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On the Bounds for the Complex Wave Velocity for Growing Perturbations

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Abstract

For the instability problem of shear flow in sea straits of arbitrary cross section known as extended Taylor-Goldstein problem, we obtained an instability region for the complex wave velocity $c = c_r+ic_i$ with positive imaginary part in the upper half of the (c_r, c_i) plane. Furthermore, we derived a necessary condition for instability and upper bound for wave number.

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1. Introduction

The standard Taylor-Goldstein problem of hydrodynamic stability deals with the stability of shear flow of an incompressible, inviscid but density stratified fluids to infinitesimal normal mode disturbances and this problem has been studied by Drazin and Reid (1981). Recently Pratt et al. (2000) have studied an extended version of Taylor-Goldstein problem which considers flow domain with variable bottom as this is necessary for the study of shear instabilities in sea straits like Bab al Mandab, a strait connecting the Red sea and the Indian Ocean.
We have the following known results for the extended Taylor-Goldstein problem:

(i) A sufficient condition for stability is that $J_0 = Min\left[\frac{N^2}{\left(U_0^{'}\right)^2}\right] \geq \frac{1}{4}$ (cf. Deng et al. (2003)).

(ii) The instability region is a semi-circle in the upper half of c_r–c_i plane. (cf. Deng et al. (2003)).

(iii) The instability region is a semi ellipse lying inside the semicircle (cf. Subbiah and Ganesh (2008)).

(iv) Howards conjecture namely, $kc_i \to 0$ as $k \to \infty$ for special classes of basic flows. (cf. Ganesh and Subbiah (2009)).

In this present work, we have obtained an instability region for the extended Taylor-Goldstein problem of hydrodynamic stability. The major axis of the instability region is same as the range of the basic velocity profile while the minor axis depends on stratification parameter N^2, breadth b(z), basic velocity profile $U_0(z)$, wave number k and minimum Richardson number J_0. Also, we derived a necessary condition for instability and bounds for the wave number.

2. Extended Taylor – Goldstein Problem

The extended Taylor – Goldstein Problem (cf. Deng et al. (2003)) is given by the equation

$$\left[\frac{(bW)'}{b}\right]' + \left[\frac{N^2}{(U_0-c)^2} - \frac{b\left(\frac{U_0^{'}}{b}\right)'}{U_0-c} - k^2\right]W = 0, \tag{1}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{2}$$

Here W is the complex eigen function, $c=c_r+ic_i$ is the complex phase velocity, b(z) is the breadth function, $N^2 \geq 0$ is the square of Bruat-Väisäla frequency, $U_0(z)$ is the basic velocity profile, $k > 0$ is the wave number.

If we introduce the function F by the transformation $W = (U_0 - c)F$ then we can get a equation satisfied by F to be

$$\left[\left(U_0-c\right)^2 \frac{(bF)'}{b}\right]' + \left[N^2 - k^2\left(U_0-c\right)^2\right]F = 0, \tag{3}$$

with boundary conditions
F(0) = 0 = F(D). (4)

Following standard procedure of Howard (1961), one can show that

$$\left[\left[c_r-\left(\frac{U_{0\min}+U_{0\max}}{2}\right)\right]^2+c_i^2-\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2\right]\int Qdz+\int N^2b\left|F^2\right|dz\le 0, \tag{5}$$

(cf. Deng et al. (2003)) where U_{0min}, U_{0max} are minimum and maximum of the velocity profile respectively and $Q=\frac{\left|(bF)'\right|^2}{b}+k^2b|F|^2$.

Deng et al. (2003) dropped the last term to derive the semicircle instability region. In this paper we find suitable estimate for the last term and use it to find an improved instability region.

3. Instability Region

Theorem 3.1:
For the extended Taylor – Goldstein Problem, an instability region is given by

$$\left[c_r-\left(\frac{U_{0\max}+U_{0\min}}{2}\right)\right]^2+c_i^2+J_0c_i^2\left[1+\left[\frac{\left|b\left(\frac{U_0'}{b}\right)'_{\max}\right|\left(U_{0\max}-U_{0\min}\right)+N_{\max}^2}{c_i^2\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\right]}\right]^{\frac{1}{2}}\right]^2\le\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2$$

Proof:
From W = $(U_0 - c)F$, we have
$(bW)' = (U_0 - c)(bF)' + U_0'(bF)$.
This implies that

$$\frac{\left|(bW)'\right|^2}{b}\ge\left|U_0-c\right|^2\frac{\left|(bF)'\right|^2}{b}-2\left|U_0-c\right|\left|(bF)'\right|\left|U_0'\right||F|+\left|U_0'\right|^2b|F|^2. \tag{6}$$

