Adomian Decomposition Method with Green’s 

Function for Solving Twelfth-Order Boundary Value Problems

Waleed Al-Hayani

Department of Mathematics
College of Computer Science and Mathematics
University of Mosul, Iraq

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Abstract

In this paper, the Adomian Decomposition Method with Green’s function (Standard Adomian and Modified Technique) is applied to solve linear and nonlinear twelfth-order boundary value problems with boundary conditions defined at even-order and odd-order derivatives as well. The numerical results obtained with a small amount of computation are compared with the exact solutions to show the efficiency of the method. The results show that the decomposition method is of high accuracy, more convenient and efficient for solving high-order boundary value problems.

Keywords: Adomian Decomposition Method; Adomian’s polynomials; Twelfth-order boundary value problems; Green’s function

1. Introduction

In the beginning of the 1980’s, Adomian [1-4] proposed a new and fruitful method (hereafter called the Adomian Decomposition Method or ADM) for solving linear and nonlinear (algebraic, differential, partial differential, integral, etc.) equations. It has been shown that this method yields a rapid convergence of the solutions series to linear and nonlinear deterministic and stochastic equations.

A class of characteristic-value problems of higher order (as higher as 24) is known to arise in hydrodynamic and hydromagnetic stability [5, 6]. Furthermore,
it is widely well known that when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability is as ordinary convection the ordinary differential equation is sixth order; when the instability sets in as overstability, it is modelled by an eighth-order ordinary differential equation. Suppose, now, that a uniform magnetic field is also applied across the fluid in the same direction as gravity. When instability sets in now as ordinary convection, it is modelled by a tenth-order boundary value problem; when instability sets in as overstability, it is modelled by a twelfth-order boundary value problem (for details, see [5]). Agarwal's book [7] contains theorems which detail the conditions for existence and uniqueness of solutions of the twelfth-order boundary value problems.

Different numerical and semi analytical methods have been proposed by various authors to solve twelfth-order boundary value problems. A few of them are: Twelfth-degree spline method [8], Modified Decomposition Method with the inverse operator (MDM) [9], Thirteen-degree spline method [10], Non-polynomial spline technique [11], Variational Iteration Method (VIM) [12,15], Differential Transform Method (DTM) [13] and Homotopy Perturbation Method (HPM) [14].

The main objective of this paper is to apply the Standard Adomian with Green's function (SAwGF) and Modified Technique with Green's function (MTwGF) to linear and nonlinear twelfth-order boundary value problems with boundary conditions defined at even-order and odd-order derivatives as well.

2. Analysis of the method

Let us consider the general BVP of twelfth-order

\[ y^{(12)}(x) + g(x,y) = f(x), \quad a \leq x \leq b \]  \hspace{1cm} (1)

with boundary conditions

\[ y^{(i)}(a) = \alpha_i, \quad y^{(i)}(b) = \beta_i, \quad i = 0,1,2,3,4,5 \]  \hspace{1cm} (2)

where \( y = y(x), \ g(x,y) \) is a linear or nonlinear function of \( y \), and \( f(x) \) are continuous functions defined in the interval \( x \in [a,b] \) and \( \alpha_i, \beta_i; \ (i = 0,1,2,3,4,5) \) are finite real constants.

Applying the decomposition method as in [1-4], Eq. (1) can be written as

\[ Ly = f(x) - Ny, \]

where \( L = \frac{d^{12}}{dx^{12}} \) is the linear operator and \( Ny = g(x,y) \) is the nonlinear operator. Consequently,
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\[ y(x) = h(x) + \int_{a}^{b} G(x, \xi) f(\xi) d\xi \]
\[ - \int_{a}^{b} G(x, \xi) N y d\xi, \quad (3) \]

where \( h(x) \) is the solution of \( Ly = 0 \) with the boundary conditions (2) and \( G(x, \xi) \) is the Green’s function [15] given by

\[ G(x, \xi) = \begin{cases} 
  g_2(x, \xi) & \text{if } a \leq x \leq \xi \leq b \\
  g_1(x, \xi) & \text{if } a \leq \xi \leq x \leq b
\end{cases} \]

