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Fekete-Szegö Coefficient for a General Class of Spirallike Functions in Unit Disk

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Abstract

The aim of this paper is to give sharp bound for the Fekete-Szegö coefficient functional of normalized analytic functions $f(z)$ defined on the open unit disk in the complex plane are derived for the general class of spirallike functions in unit disk.

Mathematics Subject Classification: 30C45

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1 Introduction

Let $\mathcal{A}_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1.1}$$

defined on the open unit disk

$$\mathbb{U} = \{z \ : \ z \in \mathbb{C} \ : \ |z| < 1\},$$

and let $\mathcal{A}_1 := \mathcal{A}$. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

With a view to recalling the principle of subordination between analytic functions, let the functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}),$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the open unit disk $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Following Robertson[16], we introduce below the familiar class of starlike functions of order α in $\mathbb{U}$.

Definition 1.1. *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also Let $\alpha \in [0, 1)$ and*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \tag{1.2}$$

We then say that the function $f(z)$ is starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike function of order α in $\mathbb{U}$.

Spaček[20] extended the class of starlike functions by introducing the class of spirallike functions of type β in $\mathbb{U}$ and gave the following analytical characterization of spirallikeness functions of type β in $\mathbb{U}$.

Theorem 1.2. *(see Spaček[20]) Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f(z)$ is a spirallike function of type β in $\mathbb{U}$ if and only if*

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}; -\frac{\pi}{2} < \beta < \frac{\pi}{2}). \tag{1.3}$$

Henceforth we denote the class of spirallike functions of type β in $\mathbb{U}$ by $\mathcal{S}_\beta$.

Libera[8] unified and extended the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{S}_\beta$ by introducing the analytic function class $\mathcal{S}_\alpha^\beta$ in $\mathbb{U}$ as follows.

Definition 1.3. *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also Let $\alpha \in [0, 1)$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We then say that $f \in \mathcal{S}_\alpha^\beta$ if and only if*

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \quad (z \in \mathbb{U}; 0 \le \alpha < 1; -\frac{\pi}{2} < \beta < \frac{\pi}{2}) \tag{1.4}$$

Obviously, we find from Definition 1.2 and 1.3 that $\mathcal{S}_\alpha^0 = \mathcal{S}^*(\alpha)$ and $\mathcal{S}_0^\beta = \mathcal{S}_\beta$ Qing-Hua Xu et al.[13] gave the following Definition.

Definition 1.4. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^\beta(A, B)$ if it satisfies the following subordination condition:*

$$(1 + i\tan\beta)\left(\frac{zf'(z)}{f(z)}\right) - i\tan\beta \prec \frac{1 + Az}{1 + Bz} \tag{1.5}$$

$(z \in \mathbb{U}; -\frac{\pi}{2} < \beta < \frac{\pi}{2})$.

Remark 1.5. *For suitable choices of A and B, $\mathcal{S}^\beta(A, B)$ provides interesting sub classes of the class of analytic functions. For example, by setting*

$$A = 1 - 2\alpha \quad (0 \le \alpha < 1) \quad \textit{and} \quad B = -1$$

in Definition 1.4, we easily observe that the general function class $\mathcal{S}^\beta(A, B)$ becomes the function class $\mathcal{S}_\alpha^\beta$, which is involved in Definition 1.3.

Remark 1.6. *After some simplification of (1.5), we get*

$$e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right) = q(z).\cos\beta + i\sin\beta$$

where $q(z) \in \mathcal{P}(A, B)$, the family of functions $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ regular in $\mathbb{U}$ if and only if $q(z) = \frac{1+A\phi(z)}{1-B\phi(z)}$, $-1 \le B < A \le 1$. Yasar Polatoglu[10] proved the following

$$f(z) \in S_\alpha^*(A, B) \Leftrightarrow \frac{zf'(z)}{f(z)} - 1 \prec \begin{cases} \frac{e^{-i\alpha}(A-B)cos\alpha z}{1+Bz} & \textit{if } B \neq 0, \\ e^{-i\alpha}(Acos\alpha)z & \textit{if } B = 0 \end{cases} \tag{1.6}$$

also Annamalai et al.[2] found the Fekete-Szegö coefficient for (1.6).

