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# Stationary Analysis of a Fluid Queue Driven by a Two Processor Computer System

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**Abstract**

In this paper, we analyse a fluid queue modulated by a computer system with two processors. The modulating process is modelled as a quasi birth and death process and the steady state probabilities are determined by standard methods. Further, for the fluid model the explicit expression for the joint steady state distribution of the content of the buffer and the number of tasks in the computer system is presented using matrix analytic methods as a tool to solve the underlying system of governing equations.

**Keywords:** Quasi Birth and Death Process, Matrix Analytic Method, Laplace Transform, Steady State Probabilities, Buffer Content Distribution

# 1 Introduction

Fluid queues has became a fascinating area of research, in recent years due to its wide spread applicability in computer and communication systems [1, 3], manufacturing systems [6] etc. A stochastic fluid flow model is an input-output system where the input is modelled as a continuous fluid that enters and leaves the storage devices called a buffer according to randomly varying rates. In these models, the fluid buffer is either filled or depleted or both at rates determined by the current state of the background Markov process. *Markov Modulated Fluid Queues* are particular class of fluid models useful for modelling many physical phenomenon and they often allow tractable analysis [2].Certain interesting real world applications of Markov Modulated Fluid Flow models can be found in [9] and [12]. Besides, fluid queues have successful applications in the field of congestion control [10] and risk processes [7].

In recent years, Silver *etal* [8] expressed the stationary distribution of a fluid queue with finite buffer as a linear combination of matrix exponential terms using matrix analytic methods. Fluid models driven by an $M/M/1/N$ queue with single and multiple exponential vacations was recently studied by Mao *etal* [4, 5] using spectral method. Xu *etal* [11] analyzed a fluid queue driven by a multi server queueing model with working vacation and presented a matrix geometric solution method for the stationary buffer content distribution.

## 2 Model Description

Consider a fluid queue model driven by a computer system with two processor which can also be viewed as a special case of an $M/M/1$ queueing model with heterogenous service. The modulating process consists of two processors namely, a main processor and a backup processor. Jobs arrive according to a Poisson process with rate $\lambda$ to the main processor. When the main processor fails the system immediately switches over to the backup processor. It is assumed that the backup processor operates at a slower rate than the main processor. Tasks executed on the main processor are assumed to be exponential with rate $\mu_1$, whereas for the backup processor the rate reduces to $\mu_2$ where $\mu_2 < \mu_1$. The failures to the main processor are assumed to occur at the rate of $\gamma$ and the main processor is repaired immediately, when the system empties.The state transition diagram for the model is given by Figure 1. Let $\{N(t), t \geq 0\}$ denote the number of jobs in the system at time $t$. Define,

$$J(t) = \begin{cases} 1, & \text{if the job is with the main processor.} \\ 2, & \text{if the job is with the backup processor.} \\ 0, & \text{if the system is empty.} \end{cases}$$

It is well known the $\{(N(t), J(t)), t \geq 0\}$ is a Markov process with the state space

$$\Omega = \{(0,0) \bigcup (k,j), k = 1, 2, \cdots, j = 1, 2\}.$$

Using the lexicographical for the states, the infinitesimal generator $Q$ can be written in the form of block tridiagonal matrix as

$$Q = \begin{pmatrix} A_0 & C_0 & & & \\ B_0 & A_1 & C_1 & & \\ & B_1 & A_1 & C_1 & \\ & \ddots & \ddots & \ddots & \end{pmatrix}$$

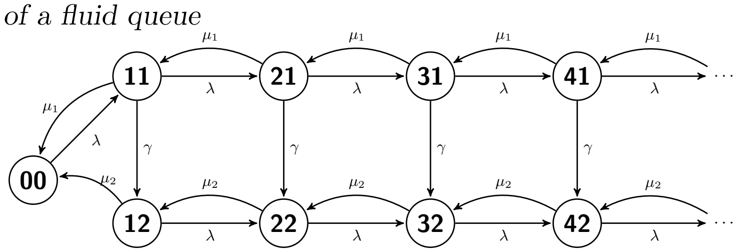


Figure 1: State Transition Diagram

where $A_0 = (-\lambda)$, $C_0 = \begin{pmatrix} \lambda & 0 \end{pmatrix}$, $B_0 = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $A_1 = \begin{pmatrix} -(\lambda+\mu_1+\gamma) & \gamma \\ 0 & -(\lambda+\mu_2) \end{pmatrix}$,
$C_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, and $B_1 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$.

