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Bayesian Queueing Model with Multiple Working Vacations under Gumbel Distribution

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Abstract

According to Paxson & Floyd[8], the regular assumption of the exponential times of the inter-arrival time and service time are vanished when heavy tailed scenaro. Consider the single server queue with multiple working vacation and the regular service time which follows Gumbel distribution. This paper exhibits the estimation of the parameters of queueing model under Bayesian procedure based on gibbs sampling algorithm through Markov Chain Monte Carlo(MCMC) technique. The performances of the empirical estimates of the model parameters and traffic intensity are discussed.

Mathematics Subject Classification: 60K25, 62C10, 62C12, 11K45

Keywords: Queue, working vacation, Gumbel distribution, Bayesian estimation, Markov Chain Monte Carlo

1 Introduction

The internet has developed into a global system of interconnected computer networks that follows the exchange of data between the millions of people in various organizations. In this connection, the tele-traffic of internet service increased day-to-day life, moreover the behavior of the service time differing very much from a Poisson process[8] because of the high variability is observed in inter-arrival time and service time, most of the times are smaller than the minor proportion of the time and this leads not only exponentially distributed but also characterization of the heavy tails. The Gumbel distribution is an appropriate for modeling strength, which is sometimes skewed to the left. In this regard, the several authors have been devoted to the service time of the system which follows a heavy tailed distribution[3],[4]. For this purpose, this paper considers the service time which follows Gumbel distribution. The Gumbel distribution is a particular case of the generalized extreme value distribution(also known as the Fisher-Tippett distribution,log-Weibull distribution and the double exponential distribution).

The probability generating function and cumulative distribution function of the Gumbel distribution are based on the location parameter, α and the scale parameter, β, respectively,

$$f(x:\alpha,\beta)=\frac{1}{\beta}e^{-\frac{(x-\alpha)}{\beta}}e^{-e^{-\frac{(x-\alpha)}{\beta}}} \quad for \quad x\in\Re,\alpha\in\Re;\beta>0 \quad and \tag{1}$$

$$F(x)=e^{-e^{-\frac{(x-\alpha)}{\beta}}} \tag{2}$$

with the mean $\alpha+\beta\gamma$ where $\gamma=0.5722...$ is the Euler's constant.

Over the last two decades, the queueing systems with vacations have been well studied because of their applications in modeling the computer networks, communication and manufacturing/ service systems. In the previous studies of the classical vacation queueing models, it is assumed that the server stops the primary service completely but the concept of working vacation on which the server works at the lower rate rather than completely stopping the primary service during the vacation period which was introduced by [9] and derived the probability generating function of the number of customers in the system and waiting time distribution for $M/M/1$ multiple working vacation model.

This paper mainly focuses for estimating the parameters of queueing model under statistical procedures. The problem of estimation of concerning the parameters of queueing process such as arrival rate, service rate and traffic intensity is the most important thing in the queueing systems. The pioneer investigators have derived the Maximum Likelihood(ML) estimators for the arrival and service parameters of an $M/M/1$ queueing model[1]. The Bayesian estimation for the double Pareto lognormal (dPLN) distribution which has been proposed by the model in the queueing system for the heavy-tailed phenomena in[2]. The evaluation of queueing model with the service time follows the Gumbel distribution has been explained in [6] based on the various combinations of the arbitrary values along with the different samples. Recently, the single server queueing model with working vacations is considered based on MLE approaches and simulation studies are carried out by the performance measures of the model[5].

The objective of this paper is to estimate the model parameters of queueing system with working vacation and the service time as follows Gumbel distribution based on informative and non-informative priors under Bayesian approach through MCMC technique. This paper is organized into five sections, this is being the first. Section 2 contains model descriptions. The frame work of Bayesian estimation of model parameters is presented in section 3. The simulation studies for empirical Bayesian estimates of the model are discussed in section 4 and section 5 provides the summary and conclusion of this work.