Definition 1.7. *Let $\phi(z)$ be a univalent starlike function function with respect to 1 which maps the open unit disk $\mathbb{U}$ onto a region in the half-plane and is symmetric with respect to the real axis, $\phi(0) = 1$, $\phi'(0) > 0$. A function $f \in \mathcal{A}_p$ is in the class $\mathcal{S}_{p,b}^{\beta}(\phi)$ if*

$$1 + \frac{1}{b}\left[(1 + i\tan\beta)\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) - i\tan\beta - 1\right] \prec \phi(z) \tag{1.7}$$

$(z \in \mathbb{U}; b \neq 0; -\frac{\pi}{2} < \beta < \frac{\pi}{2})$.

The class $\mathcal{S}_{p,b}^{\beta}(\phi)$ reduces to the following earlier classes:

1. $\mathcal{S}_{p,b}^{0}(\phi) = \mathcal{S}_{p,b}^{*}(\phi)$ introduced and studied by Ali et al.[1].

2. For $\phi(z) = \frac{1+Az}{1+Bz}$, the class $\mathcal{S}_{1,1}^{\beta}(\phi)$ reduces to $\mathcal{S}_{A,B}^{\beta}(\phi)$ introduced and studied by Qing-Hua Xu et al.[13].

3. For $\phi(z) = \frac{1+Az}{1+Bz}$, $\alpha = 0$, the class $\mathcal{S}_{p,1}^{0}(\phi)$ reduces to $\mathcal{S}(A, B, p, \alpha)$ introduced and studied by M.K.Aouf in [3].

4. For $\phi(z) = \frac{1+Az}{1+Bz}$, the class $\mathcal{S}_{1,1}^{0}(\phi)$ reduces to $\mathcal{S}(A, B)$ introduced and studied by Janowski in [6].

5. For $\phi(z) = \frac{1+Az}{1+Bz}$, $A = 1$, $B = -1$, the class $\mathcal{S}_{p,1}^{0}(\phi)$ reduces to $\mathcal{S}_{\alpha}(p)$ introduced and studied by Golzina in [5].

6. For $\phi(z) = \frac{1+Az}{1+Bz}$, $\alpha = 0$, the class $\mathcal{S}_{p,1}^{0}(\phi)$ reduces to $\mathcal{S}_{p}^{\lambda}(A, B, b)$ introduced and studied by Ganesan in [4].

7. For $\phi(z) = \frac{1+Az}{1+Bz}$, $B = 0$ the class $\mathcal{S}_{1,b}^{0}(\phi)$ reduces to $\mathcal{S}_{A,b}$ introduced and studied by Silverman in [19].

8. For $\phi(z) = \frac{1+Az}{1+Bz}$,$\beta = \lambda$ the class $\mathcal{S}_{1,b}^{\beta}(\phi)$ reduces to $\mathcal{S}_{p}^{\lambda}(A, B, b)$ introduced and studied by Polatoglu et al. in [11].

9. For $\phi(z) = \frac{1+Az}{1+Bz}$, $A = 1 - 2\gamma$, $B = 1$, the class $\mathcal{S}_{p,b}^{\beta}(\phi)$ reduces to

$$1 + \frac{1}{b}\left[(1 + i\tan\beta)\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) - i\tan\beta - 1\right] \prec \frac{1 + (1 - 2\gamma)z}{1 - z}$$

There are various sub class of Spirallike function related to the study of Geometric Function Theory, one of these class of spirallike function in multivanent function which generalised and studied by Ramachandran et al.[14], Ramachandran and Swaminathan [15], Shanmugam et al.[18] and Selvaraj et al.[17]. Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \tag{1.8}$$

in the open unit disk $\mathbb{U}$ satisfying $|w(z)| < 1$. To prove our main result, we need the following.

