**Applied Mathematical Sciences, Vol. 9, 2015, no. 46, 2265 - 2272**
**HIKARI Ltd, www.m-hikari.com**
**http://dx.doi.org/10.12988/ams.2015.53207**

# On the Instability Region for the Extended Rayleigh Problem of Hydrodynamic Stability

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**Abstract**

We consider extended Rayleigh problem of hydrodynamic stability dealing with homogeneous shear flows in sea straits of arbitrary cross section. For this problem, we obtained a parabolic instability region, which intersects the semicircle instability region under some conditions for a class of flows. Also we derived an estimate for the growth rate.

**Mathematics Subject Classification:** 76E05

**Keywords:** Hydrodynamic Stability; Shear flows; Variable bottom; Sea straits

## 1 Introduction

The extended Rayleigh problem of hydrodynamic stability deals with homogeneous shear flows in sea straits of arbitrary cross section. This problem is a special case of extended Tayler Goldstein problem which deals with the stability of incompressible, inviscid but density stratified fluid in sea straits of arbitrary cross section. This problem has been initiated by Pratt et al. (2000) and mathematical formulation has been given by Deng et al. (2003). For the extended Rayleigh problem, the following statements are already proved:

1. The wave velocity of neutral mode is bounded (cf: Subbiah and Ganesh (2007)).
2. The growth rate of an unstable mode tends to zero as the wave number tends to infinity. (cf: Subbiah and Ganesh (2007)).
3. Parabolic instability region which intersects the semicircle region under some conditions (cf: Subbiah and Ganesh (2007)).
4. Sufficient condition for the stability of basic flow is derived (cf: Subbiah and Ganesh (2010)).
5. Series solutions and Lin's perturbation formula are derived. (cf: Ganesh and Subbiah (2013)).
6. Short waves are stable (Dou and Ganesh (2014))

In this paper, we have obtained parabolic instability region for a class of flows which does not depend upon any conditions like f(z) or g(z) given in Subbiah and Ganesh (2007). Also we derived an estimate for the growth rate.

## 2 Extended Rayleigh Problem

The extended Rayleigh problem (cf: Deng et al.(2003)) is given by the second order ordinary differential equation.

$$\left[\frac{(bW)'}{b}\right]' - \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c}\right] W = 0, \tag{1}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{2}$$

Here $k > 0$ is the wave number, $U_0(z)$ is the basic velocity profile, $c = c_r + ic_i$ is the complex phase velocity, Real part of $W(z)e^{ik(x-ct)}$ is the vertical velocity of normal mode disturbance, $b(z)$ is the width function, ' a prime denotes differentiation with respect to the vertical co-ordinate. z varying over [0, D].

Introducing the transformation $W = (U_0 - c)^{1/2}G$, we can get the equation satisfied by G to be

$$\left[(U_0 - c)\frac{(bG)'}{b}\right]' - \frac{1}{2}b\left(\frac{U_0'}{b}\right)' G - k^2(U_0 - c)G - \frac{\frac{(U_0')^2}{4}}{(U_0 - c)}G = 0, \tag{3}$$

with boundary conditions

$$G(0) = 0 = G(D). \tag{4}$$

# 3 Instability Regions

Following the procedure of Sharma and Sharma(1992)for the standard Taylor-Goldstein problem,we can prove the following theorem.

**Theorem 3.1** *A necessary condition for the existence of non-trivial solution with $c_i > 0$ is that the following integral relations are true.*

**(i)** $\int U_0 Q dz - \int c_r Q dz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz + \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}(U_0 - c_r) b|G|^2 dz = 0,$

**(ii)** $\int Q dz - \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 dz = 0.$

***Proof:***
*Multiplying (3) by $(bG^*)$, integrating over [0, D] and using (4), we get*

$$\int (U_0 - c)\left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] dz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz + \int \frac{\frac{(U_0')^2}{4}}{(U_0 - c)} b|G|^2 dz = 0.$$