2 Model Descriptions

Consider an queueing model with working vacations and unknown mean arrival rate(λ), mean service rate for the sever when it is not in the working vacation(μ_b), and mean service rate for the server when it is working vacation(μ_v), where $\mu_b > \mu_v$. The regular service time of the system follows Gumbel distribution. The vacation service and working vacation times are exponentially distributed with rates μ_v & η respectively. If there are customers in the system, at the vacation period, instantly the server will have the new busy period. Otherwise, the server takes another working vacation. The inter-arrival time, service times and the working vacation time are mutually independent of each other. Here, consider the single stage service is provided in First In First Out(FIFO) discipline with infinite capacity.

In order to learn about the congestion of the system, the inference about the parameters governing the whole system $\theta = \{\lambda, \mu_b, \mu_v, \eta\}$ is considered. The statistical point of view, the queue is consolidated and operated for a long time which indicates that it is working at equilibrium and satisfies the ergodic condition. Note that, the ergodic assumption implies that the parameters can only move freely in the reduced parametric space $\Theta_e = \{\theta : \lambda < \mu_b, \mu_v; \mu_b > \mu_v; \lambda, \mu_b, \mu_v, \eta > 0\}$.

3 Estimation on M/Gumbel/1/WV Model

The Bayesian methodology consists of the sample information along with the prior information available about the parameter before the sample has been observed. A suitable probability distribution is determined for the model parameter for queueing system say $\tau(\theta)$ with reference to the prior information. The information about the parameter given by the sample $\underline{x}$ is obtained from the likelihood, $L(\theta|\underline{x})$. Combining these two information, the updated information about the parameter is obtained as the posterior distribution,$\tau(\theta|\underline{x})$. The inference about θ are drawn from this posterior distribution.

As mentioned earlier, consider an $M/Gumbel/1/WV$ queueing system under working vacation period. In queueing model, n_a be the number of customers in inter-arrival times, n_b be the number of customers in regular service time and n_v, number of customers in vacation service time are as follows Exponential(λ), Gumbel(α, β) and Exponential(μ_v) respectively, where n_a, n_b, n_v are fixed in advance. The working vacations occurs in n_w times which is distributed according to Exponential(η). The total service times of the queue is $n_s = n_b + n_v$. The joint probability generating function of this model is

$$f(\theta) = \lambda e^{-\lambda t_a} \frac{1}{\beta} e^{\frac{(t_s-\alpha)}{\beta}} e^{e^{\frac{(t_s-\alpha)}{\beta}}} \mu_v e^{\mu_v t_v} \eta e^{\eta t_w} \tag{3}$$

where $\theta = \{\lambda < \mu_b, \mu_v; \mu_b > \mu_v; \lambda, \mu_b, \mu_v, \eta > 0; \mu_b \in (\alpha, \beta)\}$ & $\rho = \frac{\lambda}{\mu_b+\mu v} < 1$.

From Eqn. 3, the corresponding likelihood equation as follows,

$$L(\theta) = \Pi_{i=1}^{n_a} \lambda^{n_a} e^{-\lambda t_{ai}} \Pi_{j=1}^{n_b} \frac{1}{\beta^n} e^{\frac{(t_{bj}-\alpha)}{\beta}} e^{e^{\frac{(t_{bj}-\alpha)}{\beta}}} \Pi_{k=1}^{n_v} \mu_v^{n_v} e^{\mu_v t_{vk}} \Pi_{l=1}^{n_w} \eta^{n_w} e^{\eta t_{wl}} \tag{4}$$

Note that, the restriction in the domain of the likelihood in Eqn. 4 corresponding to the ergodic condition of the queue.