The Adomian’s technique consists of approximating the solution of (1) as an infinite series

\[ y = \sum_{n=0}^{\infty} y_n, \quad (4) \]

and decomposing the nonlinear operator \( N \) as

\[ Ny = \sum_{n=0}^{\infty} A_n, \quad (5) \]

where \( A_n \) are polynomials (called Adomian polynomials) of \( y_0, y_1, ..., y_n \) [1-4] given by

\[ A_n = \frac{1}{n! \lambda^n} d^n N \left( \sum_{i=0}^{\infty} \lambda^i y_i \right) \bigg|_{\lambda=0}, \quad n = 0, 1, 2, ... \]

The proofs of the convergence of the series \( \sum_{n=0}^{\infty} y_n \) and \( \sum_{n=0}^{\infty} A_n \) are given in [3, 17-21]. Substituting (4) and (5) into (3) yields

\[ \sum_{n=0}^{\infty} y_n = h(x) + \int_{a}^{b} G(x, \xi) f(\xi) d\xi \]
\[ - \int_{a}^{b} G(x, \xi) \sum_{n=0}^{\infty} A_n d\xi. \quad (6) \]

From (6), the iterates defined using the Standard Adomian Method are determined in the following recursive way:
\[ y_0 = h(x) + \int_a^b G(x, \xi)f(\xi)d\xi, \]
\[ y_{n+1} = -\int_a^b G(x, \xi)A_nd\xi, \quad n = 0, 1, 2, \ldots \]

and the iterates defined using the Modified Technique [22] are determined in the following recursive way:

\[ y_0 = h(x), \]
\[ y_1 = \int_a^b G(x, \xi)f(\xi)d\xi - \int_a^b G(x, \xi)A_0d\xi, \]
\[ y_{n+2} = -\int_a^b G(x, \xi)A_{n+1}d\xi, \quad n = 0, 1, 2, \ldots. \]

Thus all components of \( y \) can be calculated once the \( A_n \) are given. We then define the \( n \)-term approximant to the solution \( y \) by \( \phi_n[y] = \sum_{i=0}^{n-1} y_i \) with \( \lim_{n \to \infty} \phi_n[y] = y \).

3. Applications and numerical results

In this section, the ADM with the Green’s function (Standard Adomian and Modified Technique) for solving linear and nonlinear twelfth-order boundary value problems is illustrated in the following examples. The computations associated with the examples were performed using a Maple 13 package with a precision of 40 digits.

Example 1

Consider the following linear BVP of twelfth-order [10, 11, 13, 15]:

\[ y^{(12)}(x) + xy(x) = -(120 + 23x + x^3)e^x, \quad 0 \leq x \leq 1 \]  \hfill (7)

with boundary conditions

\[ y^{(i)}(0) = i(2 - i), \quad y^{(i)}(1) = -i^2 e, \quad i = 0, 1, 2, 3, 4, 5. \]  \hfill (8)

The exact solution of (7), (8) is

\[ y_{Exact}(x) = x(1 - x)e^x. \]

Applying the decomposition method, Eq. (7) can be written as

\[ Ly = -(120 + 23x + x^3)e^x - xy(x), \]

where \( L = \frac{d^{12}}{dx^{12}} \) is the linear operator. Consequently,
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\[ y = h(x) - \int_{0}^{1} G(x, \xi)(120 + 23\xi + \xi^3)e^\xi d\xi \]

\[ - \int_{0}^{1} G(x, \xi)y(\xi)d\xi, \]

(9)

where \( h(x) \) is the solution of \( Ly = 0 \) with the boundary conditions (8) given by

\[ h(x) = \left( \frac{907}{8} - \frac{1001}{24} e \right)x^{11} - \left( \frac{7531}{12} - \frac{1847}{8} e \right)x^{10} + \left( \frac{11241}{8} - \frac{6203}{12} e \right)x^9 \\
- \left( \frac{3189}{2} - \frac{7039}{12} e \right)x^8 + \left( \frac{22075}{24} - \frac{2707}{8} e \right)x^7 - \left( \frac{865}{4} - \frac{1909}{24} e \right)x^6 \\
- \frac{1}{8}x^5 - \frac{1}{3}x^4 - \frac{1}{2}x^3 + x \]

