mathbf{S}. \tag{9}$$

or, equivalently,

$$\frac{\partial \mathbf{W}^0}{\partial t} + \Gamma_0^{-1} A_i^0 \frac{\partial \mathbf{W}^0}{\partial x_i} = \Gamma_0^{-1} N^{-1}M^{-1}\mathbf{S}. \tag{10}$$

This modification causes that the time derivative no longer represents physical time. However, if the flow converges to the steady state ($\partial W^0/\partial t \to 0$) the effect of preconditioning is removed and the solution is consistent with the time accurate solution. For unsteady flows the method of dual–time stepping can be employed, cf. [9].

The eigenvalues of the matrix $\Gamma_0^{-1} A_i^0$ are different from those of A_i^0. If we define

$$\Gamma_0 = \begin{bmatrix} \frac{a^2}{\beta} & 0 & 0 & 0 & \delta \\ \frac{\alpha w_1}{\rho\beta} & 1 & 0 & 0 & 0 \\ \frac{\alpha w_2}{\rho\beta} & 0 & 1 & 0 & 0 \\ \frac{\alpha w_3}{\rho\beta} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \Gamma_0^{-1} = \begin{bmatrix} \frac{\beta}{a^2} & 0 & 0 & 0 & -\frac{\delta\beta}{a^2} \\ -\frac{\alpha w_1}{\rho a^2} & 1 & 0 & 0 & \frac{\alpha w_1 \delta}{\rho a^2} \\ -\frac{\alpha w_2}{\rho a^2} & 0 & 1 & 0 & \frac{\alpha w_2 \delta}{\rho a^2} \\ -\frac{\alpha w_3}{\rho a^2} & 0 & 0 & 1 & \frac{\alpha w_3 \delta}{\rho a^2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

we find that:

Lemma 3.1. *The matrix $\Gamma_0^{-1} A_i^0$ has the eigenvalues $\lambda_{1,2,3} = w_i$ and*

$$\lambda_{4,5} = 0.5 \left[w_i \left(1 - \alpha + \frac{\beta}{a^2} \right) \pm \sqrt{ w_i^2 \left(1 - \alpha + \frac{\beta}{a^2} \right)^2 + 4\beta \left(1 - \frac{w_i^2}{a^2} \right) } \right]$$

Proof: Since, e. g.,

$$\Gamma_0^{-1} A_1^0 = \begin{bmatrix} \frac{\beta\, w_1}{a^2} & \rho\,\beta & 0 & 0 & -\frac{\delta\,\beta\, w_1}{a^2} \\ \frac{-\alpha\, {w_1}^2 + a^2}{\rho\, a^2} & (1-\alpha) w_1 & 0 & 0 & \frac{\alpha\, {w_1}^2 \delta}{\rho\, a^2} \\ -\frac{\alpha\, w_1 w_2}{\rho\, a^2} & -\alpha\, w_2 & w_1 & 0 & \frac{\alpha\, w_2 \delta\, w_1}{\rho\, a^2} \\ -\frac{\alpha\, w_3 w_1}{\rho\, a^2} & -\alpha\, w_3 & 0 & w_1 & \frac{\alpha\, w_3 \delta\, w_1}{\rho\, a^2} \\ 0 & 0 & 0 & 0 & w_1 \end{bmatrix}$$

we can easily obtain the first three eigenvalues via row/column reduction of the determinant. Now we have to solve the upper left 2×2 submatrix

$$\begin{vmatrix} \frac{\beta\, w_1}{a^2} - \lambda & \rho\,\beta \\ \frac{-\alpha\, {w_1}^2 + a^2}{\rho\, a^2} & (1-\alpha) w_1 - \lambda \end{vmatrix} = \lambda^2 - \lambda w_1 \left(1 - \alpha + \frac{\beta}{a^2} \right) - \beta \left(1 - \frac{w_1^2}{a^2} \right),$$

from which we easily confirm the result. □

The form of the preconditioning matrix (11) is similar to [8]. However, due to rotation the definition is based on the relative velocity vector. The preconditioning matrix can be expressed for other sets of variables, e. g.,

$$\Gamma_Q = N \Gamma_0 N^{-1} \qquad \text{and} \qquad \Gamma_W = M \Gamma_Q M^{-1}.$$

The matrices Γ_Q and Γ_W can be used to multiply the time derivative in (6) and (1), respectively.