The corresponding informative prior about model parameters $\lambda, \alpha, \beta, \mu_v, \eta$ of the joint density function in Eqn. 3 are considered as Gamma (a_1, b_1), Inverted Gamma(c_1, d_1), Inverted Gamma(c_2, d_2), Gamma(a_2, b_2), Gamma(a_3, b_3) respectively. And the non-informative prior information about the model parameters $\lambda, \alpha, \beta, \mu_v, \eta$ are taken as uniform distribution with the evidence of [7]. The probability density functions of Gamma and Inverted Gamma distributions are

$$\tau_1(a, b) = \frac{1}{\Gamma(b)} a^b e^{-x} x^{n-1}; a, b > 0; x = (0, \infty) \tag{5}$$

$$\tau_2(c, d) = \frac{d}{\Gamma(c)} x^{-(c+1)} e^{-c/x}; c, d > 0; x = (0, \infty) \tag{6}$$

$$\tau_3(\phi) = \frac{1}{q-p}; 0 < p \leq \phi \leq q \tag{7}$$

The combination of the likelihood Eqn. 4 and priors in Eqn. 5, 6, 7, the updated informations of the posterior distribution is obtained for the model parameters. The posterior distribution is not attained in the closed form expression then Markov Chain Monte Carlo(MCMC) simulation technique is more appropriate to deal with the empirical estimates of the model parameters.

4 Simulation Study

The MCMC method provides an alternative method for empirical parameter estimates of the model and more flexible when compared with the traditional methods. Moreover, the exact probability intervals are available rather than relying on estimates of the asymptotic variances. The samples are generated under Gibbs sampling algorithm from posterior distribution for given informative and non-informative priors for obtaining the Bayes estimates of the model and run the markov chain in R2OpenBugs for 10,000 number of iterations. From the outputs in R2OpenBugs, the Markov chain has converged in both informative and non-informative priors since it likely to be sampling from the stationary distribution and horizontal band, with no long upward or downward trends. Moreover, the autocorrelation is almost negligible for all the model parameters then the generated samples from posterior density in each iterations which are independent to each other. The kernal densities of model parameters are satisfied for the symmetric property of the posterior distribution. The Monte Carlo(MC) Error is an important one for observing the convergence which should be minimum. The posterior mean along with the corresponding minimum MC error of $M/Gumbel/1/WV$ queueing model parameters are presented in Table.[1 & 2]. Meanwhile, the empirical Bayesian estimate of traffic intensity of the model are computed from the posterior mean of corresponding parameter and it is to be observed that in Figure.[1], the stable intensity level has been maintained for the various samples 50,100,150,200,250 and various arbitrary values ($\lambda = 0.3, 0.5, 0.7, 0.9$, $\alpha = 0, 1, 0.2, 0.3, 0.4$, $\beta = 0.2, 0.3, 0.4, 0.5$, $\mu_2 = 0.15, 0.25, 0.35, 0.45$, $\&\eta = 0.5$) of the parameters.

5 Summary and Conclusion

This paper deals with the Bayesian empirical estimates of $M/Gumbel/1/WV$ queueing model with multiple working vacation under informative and non-informative priors through MCMC technique using R2OpenBugs software. The empirical bayesian estimates are presented for various samples and arbitrary values. From those results, the traffic intensities are observed which it is increased when the inter-arrival time and service time are increased but the sample sizes are not increased. Meanwhile, the stable intensity level has been maintained when the samples and arbitrary values are increased in both priors. Hence, this model is appropriate for studying the service time that follows the heavy tailed particularly, Gumbel distribution. The developed model has the applications in teletraffic communication networks for the controlling the congestion of the system.