and \( G(x, \xi) \) is the Green’s function given by

\[ G(x, \xi) = \begin{cases} 
  g_2(x, \xi) & \text{if } 0 \leq x \leq \xi \leq 1 \\
  g_1(x, \xi) & \text{if } 0 \leq \xi \leq x \leq 1
\end{cases} \]

where

\[ g_1(x, \xi) = \left( - \frac{\xi^{11}}{158400} + \frac{\xi^{10}}{28800} - \frac{\xi^9}{12960} + \frac{\xi^8}{11520} - \frac{\xi^7}{20160} + \frac{\xi^6}{86400} \\
- \frac{1}{11!} \xi^{11} \right)
\]

\[ + \left( \frac{\xi^{11}}{28800} - \frac{\xi^{10}}{5184} + \frac{\xi^9}{2304} - \frac{\xi^8}{2016} + \frac{\xi^7}{3456} - \frac{\xi^6}{14400} + \frac{\xi^5}{10!} \right) \xi^{10} \\
+ \left( - \frac{\xi^{11}}{12960} + \frac{\xi^{10}}{2304} - \frac{\xi^9}{2016} + \frac{\xi^8}{3456} - \frac{\xi^7}{14400} + \frac{\xi^6}{5760} - \frac{\xi^5}{725760} \right) \xi^9 \\
+ \left( \frac{\xi^{11}}{11520} - \frac{\xi^{10}}{2016} + \frac{\xi^9}{864} - \frac{\xi^8}{720} + \frac{\xi^7}{1152} - \frac{\xi^6}{4320} + \frac{\xi^5}{241920} \right) \xi^8 \\
+ \left( - \frac{\xi^{11}}{20160} + \frac{\xi^{10}}{3456} - \frac{\xi^9}{1440} + \frac{\xi^8}{1152} - \frac{\xi^7}{1728} + \frac{\xi^6}{5760} - \frac{\xi^5}{120960} \right) \xi^7 \\
+ \left( \frac{\xi^{11}}{86400} - \frac{\xi^{10}}{14400} + \frac{\xi^9}{5760} - \frac{\xi^8}{4320} + \frac{\xi^7}{14400} - \frac{\xi^6}{86400} \right) \xi^6 \]

\[ g_2(x, \xi) = \left( - \frac{\xi^{11}}{158400} + \frac{\xi^{10}}{28800} - \frac{\xi^9}{12960} + \frac{\xi^8}{11520} - \frac{\xi^7}{20160} + \frac{\xi^6}{86400} \\
- \frac{1}{11!} \xi^{11} \right)
\]

\[ + \left( \frac{\xi^{11}}{28800} - \frac{\xi^{10}}{5184} + \frac{\xi^9}{2304} - \frac{\xi^8}{2016} + \frac{\xi^7}{3456} - \frac{\xi^6}{14400} + \frac{\xi^5}{10!} \right) \xi^{10} \]
$+ \left( -\frac{\xi^{11}}{12960} + \frac{\xi^{10}}{2304} - \frac{\xi^9}{1008} + \frac{\xi^8}{864} - \frac{\xi^7}{1440} + \frac{\xi^6}{5760} - \frac{\xi^2}{725760} \right) x^9$

$+ \left( \frac{\xi^{11}}{11520} - \frac{\xi^{10}}{2016} + \frac{\xi^9}{864} - \frac{\xi^8}{720} + \frac{\xi^7}{1152} - \frac{\xi^6}{4320} + \frac{\xi^4}{241920} \right) x^8$

$+ \left( -\frac{\xi^{11}}{20160} + \frac{\xi^{10}}{3456} - \frac{\xi^9}{1440} + \frac{\xi^8}{1152} - \frac{\xi^7}{1728} + \frac{\xi^6}{5760} - \frac{\xi^5}{120960} \right) x^7$

$+ \left( \frac{\xi^{11}}{86400} - \frac{\xi^{10}}{14400} + \frac{\xi^9}{5760} - \frac{\xi^8}{4320} + \frac{\xi^7}{5760} - \frac{\xi^6}{14400} + \frac{\xi^5}{86400} \right) x^6.$

Substituting (4) in (9), the iterates defined using the Standard Adomian Method are determined in the following recursive way:

