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Higher Order Multi-Series Arising from Generalized Alpha-Difference Equation

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Abstract

In this paper, the authors extend the theory and $m-$series of the generalized difference equation to $m(\alpha)-$series of its $\alpha-$difference equation. We also investigate the complete and summation solutions of α-difference equation. Suitable examples are provided to illustrate the main results.

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1 Introduction

In 1984, Jerzy Popenda [3] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989 Miller and Rose [8] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [1, 2]. As application of $\Delta_h^{-\nu}$, by taking $\nu = m$ (positive integer) and $h = \ell$, the sum of m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-m} u(k)$ [9].

In 2011, M.Maria Susai Manuel, et.al, [5], have extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ which is defined as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$, $\ell \in (0,\infty)$. In [6], the authors have used the generalized α-difference equation;

$$v(k+\ell) - \alpha v(k) = u(k),\ k \in [0,\infty),\ \ell \in (0,\infty) \quad (1)$$

and obtained a summation solution of the above equation in the form

$$v(k) = \Delta_{\alpha(\ell)}^{-1} u(k) - \alpha^{[\frac{k}{\ell}]} \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1} u(k - r\ell), \quad \hat{\ell}(k) = k - \left[\frac{k}{\ell}\right]\ell. \quad (2)$$

The higher order generalized $\alpha-$difference equation is defined as

$$\Delta_{\alpha(\ell_1)}(\Delta_{\alpha(\ell_2)}(\cdots \Delta_{\alpha(\ell_n)}(v(k)) \cdots)) = u(k),\ k \in [0,\infty),\ \ell_i > 0. \quad (3)$$

There are two types of solutions for the equation (3): one is summation form and another one is closed form. If we are able to find a closed form solution which is coinciding with the summation solution of the equation (3), then we can obtain a formula for finding the values of higher order multi-alpha series of $u(k)$. Hence in this paper, we obtain higher order multi-alpha series to $u(k)$ with respect to ℓ by equating summation and closed form solutions of equation (3).

2 Preliminaries

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be useful for further subsequent discussions. The polynomial factorial $k_\ell^{(n)} = \prod_{r=0}^{n-1}(k-r\ell)$ and $k^{(n)} = \prod_{r=0}^{n-1}(k-r)$. Let $\ell_i > 0$ be fixed, $k \in [0,\infty)$, $\hat{\ell}_i(k) = k - \left[\frac{k}{\ell_i}\right]\ell_i$ for $i = 1,2,3\cdots n$, $\left[\frac{k}{\ell_i}\right]$ denotes the integer part of $\frac{k}{\ell_i}$. Note that $\hat{\ell}(k) = \hat{\ell}(k+\ell) = \hat{\ell}(k+2\ell)$, etc.

Lemma 2.1 [7] *If $\Delta_{\alpha(\ell)}v(k) = u(k)$, then $v(k) - \alpha^{\left[\frac{k}{\ell}\right]}v(\hat{\ell}(k))$ is also a solution of the equation (3) when $n = 1$, $\ell_1 = \ell$ and we can write $v(k) = \Delta^{-1}_{\alpha(\ell)}u(k)$.*

Lemma 2.2 [4] *Let s_q^m and S_q^m are the Stirling numbers of first and second kinds respectively, $m \in N(1)$. Then*

$$k_\ell^{(m)} = \sum_{q=1}^{m} s_q^m \ell^{m-q} k^q,\ \ k^m = \sum_{q=1}^{m} S_q^m \ell^{m-q} k_\ell^{(q)}, \tag{4}$$

$$\Delta_\ell k_\ell^{(m)} = (m\ell)k_\ell^{(m-1)} \ \ \text{and}\ \ \Delta_\ell^{-n} k_\ell^{(m)} = \frac{k_\ell^{(m+n)}}{\ell^n (m+n)^{(n)}}. \tag{5}$$