The use of Cauchy-Schwartz equality gives

$$\int\left|U_0-c\right|\left|U_0'\right|\left|(bF)'\right||F|dz\le\left[\int\left|U_0'\right|^2b|F|^2dz\right]^{\frac{1}{2}}\left[\int\left|U_0-c\right|^2\frac{\left|(bF)'\right|^2}{b}dz\right]^{\frac{1}{2}}$$

$$=BE, \tag{7}$$

where $$B^2 = \int \left| U_0' \right|^2 b\left| F \right|^2 dz,$$

$$E^2 = \int \left| U_0 - c \right|^2 Qdz,$$

$$Q = \frac{\left| (bF)' \right|^2}{b} + k^2 b \left| F \right|^2. \tag{8}$$

Using (6) and (7), we have

$$\int \left[\frac{\left| (bW)' \right|^2}{b} + k^2 b \left| W \right|^2 \right] dz \geq (B - E)^2. \tag{9}$$

Multiplying each term of (1) by (bW*), integrating over [0, D] and applying (2), we get

$$\int \left[\frac{\left| (bW)' \right|^2}{b} + k^2 b \left| W \right|^2 \right] dz + \int \frac{b\left(\frac{U_0'}{b} \right)'}{U_0 - c} b\left| W \right|^2 dz - \int \frac{N^2}{(U_0 - c)^2} b\left| W \right|^2 dz = 0.$$

Separating the real parts, we get

$$\int \left[\frac{\left| (bW)' \right|^2}{b} + k^2 b \left| W \right|^2 \right] dz + \int \frac{b\left(\frac{U_0'}{b} \right)'}{\left| U_0 - c \right|^2} (U_0 - c_r) b\left| W \right|^2 dz - \int \frac{N^2}{\left| U_0 - c \right|^4} \left[(U_0 - c_r)^2 - c_i^2 \right] b\left| W \right|^2 dz = 0. \tag{10}$$

Since $\left| U_0 - c_r \right| \leq U_{0\max} - U_{0\min}$, $(U_0 - c_r)^2 - c_i^2 \leq \left| U_0 - c \right|^2$, we have

$$\int \left[\frac{\left| (bW)' \right|^2}{b} + k^2 b \left| W \right|^2 \right] dz \leq \left[\left| b\left(\frac{U_0'}{b} \right)' \right|_{\max} (U_{0\max} - U_{0\min}) + N_{\max}^2 \right] \int b\left| F \right|^2 dz. \tag{11}$$

Using (11) in (9), we have

$$(B - E)^2 \leq \left[\left| b\left(\frac{U_0'}{b} \right)' \right|_{\max} (U_{0\max} - U_{0\min}) + N_{\max}^2 \right] \int b\left| F \right|^2 dz. \tag{12}$$

It is well known that $\left|(U_0 - c)^2\right| \geq c_i^2$

Therefore (8) becomes

$$E^2 \geq c_i^2 \int Q dz.$$

By Rayleigh Ritz inequality, we have

$$E^2 \geq c_i^2 \left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2 \right] \int b |F|^2 dz ;$$

i.e., $$\int b |F|^2 dz \leq \frac{E^2}{c_i^2 \left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2 \right]}. \tag{13}$$

Substituting (13) in (12), we get

$$\left(\frac{B}{E} - 1\right)^2 \leq \frac{\left[\left| b \left(\frac{U_0'}{b} \right)' \right|_{\max} (U_{0\max} - U_{0\min}) + N_{\max}^2 \right]}{c_i^2 \left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2 \right]}.$$

Let $\frac{B}{E} - 1 \leq A$;

i.e., $$\frac{B}{E} \leq 1 + A; \tag{14}$$

where $$A^2 = \frac{\left| b \left(\frac{U_0'}{b} \right)' \right|_{\max} \left[U_{0\max} - U_{0\min} \right] + N_{\max}^2}{c_i^2 \left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}} + k^2 \right]},$$

and $$\int N^2 b |F|^2 dz \geq \left[\frac{N^2}{\left(U_0' \right)^2} \right]_{\min} \int \left(U_0' \right)^2 b |F|^2 dz;$$

i.e., $$\int N^2 b |F|^2 dz \geq J_0 B^2. \tag{15}$$

Using (14) in (15)

$$\int N^2 b |F|^2 dz \geq J_0 (1 + A)^2 c_i^2 \int Q dz \tag{16}$$

Substituting (16) in (5), we get

$$\left[\left[c_r-\left(\frac{U_{0\max}+U_{0\min}}{2}\right)\right]^2+c_i^2-\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2+J_0(1+A)^2c_i^2\right]\int Qdz\le 0\,;$$

i.e.,

$$\left[c_r-\left(\frac{U_{0\max}+U_{0\min}}{2}\right)\right]^2+c_i^2+J_0c_i^2\left[1+\left[\frac{\left|b\left(\frac{U_0'}{b}\right)'_{\max}\right|\left(U_{0\max}-U_{0\min}\right)+N_{\max}^2}{c_i^2\left[\frac{\pi^2 b_{\min}}{D^2 b_{\max}}+k^2\right]}\right]^{\frac{1}{2}}\right]^2\le\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2$$

Remark:
When $J_0 = 0$ then the instability region reduces to Deng et al. (2003) semicircle instability region.