Lemma 1.8. *[1] If $w \in \Omega$, then*

$$\left|w_2 - tw_1^2\right| \leq \begin{cases} -t & if \quad t < -1 \\ 1 & if \quad -1 \leq t \leq 1 \\ t & if \quad t > 1. \end{cases} \tag{1.9}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\lambda + z}{1 + \lambda z} \qquad (0 \leq \lambda \leq 1) \tag{1.10}$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\lambda + z}{1 + \lambda z} \qquad (0 \leq \lambda \leq 1) \tag{1.11}$$

or one of its rotations.

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$\begin{aligned} &|w_2 - tw_1^2| + (1+t)\left|w_1\right|^2 \leq 1 \qquad (-1 < t \leq 0), \\ &|w_2 - tw_1^2| + (1-t)\left|w_1\right|^2 \leq 1 \qquad (0 < t < 1). \end{aligned} \tag{1.12}$$

Lemma 1.9. *[7] If $w \in \Omega$, then for any complex number t,*

$$\left|w_2 - tw_1^2\right| \leq \max\{1; |t|\}. \tag{1.13}$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

Lemma 1.10. *[12] If $w \in \Omega$, then for any real numbers q_1 and q_2 the following sharp estimate holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leq H(q_1, q_2) \tag{1.14}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \textit{for } (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| & \textit{for } (q_1,q_2) \in \cup_{k=3}^{7} D_k \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \textit{for } (q_1,q_2) \in D_8 \cup D_9 \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \textit{for } (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\} \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \textit{for } (q_1,q_2) \in D_{12}. \end{cases} .$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \qquad w(z) = z, \qquad w(z) = w_0(z) = \frac{(z[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z)=w_1(z)=\frac{z(t_1-z)}{1-t_1z},\ w(z)=w_2(z)=\frac{z(t_2+z)}{1+t_2z},$$

$$|\varepsilon_1|=|\varepsilon_2|=1,\ \varepsilon_1=t_0-e^{\frac{-i\theta_0}{2}}(a\mp b),\ \varepsilon_2=-e^{\frac{-i\theta_0}{2}}(ia\pm b),$$

$$a=t_0\cos\frac{\theta_0}{2},\ b=\sqrt{1-t_0^2\sin^2\frac{\theta_0}{2}}\quad \lambda=\frac{b\pm a}{2b},$$

$$t_0=\left[\frac{2q_2(q_1^2+2)-3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}},\quad t_1=\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2=\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}},\quad \cos\frac{\theta_0}{2}=\frac{q_1}{2}\left[\frac{q_2(q_1^2+8)-2(q_1^2+2)}{2q_2(q_1^2+2)-3q_1^2}\right].$$

The sets $D_k,\ k=1,2,\cdots,12,$ are defined as follows:

$$D_1=\left\{(q_1,\,q_2):|q_1|\le\frac{1}{2},\,|q_2|\le 1\right\},$$

$$D_2=\left\{(q_1,\,q_2):\frac{1}{2}\le|q_1|\le 2,\,\frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\le|q_2|\le 1\right\},$$

$$D_3=\left\{(q_1,\,q_2):|q_1|\le\frac{1}{2},\,|q_2|\le -1\right\},$$

$$D_4=\left\{(q_1,\,q_2):|q_1|\ge\frac{1}{2},\,|q_2|\le -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5=\{(q_1,\,q_2):|q_1|\le 2,\,|q_2|\ge 1\},$$

$$D_6=\left\{(q_1,\,q_2):2\le|q_1|\le 4,\,|q_2|\ge\frac{1}{12}(q_1^2+8)\right\},$$

$$D_7=\left\{(q_1,\,q_2):|q_1|\ge 4,\,|q_2|\ge\frac{2}{3}(|q_1|-1)\right\},$$

$$D_8=\left\{(q_1,\,q_2):\frac{1}{2}\le|q_1|\le 2,\,-\frac{2}{3}(|q_1|+1)\le|q_2|\le\frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\right\},$$

$$D_9=\left\{(q_1,\,q_2):|q_1|\ge 2,\,-\frac{2}{3}(|q_1|+1)\le|q_2|\le\frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10}=\left\{(q_1,\,q_2):2\le|q_1|\le 4,\,\frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le|q_2|\le\frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11}=\left\{(q_1,\,q_2):|q_1|\ge 4,\,\frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le|q_2|\le\frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|-4}\right\},$$

$$D_{12}=\left\{(q_1,\,q_2):|q_1|\ge 4,\,\frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|-4}\le|q_2|\le\frac{2}{3}(|q_1|-1)\right\}.$$