Preconditioning parameter

Consistently with existing literature, cf. [7], we define β to be proportional to the square of local relative velocity magnitude. To avoid $\beta = 0$, in which case Γ_0^{-1} becomes singular, a threshold based on reference velocity in the flow is used:

$$\beta = \min\left[\max\left(|\mathbf{w}|^2, K_r|\mathbf{w}_{ref}|^2\right), a^2\right], \tag{12}$$

where $\mathbf{w}_{ref} = \mathbf{u}_{ref} - \boldsymbol{\Omega} \times \mathbf{r}$ is a reference velocity (e. g., a free stream velocity) in the rotating reference frame. Also, β is bounded from above by a^2, which removes the preconditioning above the speed of sound (for α and δ equal to zero). The recommendations for K_r range usually between 0.5 to 1, but can be set even higher.

Computational method

Finite Volume (FV) methods for the solution of (1) are based on the integral identity

$$\frac{\partial}{\partial t}\int_V \mathbf{W}\, dV = -\int_\Sigma \mathbf{F}_i n_i\, d\Sigma + \int_V \mathbf{S}\, dV, \tag{13}$$

where we integrate over a control volume V and its surface Σ and the vector $\mathbf{n} = (n_1, n_2, n_3)^T$ is normal to the surface. Special properties of the inviscid fluxes $\mathbf{F}_i$, see [4], imply that $\mathbf{F}_i n_i = n_i A_i \mathbf{w}$. If the right hand side of (13) is multiplied by the matrix Γ_W^{-1} we obtain preconditioned method. Eigenvalues of the matrix $A = n_i A_i$ and $\Gamma_W^{-1}A$ are consistent with the assertions of Lemma 2.1 and 3.1 with w_i being replaced by $\mathbf{w} \cdot \mathbf{n}$. The proof is, however, much more technical.

In numerical methods eigenvalues of A or $\Gamma_W^{-1}A$ are useful to determine the time step, which is inversely proportional to the maximum eigenvalue. Preconditioned methods allow for larger time step and, hence, faster advancing towards the steady state solution.

The surface integral in (13) is approximated by the so–called numerical flux often containing artificial dissipation term, which scales with the eigenvalues.

In the case of low speed flows the maximum eigenvalue of A is proportional to the speed of sound a, which adds too much numerical dissipation to the scheme and compromises the accuracy of the solution in such a case, cf. [1]. Correct scaling of the artificial dissipation term with the eigenvalues of $\Gamma_W^{-1}A$ is was studied in the framework of the Finite Difference method in [8, 5] and then in [7] for a FV method.

4 Numerical results

The presented preconditioner was implemented into the CFD solver Edge (`www.foi.se/edge`) and two illustrative cases are presented and studied to show basic properties of the preconditioner (influence of free parameters) and consistency with the non–preconditioned case.

Rotated GAMM channel

The geometry of the GAMM channel (parallel walls with a 10% bump at the lower wall) was rotated around the axial direction. In this case we expect that rotated and steady case give identical results since the geometry is rotationally symmetric. The flow is considered inviscid and the absolute inlet velocity is set to Mach number M_{abs} between 0.001 and 0.15. Maximum of the rotation speed $|\mathbf{\Omega} \times \mathbf{r}|$ in the domain is equal to inlet velocity.

In the Figure 1 we observe that convergence towards machine precision is faster (except for few initial iterations) for preconditioned methods. Moreover, it is slightly better with increasing α. This can be explained by the fact that maximum eigenvalue is smaller allowing larger time steps. Similar picture was observed for other tested Mach numbers. The variation of δ did not have any visible effect in this case, let us note that δ does not influence the eigenvalues.

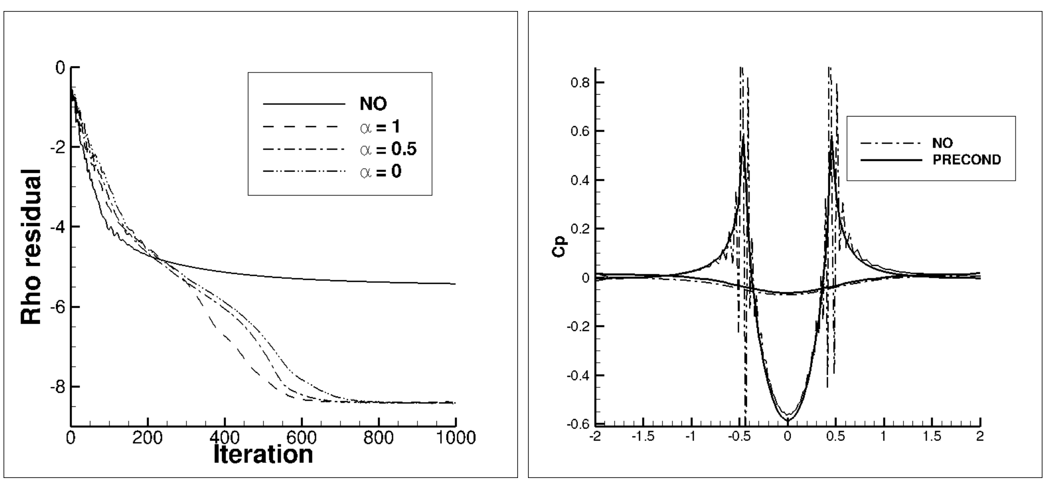

Figure 1: Convergence for $M_{abs} = 0.001$ (left) and c_p along upper and lower wall for $M_{abs} = 0.01$ (right).