Corollary 2.3 *If $2^{\ell_i} \neq \alpha$ for $i = 1,\ 2,\ \cdots,\ n$, then*

$$\prod_{i=1}^{n} \Delta^{-1}_{\alpha(\ell_i)} 2^k = \frac{2^k}{\prod_{i=1}^{n}(2^{\ell_i}-\alpha)} \tag{6}$$

and hence

$$\Delta^{-n}_{\alpha(\ell)} 2^k = \frac{2^k}{(2^\ell - \alpha)^n}. \tag{7}$$

Proof. Since $\Delta_{\alpha(\ell_1)}2^k = 2^{k+\ell_1} - \alpha 2^k = (2^{\ell_1}-\alpha)2^k$, from Lemma 2.1, we have $\Delta^{-1}_{\alpha(\ell_1)}2^k = \frac{2^k}{2^{\ell_1}-\alpha}$ which yields $\Delta^{-1}_{\alpha(\ell_2)}\Delta^{-1}_{\alpha(\ell_1)}2^k = \frac{2^k}{(2^{\ell_1}-\alpha)(2^{\ell_2}-\alpha)}$ and hence (6). Now (7) follows by taking $\ell_i = \ell$ for $i = 1,\ 2,\ \cdots,\ n$ in (6).

3 Main Results

In this section, we obtain the sum of higher order multi-alpha series by equating the closed and summation form solutions of the generalized higher kind alpha difference equation (3).

Lemma 3.1 *(Finite $\alpha-$ summation formula) For $\ell > 0$,*

$$\Delta^{-1}_{\alpha(\ell)}u(k+\ell) - \alpha^{[\frac{k}{\ell}]+1}\Delta^{-1}_{\alpha(\ell)}u(\hat{\ell}(k)) = \sum_{r=0}^{[\frac{k}{\ell}]}\alpha^r u(k-r\ell) \tag{8}$$

and hence

$$\alpha^m\Delta^{-1}_{\alpha(\ell)}u(k+\ell-m\ell) - \alpha^{\left[\frac{k}{\ell}\right]+1}\Delta^{-1}_{\alpha(\ell)}u(\hat{\ell}(k)) = \sum_{r=m}^{[\frac{k}{\ell}]}\alpha^r u(k-r\ell) \ for \ \ m < \left[\frac{k}{\ell}\right]. \tag{9}$$

Proof. From the definition of $\Delta_{\alpha(\ell)}v(k)$ and taking $\Delta^{-1}_{\alpha(\ell)}u(k) = v(k)$, we have

$$v(k+\ell) = u(k) + \alpha v(k) \tag{10}$$

which yields the relation

$$u(k)+\alpha u(k-\ell)+\alpha^2 u(k-2\ell)+\cdots+\alpha^{[\frac{k}{\ell}]}u(\hat{\ell}(k)) = v(k+\ell)-\alpha^{[\frac{k}{\ell}]+1}v(\hat{\ell}(k)) \tag{11}$$

which is obtained by replacing k by $k-\ell$, $k-2\ell$,... $(\hat{\ell}(k))$ in equation (10) and substituting the resultant values $v(k)$, $v(k-\ell)$,... $v(\hat{\ell}(k))$ in (10).
Hence, (8) follows from (11) and $\Delta^{-1}_{\alpha(\ell)}u(k+\ell) = v(k+\ell)$ and $\Delta^{-1}_{\alpha(\ell)}u(\hat{\ell}(k)) = v(\hat{\ell}(k))$. Now (9) can be obtained by replacing k by $k-m\ell$ in (8) and then multiplying both sides by α^m.

Theorem 3.2 *($\alpha-$ higher order summation formula) For $k \in \big[\sum\limits_{i=1}^{n}\ell_i, \infty\big)$,*

$$\sum_{i=1}^{n}\Big\{\sum_{(r\ell)_{1\to i}}^{[k]}\alpha^{\left[\frac{k-\sum\limits_{t=1}^{i}r_t\ell_t+\ell_{i+1}}{\ell_{i+1}}\right]+\sum\limits_{t=1}^{i}r_t}\Delta^{-1}_{\alpha(\ell_{[i+1\to n]})}u\big(\hat{\ell}_{i+1}(k - \sum_{t=1}^{i}r_t\ell_t + \ell_{i+1}) + \sum_{t=i+2}^{n}\ell_t\big)\Big\} = \Delta^{-1}_{\alpha(\ell_{[1\to n]})}u(k+\sum_{t=1}^{n}\ell_t) - \alpha^{[\frac{k+\ell_1}{\ell_1}]}\Delta^{-1}_{\alpha(\ell_{[1\to n]})}u\big(\hat{\ell_1}(k+\ell_1)+\sum_{t=2}^{n}\ell_t\big)$$