*Let* $Q = \frac{|(bG)'|^2}{b} + k^2 b|G|^2$ *then the above equation becomes*

$$\int (U_0 - c) Q dz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz + \int \frac{\frac{(U_0')^2}{4}}{(U_0 - c)} b|G|^2 dz = 0. \tag{5}$$

*Comparing real and imaginary part of (5), we get*

$$\int U_0 Q dz - \int c_r Q dz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz + \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}(U_0 - c_r) b|G|^2 dz = 0, \tag{6}$$

*and*

$$-c_i \int Q dz + c_i \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 dz = 0.$$

*Since* $c_i > 0$,

$$\int Qdz - \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 dz = 0. \tag{7}$$

**Theorem 3.2** *If* $U_{0min} > 0$ *then a necessary condition for the existence of neutral mode is*

$$c_i^2 \le \lambda \left[c_r - U_{0min} + U_{0max}\right],$$

*where* $\lambda = \frac{\left|\frac{(U_0')^2}{4}\right|_{max}}{U_{0min}\left[\frac{b_{min}\pi^2}{b_{max}D^2} + k^2\right]}$.

***Proof:***
*Multiplying (7) by* $(U_{0min} - U_{0max})$ *and adding with (6), we get*

$$\int (U_0 - c_r + U_{0min} - U_{0max})Qdz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz + \int \frac{\frac{(U_0')^2}{4}}{|U_0-c|^2}[U_0 - c_r - U_{0min} + U_{0max}]b|G|^2 dz = 0.$$

*Since* $U_{0min} < c_r < U_{0max}$, $(U_0 - c_r + U_{0min} - U_{0max}) < 0$, *dropping the term, we get*

$$\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz \ge \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}[c_r - U_0 + U_{0min} - U_{0max}]b|G|^2 dz. \tag{8}$$

*Multiplying (7) by* $c_r$ *and adding with (6), we get*

$$\int U_0 Qdz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 dz + \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}(U_0 - 2c_r)b|G|^2 dz = 0. \tag{9}$$

*Substituting (8) in (9), we get*

$$\int U_0 Qdz + \int \frac{\frac{(U_0')^2}{4}}{|U_0-c|^2}[U_{0min} - c_r - U_{0max}]b|G|^2 dz \le 0;$$

$$i.e., \int U_0 Qdz \le \int \frac{\frac{(U_0')^2}{4}}{|U_0-c|^2}[c_r - U_{0min} + U_{0max}]b|G|^2 dz.$$

*Since* $\frac{1}{|U_0-c|^2} \le \frac{1}{c_i^2}$ *and using Rayleigh-Ritz inequlatiy* $\int \frac{|(bG)'|^2}{b} dz \ge \frac{\pi^2 b_{min}}{D^2 b_{max}} \int b|G|^2$, *we get*

$$U_{0min}\left[\frac{b_{min}\pi^2}{b_{max}D^2} + k^2\right] \le \frac{\left|\frac{(U_0')^2}{4}\right|_{max}}{c_i^2}[c_r - U_{0min} + U_{0max}];$$

$$i.e., \quad c_i^2 \le \lambda[c_r - U_{0min} + U_{0max}], \tag{10}$$

*where* $\lambda = \frac{\left|\frac{(U_0')^2}{4}\right|_{max}}{U_{0min}\left[\frac{b_{min}\pi^2}{b_{max}D^2} + k^2\right]}$.

**Theorem 3.3** *If $\lambda < \lambda_c$ where*

$$\lambda_c = 3U_{0max} - U_{0min} \pm 2\sqrt{U_{0max}(2U_{0max} - U_{0min})}$$

*then the parabola $c_i^2 \leq \lambda[c_r - U_{0min} + U_{0max}]$ intersects the semicircle*

$$\left[c_r - \left(\frac{U_{0min}+U_{0max}}{2}\right)\right]^2 + c_i^2 \leq \left(\frac{U_{0max}-U_{0min}}{2}\right)^2.$$