2 Coefficient bounds

Using the Lemmas 1.8 - 1.10, we prove the following bounds for the class $\mathcal{S}^{\beta}_{p,b}(\phi)$.

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *and*

$$\begin{aligned} \sigma_1 &= \frac{1+i\tan\beta}{2pbB_1^2}\left(-B_1 + B_2 + \frac{pbB_1^2}{1+i\tan\beta}\right), \\ \sigma_2 &= \frac{1+i\tan\beta}{2pbB_1^2}\left(B_1 + B_2 + \frac{pbB_1^2}{1+i\tan\beta}\right), \\ \sigma_3 &= \frac{1+i\tan\beta}{2pbB_1^2}\left(B_2 + \frac{pbB_1^2}{1+i\tan\beta}\right). \end{aligned}$$

If $f(z)$ *given by* (1.1) *belongs to* $S^{\beta}_{p,b}(\phi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{pbB_1}{2(1+i\tan\beta)}\left[\frac{pbB_1}{(1+i\tan\beta)} + \frac{B_2}{B_1} - \frac{2\mu pbB_1}{(1+i\tan\beta)}\right] & if \quad \mu < \sigma_1 \\ \frac{pbB_1}{2(1+i\tan\beta)} & if \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{pbB_1}{2(1+i\tan\beta)}\left[\frac{pbB_1}{(1+i\tan\beta)} + \frac{B_2}{B_1} - \frac{2\mu pbB_1}{(1+i\tan\beta)}\right] & if \quad \mu > \sigma_2. \end{cases} \tag{2.1}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1+i\tan\beta}{2pbB_1^2}\left(B_1 - B_2 - \frac{pbB_1^2}{1+i\tan\beta}\right)\left|a_{p+1}\right|^2 \leq \frac{pbB_1}{2(1+i\tan\beta)}. \tag{2.2}$$

Further, if $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1+i\tan\beta}{2pbB_1^2}\left(B_1 + B_2 - \frac{pbB_1^2}{1+i\tan\beta}\right)\left|a_{p+1}\right|^2 \leq \frac{pbB_1}{2(1+i\tan\beta)}. \tag{2.3}$$

For any complex number μ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{pB_1 b}{2(1+i\tan\beta)} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{pbB_1}{1+i\tan\beta} - \frac{2\mu B_1 pb}{1+i\tan\beta}\right|\right\}. \tag{2.4}$$

Further,

$$|a_{p+3}| \leq \frac{pbB_1}{2(1+i\tan\beta)} H(q_1, q_2) \tag{2.5}$$

where $H(q_1, q_2)$ *is as defined in Lemma 1.10,*

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{3}{2}\frac{p^2b^2B_1^2}{1+i\tan\beta} \\ q_2 &= \frac{B_3}{B_1} + \frac{3}{2}\frac{pB_1 b}{1+i\tan\beta}\left(\frac{B_2}{B_1} + \frac{pbB_1}{1+i\tan\beta}\right) - \frac{p^2b^2B_1^2}{(1+i\tan\beta)^2}. \end{aligned}$$

These results are sharp.