$y_0 = h(x) - \int_0^1 G(x, \xi)(120 + 23\xi + \xi^3)e^{\xi}d\xi,$

$y_{n+1} = -\int_0^1 G(x, \xi)\xi y_n(\xi)d\xi, \quad n = 0,1,2, ...$

and the iterates defined using the Modified Technique [22] are determined in the following recursive way:

$y_0 = h(x),$  

$y_1 = -\int_0^1 G(x, \xi)(120 + 23\xi + \xi^3)e^{\xi}d\xi - \int_0^1 G(x, \xi)\xi y_0(\xi)d\xi,$

$y_{n+2} = -\int_0^1 G(x, \xi)\xi y_{n+1}(\xi)d\xi, \quad n = 0,1,2, ... .$

In Table 1, we list the absolute errors obtained by SAwGF and MTwGF. Comparing them with the Thirteen-degree spline method [10], Non-polynomial spline technique [11], DTM [13] and VIM [15] results. In [10] the maximum absolute error is 7.38E-09 with $\mu = 0, k = 22$. In [11] the maximum absolute error is 4.72E-06. It can be noticed that the result obtained by the present method (SAwGF) is very superior (lower error combined with less number of iterations) to that obtained by the other mentioned methods. From Table 1, it can be deduced that, the error decreased monotonically with the increment of the integer $n$. 
where

\[ y = h(x) - 12 \int_{-1}^{1} G(x, \xi)(2 \xi \cos \xi + 11 \sin \xi) d\xi + \int_{-1}^{1} G(x, \xi) y(\xi) d\xi, \]

where \( h(x) \) is the solution of \( Ly = 0 \) with the boundary conditions (11) given by

**Example 2**

Consider the following linear BVP of twelfth-order [10, 11, 13, 15]:

\[ y^{(12)}(x) - y(x) = -12(2x \cos x + 11 \sin x), \quad -1 \leq x \leq 1 \]  

with boundary conditions

\[
\begin{align*}
  y(-1) &= y(1) = 0, \\
  y^{(1)}(-1) &= y^{(1)}(1) = 2 \sin(1), \\
  y^{(2)}(-1) &= -y^{(2)}(1) = -4 \cos(1) - 2 \sin(1), \\
  y^{(3)}(-1) &= y^{(3)}(1) = 6 \cos(1) - 6 \sin(1), \\
  y^{(4)}(-1) &= -y^{(4)}(1) = 8 \cos(1) + 12 \sin(1), \\
  y^{(5)}(-1) &= y^{(5)}(1) = -20 \cos(1) + 10 \sin(1). 
\end{align*}
\]

The exact solution of (10), (11) is

\[ y_{Exact}(x) = (x^2 - 1) \sin x. \]

Applying the decomposition method, Eq. (10) can be written as

\[ Ly = -12(2x \cos x + 11 \sin x) + y(x), \]

where \( L = \frac{d^{12}}{dx^{12}} \) is the linear operator. Consequently,
\[ h(x) = \left( \frac{61}{384} \sin 1 - \frac{95}{384} \cos 1 \right) x^11 - \left( \frac{377}{384} \sin 1 - \frac{587}{384} \cos 1 \right) x^9 \\
+ \left( \frac{497}{192} \sin 1 - \frac{257}{192} \cos 1 \right) x^7 - \left( \frac{761}{192} \sin 1 - \frac{1123}{192} \cos 1 \right) x^5 \\
+ \left( \frac{1649}{384} \sin 1 - \frac{1739}{384} \cos 1 \right) x^3 - \left( \frac{805}{384} \sin 1 - \frac{181}{128} \cos 1 \right) x \]

and \( G(x, \xi) \) is the Green’s function given by

\[ G(x, \xi) = \begin{cases} 
  g_2(x, \xi) & \text{if } -1 \leq x \leq \xi \leq 1 \\
  g_1(x, \xi) & \text{if } -1 \leq \xi \leq x \leq 1 
\end{cases} \]