Table 1: Empirical Bayesian Estimates for Model Parameters under Informative Prior

Parameters	*Samples*	$\hat{\lambda}$	*MC.E*	$\hat{\alpha}$	*MC.E*	$\hat{\beta}$	*MC.E*	$\hat{\mu}_2$	*MC.E*	$\hat{\eta}$	*MC.E*
$\lambda = 0.3$,	50	0.2340	0.00034	0.1641	0.00024	0.1343	0.00025	0.1336	0.00035	0.4744	0.00127
$\alpha = 0.1$,	100	0.3233	0.00030	0.1864	0.00022	0.1482	0.00018	0.2035	0.00033	0.5172	0.00099
$\beta = 0.2$,	150	0.3015	0.00026	0.1966	0.00020	0.1844	0.00015	0.1492	0.00021	0.5809	0.00120
$\mu_2 = 0.15$,	200	0.2915	0.00021	0.2004	0.00017	0.1826	0.00015	0.1572	0.00019	0.6073	0.00103
$\eta = 0.5$	250	0.3193	0.00022	0.1889	0.00016	0.1644	0.00010	0.1415	0.00014	0.5061	0.00092
$\lambda = 0.5$,	50	0.4541	0.00056	0.3403	0.00061	0.3513	0.00055	0.3001	0.00064	0.4762	0.00137
$\alpha = 0.2$,	100	0.5015	0.00046	0.3128	0.00043	0.2838	0.00034	0.2425	0.00037	0.5147	0.00120
$\beta = 0.3$,	150	0.4557	0.00030	0.3032	0.00027	0.2442	0.00021	0.2678	0.00041	0.5824	0.00117
$\mu_2 = 0.25$,	200	0.4860	0.00033	0.3126	0.00026	0.2906	0.00023	0.2459	0.00029	0.6093	0.00102
$\eta = 0.5$	250	0.4874	0.00032	0.2943	0.00018	0.2344	0.00015	0.2571	0.00025	0.5068	0.00095
$\lambda = 0.7$,	50	0.6200	0.00067	0.3957	0.00068	0.3735	0.00065	0.5518	0.00110	0.4750	0.00136
$\alpha = 0.3$,	100	0.6945	0.00070	0.4229	0.00050	0.3949	0.00039	0.4003	0.00066	0.5162	0.00130
$\beta = 0.4$,	150	0.6776	0.00048	0.4267	0.00039	0.3780	0.00032	0.3494	0.00048	0.5790	0.00095
$\mu_2 = 0.35$,	200	0.6862	0.00044	0.4567	0.00037	0.3652	0.00027	0.3443	0.00040	0.6088	0.00113
$\eta = 0.5$	250	0.6876	0.00044	0.3825	0.00028	0.2983	0.00021	0.3930	0.00043	0.5057	0.00081
$\lambda = 0.9$,	50	0.7699	0.00074	0.5037	0.00085	0.5348	0.00081	0.4411	0.00098	0.4751	0.00127
$\alpha = 0.4$,	100	0.7893	0.00059	0.551	0.00067	0.4553	0.00058	0.4240	0.00065	0.5175	0.00113
$\beta = 0.5$,	150	0.8530	0.00045	0.5559	0.00050	0.4348	0.00037	0.5559	0.00077	0.5830	0.00112
$\mu_2 = 0.45$,	200	0.8198	0.00046	0.4915	0.00041	0.4394	0.00034	0.4842	0.00059	0.6098	0.00104
$\eta = 0.5$	250	0.8834	0.79830	0.5172	0.00040	0.4214	0.00026	0.4338	0.00053	0.5060	0.00083

Table 2: Empirical Bayesian Estimates for Model Parameters under Non-informative Prior