***Proof:***
*The semi-circle given in Deng et al (2003) is*

$$\left[c_r - \left(\frac{U_{0min}+U_{0max}}{2}\right)\right]^2 + c_i^2 \leq \left(\frac{U_{0max}-U_{0min}}{2}\right)^2. \tag{11}$$

*Substituting (10) in (11), we get*

$$c_r^2 + [\lambda - U_{0max} - U_{0min}]c_r + (U_{0max}U_{0min} - \lambda U_{0min} + \lambda U_{0max}) \leq 0.$$

*It is a quadratic equation in $c_r$, hence its discriminant part is given by*

$$[\lambda - U_{0max} - U_{0min}]^2 - 4(U_{0max}U_{0min} - \lambda U_{0min} + \lambda U_{0max}) \geq 0.$$

*For the two values of $\lambda$ satisfying the equation is given by*

$$\lambda^2 + [2U_{0min} - 6U_{0max}]\lambda + (U_{0max} - U_{0min})^2 \geq 0.$$

*Solving for $\lambda$, we get*
$\lambda = (3U_{0max} - U_{0min}) \pm 2\sqrt{U_{0max}(2U_{0max} - U_{0min})}$.
*If $\lambda < \lambda_c$ where $\lambda_c = (3U_{0max} - U_{0min}) - 2\sqrt{U_{0max}(2U_{0max} - U_{0min})}$*
*then the parabola will intersect the Deng et al (2003), semicircle.*

**Theorem 3.4** *If $\left[b\left(\frac{U_0'}{b}\right)'\right]_{min} > 0$ then a necessary condition for existence of unstable mode is that*

$$c_i^2 \leq \lambda[c_r - 2U_{0min} + U_{0max}],$$

*where $\lambda = \left[\frac{\frac{(U_0')^2}{2}}{b\left(\frac{U_0'}{b}\right)'}\right]_{max}$.*

***Proof:***
*Multiplying (7) by $(U_{0max} - U_{0min})$ and adding with (6), we get*

$$\int(U_0 - c_r + U_{0max} - U_{0min})Qdz + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2dz$$
$$+ \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}[U_0 - c_r - U_{0max} + U_{0min}]b|G|^2dz \leq 0. \tag{12}$$

*Since $(U_0 - c_r + U_{0max} - U_{0min}) > 0$, dropping the term, we get*

$$\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2dz+\int\frac{\frac{(U_0')^2}{4}}{|U_0-c|^2}[U_0-c_r-U_{0max}+U_{0min}]b|G|^2dz\leq 0;$$

$$ie., \int\frac{b\left(\frac{U_0'}{b}\right)'|U_0-c|^2+\frac{(U_0')^2}{2}[U_0-c_r-U_{0max}+U_{0min}]}{2|U_0-c|^2}b|G|^2dz\leq 0.$$

*Since* $|U_0-c|^2\geq c_i^2$, *we get*

$$\int b\left(\frac{U_0'}{b}\right)' c_i^2+\frac{(U_0')^2}{2}[U_0-c_r-U_{0max}+U_{0min}]b|G|^2dz\leq 0,$$

$$c_i^2\leq\lambda[c_r-2U_{0min}+U_{0max}], \qquad (13)$$

*where* $\lambda=\left[\frac{\frac{(U_0')^2}{2}}{b\left(\frac{U_0'}{b}\right)'}\right]_{max}$.

**Theorem 3.5** *If* $\lambda<\lambda_c$ *where* $\lambda_c=3[U_{0max}-u_{0min}]\pm 2\sqrt{2}(U_{0min}-U_{0max})$ *then the parabola* $c_i^2\leq\lambda[c_r-2U_{0min}+U_{0max}]$ *intersects the semicircle*

$$\left[c_r-\left(\frac{U_{0min}+U_{0max}}{2}\right)\right]^2+c_i^2\leq\left(\frac{U_{0max}-U_{0min}}{2}\right)^2.$$