To show disputable accuracy of the compressible flow methods without preconditioning in the low speed region, the plot of the non-dimensional pressure coefficient $c_p = (p - p_{in})/(0.5\rho_{in}|\mathbf{u}_{in}|^2)$ is presented, the subscript *"in"* denotes the inlet reference quantities. Significant oscillations are observed. This behaviour is noticed in extremely low speeds and vanishes as Mach number reaches above 0.1, cf. [1]. The preconditioned case is displayed for the setting $\alpha = 0.5$ and $\delta = 0$, however, there are no visible differences for other combinations.

Axial fan rotor

For a more complex case let us compare solution of the flow through axial fan rotor which was experimentally measured in our facility. The flow is considered viscous with turbulence model added to the flow equations, however, only the conservation equations (1) are preconditioned. Since the problem is rotationally periodic, only one rotor blade is modelled in the computational domain. The hybrid (tetrahedral and hexahedral elements) computational grid was used as a primary grid with approximately 2.3 million nodes. The boundary layer was fully resolved around the rotor blade and solid walls and the gap between the blade tip and the casing was also included in the model. The layout of the simulation and the planes where the flow properties were evaluated is indicated in the Figure 2.

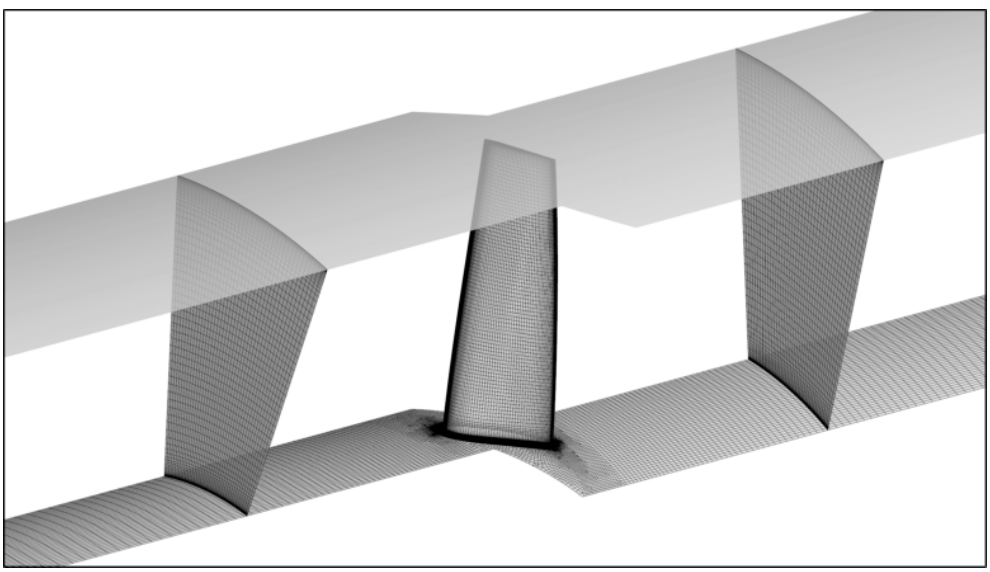

Figure 2: Layout of the axial fan rotor case.

The computation was carried out for various settings of outlet boundary condition (prescribed pressure) to simulate choking of the flow in the experimental device. The torque and the increase in the static pressure between the planes downstream and upstream of the rotor was evaluated, see Figure 3.

We observe that preconditioning for various settings of free parameters gives consistent results with the original method. Let us note that in this case the relative Mach number in the flow was above 0.2, except for the stagnation regions.