where

\[ g_1(x, \xi) = \left( \frac{10^{-2} \xi}{798336} \right)^{5} \left( \begin{array}{c}
  -10^{-2} \xi_{11} \xi^{10} + 10^{-1} \xi^{9} - 10^{-1} x^{7} + 10^{-2} x^{5} - 10^{-3} x^{3} + 10^{-2} x^{1} \\
  + \left( \begin{array}{c}
  10^{-2} x^{10} + 10^{-1} x^{8} - 10^{-2} x^{6} - 10^{-1} x^{4} - 10^{-1} x^{2} + 10^{-2} x^{1} \\
  + \left( \begin{array}{c}
  10^{-2} x^{10} + 10^{-2} x^{8} - \frac{1}{2} \xi^{8} + 10^{-1} x^{4} - 10^{-1} x^{2} + 10^{-2} x^{1}
  \end{array} \right) \xi^{9} - \\
  + \left( \begin{array}{c}
  10^{-2} x^{10} - \frac{1}{2} \xi^{8} + 10^{-1} x^{4} + \frac{1}{2} \xi^{2} - 10^{-1} x^{2} + 10^{-2} x^{1}
  \end{array} \right) \xi^{9}
\end{array} \right)
\]
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Substituting (4) in (12), the iterates defined using the Standard Adomian Method are determined in the following recursive way:

\[y_0 = h(x) - 12 \int_{-1}^{1} G(x, \xi)(2\xi \cos \xi + 11 \sin \xi)d\xi,\]
\[y_{n+1} = \int_{-1}^{1} G(x, \xi)y_n(\xi)d\xi, \quad n = 0, 1, 2, ...\]

and the iterates defined using the Modified Technique [22] are determined in the following recursive way:

\[y_0 = h(x),\]
\[y_1 = -12 \int_{-1}^{1} G(x, \xi)(2\xi \cos \xi + 11 \sin \xi)d\xi + \int_{-1}^{1} G(x, \xi)y_0(\xi)d\xi,\]
\[y_{n+2} = \int_{-1}^{1} G(x, \xi)y_{n+1}(\xi)d\xi, \quad n = 0, 1, 2, ... .\]

In Table 2, we present the absolute errors obtained by SAwGF and MTwGF. Comparing them with the Thirteen-degree spline method [10], Non-polynomial spline technique [11], DTM [13] and VIM [15] results. In [10], the maximum absolute error is 4.69E-05 with \( \mu = 0, k = 22 \). In [11] the maximum absolute error is 4.67E-07. It can be noticed that the result obtained by the present method (SAwGF) is very superior to that obtained by the four previous mentioned methods. From Table 2, it can be deduced that, the error decreased monotonically with the increment of the integer \( n \).

Table 2: Comparison of absolute errors for example 2

<table>
<thead>
<tr>
<th>( x )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>DTM [13]</th>
<th>VIM [15]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>2.78E-17</td>
</tr>
<tr>
<td>0.1</td>
<td>1.22E-11</td>
<td>8.08E-22</td>
<td>2.28E-09</td>
<td>1.52E-19</td>
<td>1.64E-15</td>
<td>1.39E-15</td>
</tr>
<tr>
<td>0.2</td>
<td>2.03E-11</td>
<td>1.34E-21</td>
<td>3.79E-09</td>
<td>2.51E-19</td>
<td>2.08E-13</td>
<td>8.33E-16</td>
</tr>
</tbody>
</table>
In results not presented here, we have seen that the absolute errors obtained by SAwGF and MTwGF in the interval $[-1,0]$ are the same as for the interval $[0,1]$.

**Example 3**

Finally, we consider the following nonlinear BVP of twelfth-order [9, 12-14]:

$$y^{(12)}(x) = 2e^x y^2(x) + y'''(x), \quad 0 < x < 1$$  \hspace{1cm} (13)

with two sets of boundary conditions

$$y^{(i)}(0) = (-1)^i, \quad y^{(i)}(1) = (-1)^i e^{-1}, \quad i = 0, 1, 2, 3, 4, 5$$  \hspace{1cm} (14)

$$y^{(2i)}(0) = 1, \quad y^{(2i)}(1) = e^{-1}, \quad i = 0, 1, 2, 3, 4, 5.$$  \hspace{1cm} (15)