Let $(N, J)$ be the stationary random vector of the process $\{(N(t), J(t)), t \geq 0\}$ with the corresponding stationary distribution given by $\pi_{k,j} = P\{N = k, J = j\} = lim_{t\to\infty} P\{N(t) = k, J(t) = j\}, (k, j) \in \Omega$.

$$\text{Define} \qquad \pi_0 = \pi_{0,0}, \quad \pi_k = (\pi_{k,1}, \pi_{k,2})^T, k \geq 1, \quad \pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, ....)^T$$

Then, the system of Kolmogrov equation satisfied by the steady state probability vector $\pi$ can be written as $\pi^{\mathbf{T}}\mathbf{Q} = \mathbf{0}$.

**Theorem 1**

The steady state probability distribution $\pi_k, k \geq 0$ are given by

$$\pi_0 = \frac{(1-r)^2(\mu_2-\lambda)}{\gamma r + (1-r)(\mu_2-\lambda)} \quad \text{and} \quad \pi_k = R\pi_{k-1} = \pi_0 R^k e_1 \quad k = 1, 2, 3, \cdots$$

$$\text{where} \qquad r = \frac{1}{2\mu_1}(\lambda + \mu_1{}^{1}+ \gamma - \sqrt{(\lambda+\mu_1+\gamma)^2 - 4\lambda\mu_1}),$$

and $e_1 = (1 \quad 0)^T$ subject to the stability condition $\rho < 1$.

The rate matrix $R$ is given by $R = \begin{pmatrix} r_{11} & 0 \\ r_{21} & r_{22} \end{pmatrix}$ with $r_{11} = r, r_{22} = \rho$ and $r_{21} = \frac{\gamma r}{\mu(1-r)}$.

# 3 Analysis of fluid model

Consider a fluid model driven by a two processor computer system. Let $C(t)$ denote the content of the buffer at time $'t'$. Furthermore, it is assumed that the content of the buffer increases at the rate of $\sigma$ when there are jobs being processed in the background queueing model, while the buffer content decreases at the rate $\sigma_0$ when the processors are empty. The dynamics of the buffer content process is given by

$$\frac{dC(t)}{dt} = \begin{cases} 0, & N(t) = 0, C(t) = 0 \\ \sigma_0, & N(t) = 0, C(t) > 0. \\ \sigma, & N(t) > 0. \end{cases}$$

where $\sigma_0 < 0$ and $\sigma > 0$. Clearly the 3-dimensional stochastic process $\{N(t), J(t), C(t)\}$ represent a fluid queue modulated by a Markov process $\{N(t), J(t)\}$ subject to the stability condition given by $d = \sigma_0\pi_0 + \sigma\sum_{k=1}^{\infty}\pi_{k,1} + \sigma\sum_{k=1}^{\infty}\pi_{k,2} < 0$.
Define the joint probability distribution functions of the Markov process $\{(N(t), J(t), C(t)), t \geq 0\}$ at time $t$ as $F_{k,j}(t,w) = Pr\{N(t) = k, J(t) = j, C(t) \leq w\}$. When the process $\{(N(t), J(t), C(t)), t \geq 0\}$ is stable, its stationary random vector is denoted by $(N, J, C)$, Under steady state conditions, let

$$\begin{aligned} F_{k,j}(w) &= \lim_{t\to\infty} Pr\{N(t) = k, J(t) = j, C(t) \leq w\} \\ &= Pr\{N = k, J = j, C \leq w\}, \qquad w > 0, (k,j) \in \Omega, \end{aligned}$$