where $[\frac{*}{\ell_{n+1}}] = 0$, $\displaystyle\sum_{(r\ell)_{1\to i}}^{[k]} = \sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\cdots\sum_{r_i=0}^{\left[\frac{k-r_1\ell_1-r_2\ell_2-\cdots-r_{i-1}\ell_{i-1}}{\ell_i}\right]}$ *and*
$\Delta^{-1}_{\alpha(\ell_{[1\to n]})} = \Delta^{-1}_{\alpha(\ell_1)}\Delta^{-1}_{\alpha(\ell_2)}\Delta^{-1}_{\alpha(\ell_3)}\cdots\Delta^{-1}_{\alpha(\ell_n)}$.

Proof. Replacing ℓ by ℓ_1 in (8), we have
$$u(k)+\alpha u(k-\ell_1)+\alpha^2u(k-2\ell_1)+\cdots+\alpha^{[\frac{k}{\ell_1}]}u(\hat{\ell_1}(k))$$
$$=\Delta^{-1}_{\alpha(\ell_1)}u(k+\ell_1)-\alpha^{[\frac{k}{\ell_1}]+1}\Delta^{-1}_{\alpha(\ell_1)}u(\hat{\ell_1}(k)). \quad (12)$$

Since $\hat{\ell_1}(k+\ell_1)=\hat{\ell_1}(k)$, replacing $u(k)$ by $\Delta^{-1}_{\alpha(\ell_2)}u(k+\ell_2)$, $u(k-\ell_1)$ by $\Delta^{-1}_{\alpha(\ell_2)}u(k-\ell_1+\ell_2)$,...,$u(\hat{\ell_1}(k))$ by $\Delta^{-1}_{\alpha(\ell_2)}u(\hat{\ell_1}(k)+\ell_2)$ in (12), we find
$$\Delta^{-1}_{\alpha(\ell_2)}u(k+\ell_2)+\alpha\Delta^{-1}_{\alpha(\ell_2)}u(k+\ell_2-\ell_1)+\cdots+\alpha^{[\frac{k}{\ell_1}]}\Delta^{-1}_{\alpha(\ell_2)}u(\hat{\ell_1}(k)+\ell_2)$$
$$=\Delta^{-1}_{\alpha(\ell_1)}\Delta^{-1}_{\alpha(\ell_2)}u(k+\ell_2+\ell_1)-\alpha^{[\frac{k}{\ell_1}]+1}\Delta^{-1}_{\alpha(\ell_1)}\Delta^{-1}_{\alpha(\ell_2)}u(\hat{\ell_1}(k)+\ell_2). \quad (13)$$

Replacing ℓ_1 *by* ℓ_2 in (12), we get
$$u(k)+\alpha u(k-\ell_2)+\alpha^2u(k-2\ell_2)+\alpha^3u(k-3\ell_2)+\cdots+\alpha^{[\frac{k}{\ell_2}]}u\left(\hat{\ell_2}(k)\right)$$
$$=\Delta^{-1}_{\alpha(\ell_2)}u(k+\ell_2)-\alpha^{[\frac{k}{\ell_2}]+1}\Delta^{-1}_{\alpha(\ell_2)}u\left(\hat{\ell_2}(k+\ell_2)\right). \quad (14)$$

For $r=1,2,3...[\frac{k}{\ell_1}]$ replacing k by $k-r\ell_1$ in (14) and multiplying both sides by α^r, we find that
$$\alpha^r\left\{u(k-r\ell_1)+\alpha u(k-r\ell_1-\ell_2)+\alpha^2u(k-r\ell_1-2\ell_2)+\cdots+\alpha^{[\frac{k-r\ell_1}{\ell_2}]}u\left(\hat{\ell_2}(k-r\ell_1)\right)\right\}$$
$$=\alpha^r\left\{\Delta^{-1}_{\alpha(\ell_2)}u(k-r\ell_1+\ell_2)-\alpha^{[\frac{k-r\ell_1}{\ell_2}]+1}\Delta^{-1}_{\alpha(\ell_2)}u\left(\hat{\ell_2}(k-r\ell_1+\ell_2)\right)\right\}. \quad (15)$$