Proof. If $f(z) \in S^{\beta}_{b,p}(\phi)$, then there is a Schwarz function $w(z) \in \Omega$ of the form (1.8) such that

$$1 + \frac{1}{b}\left[\frac{(1+i\tan\beta)}{p}\left(\frac{zf'(z)}{f(z)}\right) - i\tan\beta - 1\right] = \phi(w(z)). \tag{2.6}$$

Since

$$\frac{zf'(z)}{f(z)} = p + a_{p+1}z + (2a_{p+2} - a_{p+1}^2)z^2 + (3a_{p+3} - 3a_{p+1}a_{p+2} + a_{p+1}^3)z^3 + \dots$$

$$\phi(w(z)) = 1 + B_1w_1 + (B_1w_2 + B_2w_1^2)z^2 + (B_1w_3 + 2B_2w_1w_2 + B_3w_1^3)z^3 + \dots$$

we have from (2.6),

$$a_{p+1} = \frac{pB_1bw_1}{1+i\tan\beta}, \tag{2.7}$$

$$a_{p+2} = \frac{pB_1b}{2(1+i\tan\beta)}\left[w_2 + \left(\frac{B_2}{B_1} + \frac{pB_1b}{1+i\tan\beta}\right)w_1^2\right], \tag{2.8}$$

and

$$a_{p+3} = \frac{pB_1b}{3(1+i\tan\beta)}\left\{w_3 + \left[\frac{2B_2}{B_1} + \frac{3p^2b^2B_1}{2(1+i\tan\beta)}\right]w_1w_2\right\} + \frac{pB_1b}{3(1+i\tan\beta)}$$
$$\cdot\left\{\left[\frac{B_3}{B_1} + \frac{3p^2B_1b^2}{2(1+i\tan\beta)}\left(\frac{B_2}{B_1} + \frac{pB_1b}{1+i\tan\beta}\right) - \frac{B_1^2p^2b^2}{(1+i\tan\beta)^2}\right]\right\}w_1^3.$$

Therefore we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pB_1b}{3(1+i\tan\beta)}[w_2 - \nu w_1^2] \tag{2.9}$$

where

$$\nu = -\frac{B_2}{B_1} - \frac{pB_1b}{1+i\tan\beta} + \frac{2\mu B_1pb}{1+i\tan\beta}.$$

The results (2.1)– (2.3) are established by an application of Lemma 1.8, inequality (2.4) by Lemma 1.9 and (2.5) follows from Lemma 1.10.
To show that the bounds in (2.1)-(2.4)are sharp, we define the function $K_{\phi n}$ ($n = 2, 3, \dots$) by

$$1 + \frac{1}{b}\left[\frac{1+i\tan\beta}{p}\left(\frac{z(K_{\phi n})'(z)}{K_{\phi n}(z)}\right) - i\tan\beta - 1\right] = \phi(z^{n-1})$$

$$K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1$$

and the function F_λ and G_λ ($0 \le \lambda \le 1$) by

$$1 + \frac{1}{b}\left[\frac{1+i\tan\beta}{p}\left(\frac{z(F_\lambda)'(z)}{F_\lambda(z)}\right) - i\tan\beta - 1\right] = \phi\left(z\frac{\lambda+z}{1+\lambda z}\right)$$

$$F_\lambda(0) = 0 = [F_\lambda]'(0) - 1$$

and

$$1+\frac{1}{b}\left[\frac{1+i\tan\beta}{p}\left(\frac{z(G_\lambda)'(z)}{G_\lambda(z)}\right)-i\tan\beta-1\right]=\phi\left(-z\frac{\lambda+z}{1+\lambda z}\right)$$

$$G_\lambda(0)=0=[G_\lambda]'(0)-1.$$

Clearly the functions $K_{\phi n},\ F_\lambda,\ G_\lambda \in S^{\beta}_{p,b}(\phi)$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations. □

Remark 2.2. *For $p = 1$, $b = 1$ and $\beta = 0$ results* (2.1)–(2.3) *coincides with the results obtained for the class $S^*(\phi)$ by Ma and Minda [9].*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{2}\left[B_2+(1-2\mu)B_1^2\right] & if \quad \mu<\sigma_1 \\ \frac{B_1}{2} & if \quad \sigma_1\le\mu\le\sigma_2 \\ -\frac{1}{2}\left[B_2+(1-2\mu)B_1^2\right] & if \quad \mu>\sigma_2. \end{cases}$$

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