For convenience, we introduce the matrices $F(w) = \begin{pmatrix} F_0(w) \\ F_1(w) \\ F_2(w) \\ \vdots \end{pmatrix}$, $\Lambda = \begin{pmatrix} \sigma_0 & & & \\ & \Sigma & & \\ & & \Sigma & \\ & & & \ddots \end{pmatrix}$

where $F_0(w) = F_{0,0}(w)$, $F_k(w) = \begin{pmatrix} F_{k,1}(w) \\ F_{k,2}(w) \end{pmatrix}, k \geq 1$ and $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$.

Then the system of differential difference equation can be written in the matrix form as

$$\Lambda \frac{dF(w)}{dw} = Q^T F(w) \tag{3.1}$$

$$\text{with boundary condition } F_0(0) = a \text{ and } F_k(0) = 0, k = 1, 2, 3, \cdots \tag{3.2}$$

The constant $a$ such that $0 < a < 1$ is an unknown to be determined. Let $\hat{F}_0(s)$ and $\hat{F}_k(s)$ denote the Laplace transform of the functions $F_0(w)$ and $F_k(w)$ for $k = 1, 2, 3, \cdots$ respectively. Taking the Laplace transforms on both sides of equation (3.1) and using boundary condition equation (3.2), leads to

$$(Q^T - s\Lambda)\hat{F}(s) = -\Lambda F(0) = \begin{pmatrix} -a\sigma_0 \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \tag{3.3}$$

# 4 Steady State Distribution of the Buffer Content

This section present a explicit expression for the stationary distribution of the buffer content process.

**Lemma 2** If $\rho < 1$, the matrix quadratic equation $B_1 R(s)^2 + (A_1^T - s\Sigma)R(s) + C_1 = 0$ has the minimal nonnegative solutions given by $R(s) = \begin{pmatrix} r_0(s) & 0 \\ \delta(s) & z_0(s) \end{pmatrix}$

$$\text{where}\quad r_0(s) = \frac{1}{2\mu_1}(\lambda + \mu_1 + \gamma + s\sigma - \sqrt{(\lambda + \mu_1 + \gamma + s\sigma)^2 - 4\lambda\mu_1}) \quad (4.4)$$

$$z_0(s) = \frac{1}{2\mu_2}(\lambda + \mu_2 + s\sigma - \sqrt{(\lambda + \mu_2 + s\sigma)^2 - 4\lambda\mu_2}) \quad (4.5)$$

$$\text{and}\quad \delta(s) = \frac{\gamma r_0(s)}{(\lambda + \mu_2 + s\sigma) - \mu_2\{r_0(s) + z_0(s)\}}. \quad (4.6)$$

**Proof:** Since $B_1$ , $A_1^T - s\Sigma$ and $C_1$ are all lower triangular matrices, we can assume that the solution $R$ has the same structure as $R(s) = \begin{pmatrix} r_{11}(s) & 0 \\ r_{21}(s) & r_{22}(s) \end{pmatrix}$. which upon substituting in given matrix quadratic equation yields,

$$\begin{aligned} \mu_1 r_{11}^2(s) - (\lambda + \mu_1 + \gamma + s\sigma) r_{11}(s) + \lambda = 0 \\ \mu_2 r_{22}^2(s) - (\lambda + \mu_2 + s\sigma) r_{22}(s) + \lambda = 0 \\ \mu_2(r_{11}(s)r_{21}(s) + r_{21}(s)r_{22}(s)) - (\lambda + \mu_2 + s\sigma)r_{21}(s) + \gamma r_{11}(s) = 0 \end{aligned} \quad (4.7)$$

Solving the system of equations (4.7) and using the relation

$s\sigma + \lambda + \mu_2(1 - z_0(s)) = \dfrac{\lambda}{z_0(s)} = \mu_2 + \dfrac{s\sigma}{1 - z_0(s)}$, which gives the equations (4.4), (4.5) and (4.6).