Parameters	*Samples*	$\hat{\lambda}$	*MC.E*	$\hat{\alpha}$	*MC.E*	$\hat{\beta}$	*MC.E*	$\hat{\mu}_2$	*MC.E*	$\hat{\eta}$	*MC.E*
$\lambda = 0.3$,	50	0.2293	0.00031	0.1566	0.00027	0.1282	0.00024	0.1212	0.00029	0.5564	0.00127
$\alpha = 0.1$,	100	0.3217	0.00028	0.1825	0.00019	0.1454	0.00018	0.1968	0.00032	0.5959	0.00170
$\beta = 0.2$,	150	0.3000	0.00024	0.1940	0.00021	0.1824	0.00016	0.145	0.00021	0.6813	0.00145
$\mu_2 = 0.15$,	200	0.2904	0.00022	0.1987	0.00018	0.1808	0.00014	0.1544	0.00017	0.7052	0.00153
$\eta = 0.5$	250	0.3191	0.00019	0.1875	0.00014	0.1632	0.00012	0.1389	0.00015	0.5340	0.00105
$\lambda = 0.5$,	50	0.4594	0.00065	0.3343	0.00060	0.3484	0.00065	0.2895	0.00069	0.5596	0.00203
$\alpha = 0.2$,	100	0.5066	0.00053	0.3092	0.00033	0.2804	0.00034	0.2364	0.00040	0.5975	0.00156
$\beta = 0.3$,	150	0.4573	0.00040	0.3013	0.00028	0.2423	0.00019	0.2629	0.00033	0.6815	0.00142
$\mu_2 = 0.25$,	200	0.4876	0.00030	0.3110	0.00029	0.2891	0.00026	0.2427	0.00029	0.7068	0.00108
$\eta = 0.5$	250	0.4895	0.00027	0.2927	0.00020	0.2334	0.00017	0.2543	0.00024	0.5328	0.00103
$\lambda = 0.7$,	50	0.6613	0.00083	0.3962	0.00075	0.3738	0.00072	0.6359	0.00152	0.5603	0.00196
$\alpha = 0.3$,	100	0.7348	0.00074	0.4245	0.00055	0.3950	0.00051	0.4022	0.00058	0.5970	0.00167
$\beta = 0.4$,	150	0.6995	0.00056	0.4264	0.00044	0.3778	0.00034	0.3469	0.00048	0.6806	0.00155
$\mu_2 = 0.35$,	200	0.7048	0.00048	0.4581	0.00038	0.3652	0.00029	0.3424	0.00040	0.7071	0.00146
$\eta = 0.5$	250	0.7012	0.00046	0.3822	0.00029	0.2977	0.00021	0.3924	0.00042	0.5335	0.00096
$\lambda = 0.9$,	50	0.8762	0.00079	0.5345	0.00116	0.5644	0.00102	0.4632	0.00104	0.5610	0.00186
$\alpha = 0.4$,	100	0.8652	0.00073	0.5628	0.00067	0.4604	0.00058	0.4285	0.00075	0.5974	0.00178
$\beta = 0.5$,	150	0.9254	0.00053	0.5637	0.00044	0.4387	0.00041	0.5773	0.00079	0.6820	0.00142
$\mu_2 = 0.45$,	200	0.8739	0.00047	0.4948	0.00046	0.4417	0.00037	0.4907	0.00052	0.7062	0.00145
$\eta = 0.5$	250	0.9435	0.00035	0.5209	0.00039	0.4228	0.00027	0.4347	0.00048	0.5343	0.00108

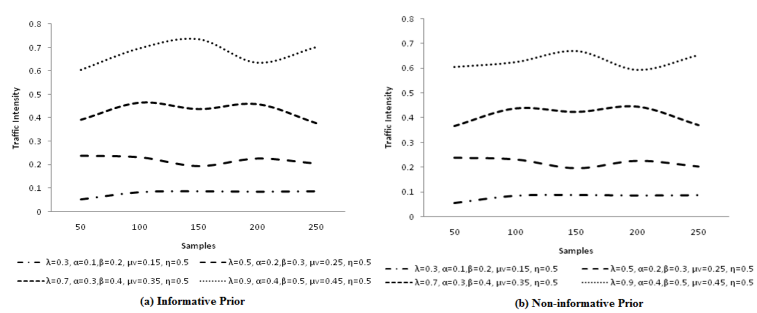

Figure 1: Empirical Estimate for Traffic Intensity of M/Gumbel/1/WV Model

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