The exact solution of (13) with ((14) or (15)) is

$$y_{Exact}(x) = e^{-x}.$$  

Applying the decomposition method, Eq. (13) can be written as

$$Ly = 2e^x Ny + y'''(x),$$

where $L = \frac{d^{12}}{dx^{12}}$ is the linear operator and $Ny = y^2$ is the nonlinear operator. Consequently,

$$y = h(x) + \int_0^1 G(x, \xi) \{2e^\xi Ny(\xi)$$

$$+ y'''(\xi)\} d\xi,$$  \hspace{1cm} (16)

where $h(x)$ is the solution of $Ly = 0$ with the boundary conditions (14) given by
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\[ h(x) = \frac{18089 - 49171e^{-1}}{120} x^{11} - \frac{98989 - 26908e^{-1}}{120} x^{10} \\
+ \frac{48797 - 132644e^{-1}}{43721 - 118846e^{-1}} x^9 \\
- \frac{24}{120} x^8 + \frac{27595 - 75011e^{-1}}{24} x^7 - \frac{31739 - 86276e^{-1}}{120} x^6 \\
- \frac{1}{5!} x^5 + \frac{1}{4!} x^4 - \frac{1}{3!} x^3 + \frac{1}{2!} x^2 - x + 1, \]

and \( G(x, \xi) \) is the Green’s function given previously in example 1.

For the boundary conditions (15), \( h(x) \) is given by

\[ h(x) = \frac{e^{-1} - \frac{1}{11!} x^{11} + \frac{1}{10!} x^{10} + \left( \frac{e^{-1}}{435456} - \frac{1}{272160} \right) x^9 + \frac{1}{8!} x^8}{11!} \\
+ \left( \frac{307e^{-1}}{1814400} - \frac{59}{226800} \right) x^7 + \frac{1}{6!} x^6 + \left( \frac{12863e^{-1}}{1814400} - \frac{1241}{113400} \right) x^5 + \frac{1}{4!} x^4 \\
+ \left( \frac{2723e^{-1}}{19200} - \frac{1034}{4725} \right) x^3 + \frac{1}{2!} x^2 + \left( \frac{14556793e^{-1}}{14556793} - \frac{614206}{467775} \right) x + 1, \]

and \( G(x, \xi) \) is the Green’s function given by

\[ G(x, \xi) = \begin{cases} 
    g_2(x, \xi) & \text{if } 0 \leq x \leq \xi \leq 1 \\
    g_1(x, \xi) & \text{if } 0 \leq \xi \leq x \leq 1 
\end{cases} \]

where

\[ g_1(x, \xi) = \left( \frac{1}{11!} x - \frac{1}{11!} \right) \xi^{11} + \left( \frac{1}{2177280} x^3 - \frac{1}{725760} x^2 + \frac{1}{1088640} x \right) \xi^9 \\
+ \left( \frac{1}{604800} x^5 - \frac{1}{120960} x^4 + \frac{1}{90720} x^3 - \frac{1}{226800} x \right) \xi^7 \\
+ \left( \frac{1}{604800} x^7 - \frac{1}{86400} x^6 + \frac{1}{43200} x^5 - \frac{1}{32400} x^3 + \frac{1}{56700} x \right) \xi^5 \\
+ \left( \frac{1}{2177280} x^9 - \frac{1}{241920} x^8 + \frac{1}{90720} x^7 - \frac{1}{32400} x^5 + \frac{1}{17010} x^3 \right. \\
\left. - \frac{1}{28350} x \right) \xi^3 \]
\[ + \left( \frac{1}{11!} x^{11} - \frac{1}{10!} x^{10} + \frac{1}{1088640} x^9 - \frac{1}{226800} x^7 + \frac{1}{56700} x^5 - \frac{1}{28350} x^3 \right. \]
\[ \left. + \frac{2}{93555} x \right) \xi \]
\[ g_2(x, \xi) = \left( \frac{1}{11!} \xi - \frac{1}{10!} \right) x^{11} + \left( \frac{1}{2177280} \xi^3 - \frac{1}{725760} \xi^2 + \frac{1}{1088640} \xi \right) x^9 \]
\[ + \left( \frac{1}{604800} \xi^5 - \frac{1}{120960} \xi^4 + \frac{1}{90720} \xi^3 - \frac{1}{226800} \xi \right) x^7 \]
\[ + \left( \frac{1}{604800} \xi^7 - \frac{1}{86400} \xi^6 + \frac{1}{43200} \xi^5 - \frac{1}{32400} \xi^3 + \frac{1}{56700} \xi \right) x^5 \]
\[ + \left( \frac{1}{2177280} \xi^9 - \frac{1}{241920} \xi^8 + \frac{1}{90720} \xi^7 - \frac{1}{32400} \xi^5 + \frac{1}{17010} \xi^3 \right. \]
\[ \left. - \frac{1}{28350} \xi \right) x^3 \]
\[ + \left( \frac{1}{11!} \xi^{11} - \frac{1}{10!} \xi^{10} + \frac{1}{1088640} \xi^9 - \frac{1}{226800} \xi^7 + \frac{1}{56700} \xi^5 - \frac{1}{28350} \xi^3 \right. \]
\[ \left. + \frac{2}{93555} \xi \right) x \]