It is seen that $R(0) = R$. Note that, $R^k(s) = \begin{pmatrix} r_0(s)^k & 0 \\ \delta(s)\sum_{i=1}^{k} r_0(s)^{k-i} z_0(s)^{i-1} & z_0(s)^k \end{pmatrix}$
This completes the proof.

**Theorem 2**
Assuming $\rho < 1$ and $d < 0$, the Laplace transform of the stable joint probability distribution functions, $\hat{F}_0(s)$ and $\hat{F}_k(s), k = 1, 2, 3\cdots$ satisfies the relations

$$\hat{F}_k(s) = R(s)\hat{F}_{k-1}(s) = \hat{F}_0(s)R^k(s)e_1 \quad k \geq 1 \quad (4.8)$$

$$\hat{F}_0(s) = \frac{a\sigma_0}{s\sigma_0 - A_0 - B_0^T R(s)e_1} \quad \text{where} \quad e_1 = (1 \quad 0)^T \quad (4.9)$$

Further $\hat{F}_0(s)$ can be simplified as, $\hat{F}_0(s) = \dfrac{a\sigma_0}{s\sigma_0 + \lambda - \{\mu_1 r_0(s) + \mu_2 z_0(s)\}}$ (4.10)

**Proof** Observe that equation (3.3) can be written equivalently as the following system of difference equations

$$\begin{aligned} (A_0 - s\sigma_0)\hat{F}_0(s) + B_0^T \hat{F}_1(s) &= -a\sigma_0, \\ C_0^T \hat{F}_0(s) + (A_1^T - s\Sigma)\hat{F}_1(s) + B_1\hat{F}_2(s) &= 0, \\ C_1\hat{F}_{k-1}(s) + (A_1^T - s\Sigma)\hat{F}_k(s) + B_1\hat{F}_{k+1}(s) &= 0 \qquad k = 2, 3, \cdots. \end{aligned} \quad (4.11)$$

When $\rho < 1$, the 3-dimensional Markov process $\{(N(t), J(t), C(t)), t \geq 0\}$ has a unique stationary probability distribution. Therefore there exists a unique solution to the above system of difference equations. Below, we verify that equation (4.8) satisfies the system of equations given by equation (4.11). For $k \geq 2$, substituting the equation (4.8) in equation (4.11) yields,

$$\begin{aligned} C_1\hat{F}_{k-1}(s) + (A_1^T - s\Sigma)\hat{F}_k(s) + B_1\hat{F}_{k+1}(s) &= [C_1 + (A_1^T - s\Sigma)R(s) + B_1R^2(s)]\hat{F}_{k-1}(s) \\ &= 0 \quad \text{(by Lemma 2).} \end{aligned}$$

Similarly way its true for $k = 1$. Finally, the first equation of the simultaneous equations given by equation (4.11) yields,

$$(A_0 - s\sigma_0)\hat{F}_0(s) + B_0^T\hat{F}_0(s)R(s)e_1 = -a\sigma_0,$$

which upon simplification leads to the (4.9). Substituting for the matrices $A_0$, $B_0^T$ and $R(s)$ in the equation (4.9) yields equation (4.10).
This completes the proof.
Having determined all the joint steady state probabilities in the Laplace domain, we now present the explicit analytical solution by inverting using transform techniques. Now taking laplace transform of the equations (4.8) and (4.9) leads to