4. Necessary condition for Instability

Theorem 4.1:

A necessary condition for instability is that $b\left(\frac{U_0'}{b}\right)(U_0-c_r)\le N^2$

Proof:
From (10), dropping the first term being non-negative, we get

$$\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{|U_0-c|^2}b|W|^2\,dz\le\int\frac{N^2\left[(U_0-c_r)^2-c_i^2\right]}{|U_0-c|^4}b|W|^2\,dz;$$

$$\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{|U_0-c|^2}b|W|^2\,dz\le\int\frac{N^2\left[(U_0-c_r)^2-c_i^2+2c_i^2\right]}{|U_0-c|^4}b|W|^2\,dz;$$

i.e., $$\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{|U_0-c|^2}b|W|^2\,dz\le\int\frac{N^2\left[(U_0-c_r)^2+c_i^2\right]}{|U_0-c|^4}b|W|^2\,dz$$

This implies that

$$b\left(\frac{U_0'}{b}\right)'(U_0-c_r)\le N^2 .$$

When b=1, our result reduces to result of Subbiah & Jain (1987) for the standard Taylor-Goldstein problem.

5. Bounds for Wave Number

Theorem 5.1:

The wave number is bounded i.e., $k^2 \le \dfrac{N^2}{(U_{0zp}-U_{0s})^2-(c_r-U_{0s})^2-c_i^2}$.

Proof:

Multiplying (3) by (bF*), integrating over (0, D) and applying (4), we get

$$\int (U_0-c)^2\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz-\int N^2b|F|^2\,dz=0.$$

Equating real and imaginary parts, we get

$$\int [(U_0-c_r)^2-c_i^2]\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz-\int N^2b|F|^2\,dz=0, \tag{17}$$

and since $c_i > 0$, we have

$$\int (U_0-c_r)\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz=0. \tag{18}$$

Equation (17) can be written as

$$\int\Big[\big[(U_0-U_{0s})-(c_r-U_{0s})\big]^2-c_i^2\Big]\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz-\int N^2b|F|^2\,dz=0\,;$$

i.e., $$\int\Big[(U_0-U_{0s})^2+(c_r-U_{0s})^2-2(U_0-U_{0s})(c_r-U_{0s})-c_i^2\Big]\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz$$
$$-\int N^2b|F|^2\,dz=0\,. \tag{19}$$

Multiplying both sides of (18) by $(c_r - U_{0s})$, we get

$$\int (U_0-c_r)(c_r-U_{0s})\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz=0\,;$$

i.e., $\int\left[(U_0-U_{0s})-(c_r-U_{0s})\right](c_r-U_{0s})\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz=0;$

i.e.,

$$\int(U_0-U_{0s})(c_r-U_{0s})\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz=\int(c_r-U_{0s})^2\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz. \tag{20}$$

Substituting (20) in (19), we get

$$\int\left[(U_0-U_{0s})^2+(c_r-U_{0s})^2-2(c_r-U_{0s})^2-c_i^2\right]\left[\frac{|(bF)'|^2}{b}+k^2b|F|^2\right]dz-\int N^2b|F|^2\,dz=0$$

i.e.,

$$\int\left[(U_0-U_{0s})^2-(c_r-U_{0s})^2-c_i^2\right]\frac{|(bF)'|^2}{b}dz+k^2\int\left[(U_0-U_{0s})^2-(c_r-U_{0s})^2-c_i^2-\frac{N^2}{k^2}\right]b|F|^2\,dz=0$$

There exist a point $z_p\in[0,D]$ such that

$(U_{0zp}-U_{0s})^2-(c_r-U_{0s})^2-c_i^2\geq 0$ and $(U_{0zp}-U_{0s})^2-(c_r-U_{0s})^2-c_i^2-\frac{N^2}{k^2}\leq 0$

This implied that $k^2\leq\frac{N^2}{(U_{0zp}-U_{0s})^2-(c_r-U_{0s})^2-c_i^2}$

Therefore k^2 is bounded.

6. Concluding remarks

In this present work we derived an instability region for the extended Taylor-Goldstein problem of hydrodynamic stability which deals with stability of stratified shear flows in the sea straits of arbitrary cross section. This instability region depends on stratification parameter, breadth function, basic velocity profile, wave number and minimum Richardson number. Also, we derived necessary condition for instability for a growing perturbation and bounds for the wave number.

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