Substituting (4) and (5) in (16), the iterates defined using the Standard Adomian Method are determined in the following recursive way:

\[ y_0 = h(x), \]
\[ y_{n+1} = \int_0^1 G(x, \xi) \left\{ 2e^\xi A_n + y'''(\xi) \right\} d\xi, \quad n = 0, 1, 2, \ldots . \]

For the nonlinear term \( Ny = y^2 = \sum_{n=0}^\infty A_n \) the corresponding Adomian polynomials are:

\[ A_0 = y_0^2, \]
\[ A_1 = 2y_0y_1, \]
\[ A_2 = 2y_0y_2 + y_1^2, \]
\[ \vdots \]
\[ A_n = \sum_{i=0}^n y_i y_{n-i}, \quad n \geq i, \quad n = 0, 1, 2, \ldots . \]

In Tables 3A and 3B we give the absolute errors for the problem (13) with boundary conditions (14) and (15) respectively obtained by SAwGF. Comparing it with the MDM [9], DTM [13] and HPM [14] results, it can be seen easily that the result obtained by the present method (SAwGF) is very superior to that obtained.
Adomian decomposition method with Green’s function

by the three previous mentioned methods. From Tables 3A and 3B, it can be deduced that, the error decreased monotonically with the increment of the integer $n$.

Table 3A: Comparison of absolute errors for example 3

<table>
<thead>
<tr>
<th>$x$</th>
<th>SAwGF, $n = 1$</th>
<th>DTM [13]</th>
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<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.1</td>
<td>6.99E-16</td>
<td>4.11E-15</td>
</tr>
<tr>
<td>0.2</td>
<td>2.19E-14</td>
<td>1.30E-13</td>
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<tr>
<td>0.3</td>
<td>1.11E-13</td>
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<tr>
<td>0.4</td>
<td>2.46E-13</td>
<td>1.53E-12</td>
</tr>
<tr>
<td>0.5</td>
<td>3.12E-13</td>
<td>1.98E-12</td>
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<tr>
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<td>2.42E-13</td>
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<tr>
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<td>1.08E-13</td>
<td>7.17E-13</td>
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<tr>
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<td>2.09E-14</td>
<td>1.42E-13</td>
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<tr>
<td>0.9</td>
<td>6.58E-16</td>
<td>4.16E-15</td>
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<tr>
<td>1.0</td>
<td>0.00</td>
<td>1.22E-15</td>
</tr>
</tbody>
</table>

Table 3B: Comparison of absolute errors for example 3

<table>
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</thead>
<tbody>
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<td>$n = 2$</td>
<td>$n = 3$</td>
<td>$n = 3$</td>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
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<td>1.14E-12</td>
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<td>1.61E-07</td>
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<tr>
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<td>2.17E-12</td>
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<td>3.07E-07</td>
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</tr>
<tr>
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<td>2.00E-10</td>
<td>1.11E-16</td>
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It is clear from the Tables 3A and 3B that the numerical results corresponding to problem (13) with boundary conditions (14) are superior to those with boundary conditions (15).

4. Conclusions

The ADM with Green’s function (Standard Adomian and Modified Technique) has been applied for solving linear and nonlinear twelfth-order
boundary value problems with boundary conditions defined at even-order and odd-order derivatives as well. Comparison of the results obtained by the present method with those obtained by the Twelfth-degree spline method, Modified decomposition method with the inverse operator, Thirteen-degree spline method, Non-polynomial spline technique, Variational iteration method, Differential transform method and Homotopy perturbation method has revealed that the present method is superior because of the lower error and fewer required iterations. It has been shown that error is monotonically reduced with the increment of the integer $n$.

References


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