$$\begin{aligned} F_0(w) &= a\left[\sum_{k=0}^{\infty}\frac{1}{\sigma_0^k}e^{-\frac{\lambda}{\sigma_0}w}\frac{w^k}{k!} * \left[\frac{\sigma}{2}e^{-(\frac{\lambda+\mu_1+\gamma}{\sigma})w}\frac{I_1(\alpha w)\alpha}{w} + \mu_2\delta(w)\right]^{*k}\right], \\ F_{k,1}(w) &= F_0(w) * \left(\frac{\sigma}{2\mu_1}\right)^k e^{-(\frac{\lambda+\mu_1+\gamma}{\sigma})w}\frac{kI_k(\alpha w)\alpha^k}{w} \quad \text{for} \quad k = 1, 2, \cdots, \end{aligned}$$

and

$$\begin{aligned} F_{k,2}(w) = \gamma\Bigg[F_0(w) * \sum_{j=0}^{\infty}\sum_{i=1}^{k}\frac{\mu_2^j}{\lambda^{j+1}}\left(\frac{\sigma}{2\mu_1}\right)^{k+j-i+1} e^{-(\frac{\lambda+\mu_1+\gamma}{\sigma})w}\frac{(k+j-i+1)I_{k+j-i+1}(\alpha w)\alpha^{k+j-i+1}}{w} \\ * \left(\frac{\sigma}{2\mu_2}\right)^{i+j} e^{-(\frac{\lambda+\mu_2}{\sigma})w}\frac{(i+j)I_{i+j}(\beta w)\beta^{i+j}}{w}\Bigg] \quad \text{for} \quad k = 1, 2, \cdots, \end{aligned}$$

where

$$\begin{aligned} \delta(w) = \gamma\Bigg[\sum_{i=0}^{\infty}\frac{\mu_2^i}{\lambda^{i+1}}\left(\frac{\sigma}{2\mu_1}\right)^{i+1} e^{-(\frac{\lambda+\mu_1+\gamma}{\sigma})w}\frac{(i+1)I_{i+1}(\alpha w)\alpha^{i+1}}{w} \\ * \left(\frac{\sigma}{2\mu_2}\right)^{i+1} e^{-(\frac{\lambda+\mu_2}{\sigma})w}\frac{(i+1)I_{i+1}(\beta w)\beta^{i+1}}{w}\Bigg] \qquad (4.12) \end{aligned}$$

$\alpha = 2\sqrt{\lambda\mu_1}$ and $\beta = 2\sqrt{\lambda\mu_2}$. Thus all the joint steady state probabilities of the state of the system and the content of the buffer in terms of modified Bessel's function of first kind.

It still remains to determine the constant $a$ which represents $F_0(0)$. Towards this end, adding the governing systems of equations for all $(k, j) \in \Omega$ and integrating from zero to infinity gives the constant $a$.

Therefore, the constant $a$ is explicitly given by

$$F_0(0) = a = \frac{(\sigma_0 - \sigma)\pi_0 + \sigma}{\sigma_0},$$

**Theorem 3**
If the fluid queue is stable (i.e., $\rho < 1$ and $d < 0$), then the stationary distribution of the content of the buffer is given by

$$\begin{aligned} H(w) &= P\{C \le w\} = F_{0,0}(w) + \sum_{k=1}^{\infty}\sum_{j=1}^{2} F_{k,j}(w) \\ H(w) &= F_0(w) * \left[\sum_{i=0}^{\infty}\left(\frac{\sigma}{2\mu_1}\right)^i e^{-(\frac{\lambda+\mu_1+\gamma}{\sigma})w}\frac{iI_i(\alpha w)\alpha^i}{w} + \delta(w) * \sum_{i=0}^{\infty}\left(\frac{\sigma}{2\mu_1}\right)^i e^{-(\frac{\lambda+\mu_1+\gamma}{\sigma})w}\right. \\ &\quad \left.\frac{iI_i(\alpha w)\alpha^i}{w} * \sum_{j=0}^{\infty}\left(\frac{\sigma}{2\mu_2}\right)^j e^{-(\frac{\lambda+\mu_2}{\sigma})w}\frac{jI_j(\beta w)\beta^j}{w}\right] \end{aligned}$$

where $\delta(w)$ is given in equation (4.12)

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