***Proof:***
*Substituting (13) in (11), we get*

$$c_r^2+[\lambda-U_{0min}-U_{0max}]c_r+(U_{0max}U_{0min}-2U_{0min}\lambda+U_{0max}\lambda)\leq 0.$$

*It is a quadratic equation in* $c_r$, *hence its discriminant part is given by*

$$[\lambda-U_{0max}-U_{0min}]^2-4(1)[U_{0min}U_{0max}-2U_{0min}\lambda+U_{0max}\lambda]\geq 0.$$

*For the two values of* $\lambda$ *satisfying the equation is given by*

$$\lambda^2+[6U_{0min}-6U_{0max}]\lambda+(U_{0max}-U_{0min})^2\geq 0.$$

*Solving for* $\lambda$, *we get*
$\lambda=3(U_{0max}-U_{0min})\pm 2\sqrt{2}[U_{0max}-U_{0min}]$.
*If* $\lambda<\lambda_c$ *then the parabola given in (13) intersects the Deng et al (2003) semicircle.*

# 4 Estimate for growth rate

**Theorem 4.1** *The growth rate of an unstable mode is given by*

$$k^2 c_i^2 \leq [U_{0max} - U_{0min}]^2 K_{max},$$

*where* $K = -\frac{b\left(\frac{U_0'}{b}\right)'}{U_0 - U_{0s}}$ *and* $U_{0s} = U_0(z_s)$

***Proof.***

*Multiplying (1) by* $(bW)'$*, integrating over [0, D] and applying (2), we get*

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right] dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0 - c)} b|W|^2 dz = 0.$$

*Equating real and imaginary part, we get*

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right] dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - c_r) b|W|^2 dz = 0 \qquad (14)$$

*and*

$$c_i \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2} b|W|^2 dz = 0. \qquad (15)$$

*Multiplying (15) by* $\left(\frac{c_r - U_{0s}}{c_i}\right)$ *and subtracting with (14), we get*

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right] dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - 2c_r + U_{0s}) b|W|^2 dz = 0.$$

*Let* $K = -\frac{b\left(\frac{U_0'}{b}\right)'}{U_0 - U_{0s}} > 0$ *and* $U_{0s} = U_0(z_s)$,
*then the above equation becomes*

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right] dz - \int \frac{(U_0 - U_{0s})(U_0 - 2c_r + U_{0s})}{|U_0 - c|^2} K b|W|^2 dz = 0 \ ;$$

$$i.e., \int \left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right] dz = \int \frac{(U_0 - c_r)^2 - (U_{0s} - c_r)^2}{|U_0 - c|^2} K b|W|^2 dz$$

*Since* $|U_0 - c_r| \leq |U_{0max} - U_{0min}|$ *and* $\frac{k^2 c_i^2}{|U_0 - c|^2} \leq k^2$*, we get*

$$\int \left[\frac{|(bW)'|^2}{b} + \frac{k^2 c_i^2}{|U_0 - c|^2} b|W|^2\right] dz \leq \int \frac{[U_{0max} - U_{0min}]^2 - (c_r - U_{0s})^2}{|U_0 - c|^2} K b|W|^2 dz.$$

*Dropping the first term being non-negative,*

$$k^2 c_i^2 \int \frac{b|W|^2}{|U_0 - c|^2} dz \leq \left[\left(U_{0max} - U_{0min}\right)^2 - \left(c_r - \frac{U_{0max} - U_{0min}}{2}\right)^2\right] K_{max} \int \frac{b|W|^2}{|U_0 - c|^2} dz;$$

*i.e.,* $k^2 c_i^2 \leq [U_{0max} - U_{0min}]^2 K_{max}$

# 5 Concluding Remarks

In this paper, we have obtained some general analytical results for the extended Rayleigh problem of hydrodynamic stability dealing with stability of homogeneous shear flows in sea straits of arbitrary cross section. For this problem, we have obtained parabolic instability region, which intersects semicircle instability region under some conditions. Also we obtained an estimate for growth rate of an unstable mode.

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**Received: December 15, 2014; Published: March 21, 2015**