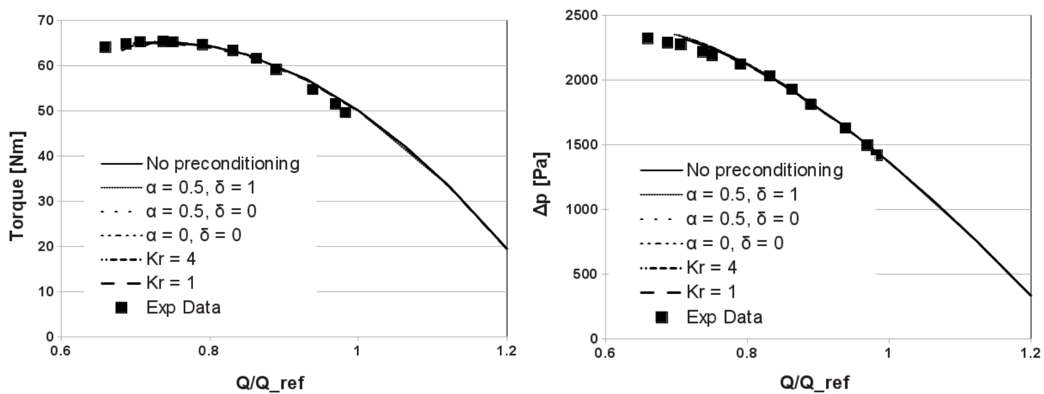

Figure 3: Torque and Δp as a function of volume flow rate.

5 Discussion and conclusion

Compared other preconditioners appearing in the literature, the presented one is an extension of classical Choi–Merkle or Turkel preconditioner, cf. [8, 9] and the references herein, for the case of rotating reference frame. Although the formulation of the governing equations is slightly different in [7], rewriting the flow equations in terms of the entropy variables we find that our preconditioner is consistent for $\alpha = 0$ and $\delta = 1$ with the mentioned reference.

We observe that for correct definition the preconditioning matrix (in the entropy variables) has to be based on relative velocity vector $\mathbf{w}$ which provides consistent results with the theory available for steady grids.

Various other preconditioners, e. g., diagonal preconditioner [6] written in the (ρ, u, v, w, p) variables, cannot be directly linked to our case. However, if written in terms of the entropy variables (9) we find that the common goal of the preconditioning is to change the coefficient in front of the pressure time derivative. Various other free parameters are used to tune the eigenvalues and/or the eigenvectors of the resulting system matrices.

The preconditioning presented here was implemented into the CFD code which already contained options for reference frame rotation and preconditioning for the steady reference frame. Various adjustments were necessary to correctly include preconditioning for the rotating reference frame consistently with the existing parts.

6 Appendix

The form of the transformation matrices can be expressed as follows:

$$M = \frac{D\mathbf{W}}{D\mathbf{Q}} = \begin{bmatrix} \frac{\rho}{p} & 0 & 0 & 0 & -\frac{\rho}{T} \\ \frac{\rho u_1}{p} & \rho & 0 & 0 & -\frac{\rho u_1}{T} \\ \frac{\rho u_2}{p} & 0 & \rho & 0 & -\frac{\rho u_2}{T} \\ \frac{\rho u_3}{p} & 0 & 0 & \rho & -\frac{\rho u_3}{T} \\ \frac{\rho E_r}{p} & \rho w_1 & \rho w_2 & \rho w_3 & -\frac{\rho}{T}\left(\mathbf{u}\cdot\mathbf{w} - \frac{|\mathbf{u}|^2}{2}\right) \end{bmatrix}, \quad (14)$$

$$M^{-1} = \frac{D\mathbf{Q}}{D\mathbf{W}} = \begin{bmatrix} \frac{\gamma_1}{2}|\mathbf{u}|^2 & -\gamma_1 w_1 & -\gamma_1 w_2 & -\gamma_1 w_3 & \gamma_1 \\ -\frac{u_1}{\rho} & \rho^{-1} & 0 & 0 & 0 \\ -\frac{u_2}{\rho} & 0 & \rho^{-1} & 0 & 0 \\ -\frac{u_3}{\rho} & 0 & 0 & \rho^{-1} & 0 \\ \frac{\gamma_1|\mathbf{u}|^2 T}{2p} - \frac{T}{\rho} & -\frac{\gamma_1 w_1 T}{p} & -\frac{\gamma_1 w_2 T}{p} & -\frac{\gamma_1 w_3 T}{p} & \frac{\gamma_1 T}{p} \end{bmatrix}, \quad (15)$$

where $\gamma_1 = \gamma - 1$. These matrices are slightly different from [8, 5] because of the definition of the conservative variables in the rotating reference frame. The matrices

$$N = \frac{D\mathbf{Q}}{D\mathbf{W}^0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{\gamma_1 T}{\gamma p} & 0 & 0 & 0 & \frac{T}{\gamma p} \end{bmatrix}, \quad N^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\gamma_1 & 0 & 0 & 0 & \frac{\gamma p}{T} \end{bmatrix} \quad (16)$$

are identical in the mentioned papers.

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