Adding (14) and (15) for $r=1,2,3...[\frac{k}{\ell_1}]$ and applying (13), we derive
$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\alpha^{r_2+r_1}u(k-r_2\ell_2-r_1\ell_1)=\Delta^{-1}_{\alpha(\ell_1)}\left(\Delta^{-1}_{\alpha(\ell_2)}u(k+\ell_2+\ell_1)\right)-\alpha^{[\frac{k}{\ell_1}]+1}$$
$$\times\Delta^{-1}_{\alpha(\ell_1)}\left(\Delta^{-1}_{\alpha(\ell_2)}u(\ell_2+\hat{\ell_1}(k))\right)-\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\alpha^{[\frac{k-r_1\ell_1}{\ell_2}]+r_1+1}\Delta^{-1}_{\alpha(\ell_2)}u\left(\hat{\ell_2}(k-r_1\ell_1+\ell_2)\right). \quad (16)$$

Replacing r_1,r_2 *by* r_2,r_3 and ℓ_1,ℓ_2 *by* ℓ_2,ℓ_3 in (16), we find
$$\sum_{r_2=0}^{[\frac{k}{\ell_2}]}\sum_{r_3=0}^{[\frac{k-r_2\ell_2}{\ell_3}]}\alpha^{(r_3+r_2)}u(k-r_3\ell_3-r_2\ell_2)=\Delta^{-1}_{\alpha(\ell_2)}\left(\Delta^{-1}_{\alpha(\ell_3)}u(k+\ell_3+\ell_2)\right)-\alpha^{[\frac{k}{\ell_2}]+1}$$
$$\times\Delta^{-1}_{\alpha(\ell_2)}\left(\Delta^{-1}_{\alpha(\ell_3)}u(\ell_3+\hat{\ell_2}(k))\right)-\sum_{r_2=0}^{[\frac{k}{\ell_2}]}\alpha^{[\frac{k-r_2\ell_2}{\ell_3}]+r_2+1}\Delta^{-1}_{\alpha(\ell_3)}u\left(\hat{\ell_3}(k-r_2\ell_2+\ell_3)\right). \quad (17)$$

Replacing k by $k-r\ell_1$ in (17) and multiplying both sides by α^r and adding the corresponding expressions for $r=0,1,2,...[\frac{k}{\ell_1}]$, we derive

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\sum_{r_3=0}^{[\frac{k-r_1\ell_1-r_2\ell_2}{\ell_3}]}\alpha^{(r_3+r_2+r_1)}u(k-r_3\ell_3-r_2\ell_2-r_1\ell_1)$$
$$=\Delta^{-1}_{\alpha(\ell_1)}\left(\Delta^{-1}_{\alpha(\ell_2)}\left(\Delta^{-1}_{\alpha(\ell_3)}u(k+\ell_3+\ell_2+\ell_1)\right)\right)$$
$$-\alpha^{[\frac{k}{\ell_1}]+1}\Delta^{-1}_{\alpha(\ell_1)}\left(\Delta^{-1}_{\alpha(\ell_2)}\left(\Delta^{-1}_{\alpha(\ell_3)}u(\hat{\ell}_1(k)+\ell_3+\ell_2)\right)\right)$$
$$-\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\alpha^{[\frac{k-r_1\ell_1}{\ell_2}]+r_1+1}\Delta^{-1}_{\alpha(\ell_2)}\left(\Delta^{-1}_{\alpha(\ell_3)}u\left(\hat{\ell}_2(k-r_1\ell_1+\ell_2)+\ell_3\right)\right)$$
$$-\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\alpha^{[\frac{k-r_2\ell_2-r_1\ell_1}{\ell_3}]+r_2+r_1+1}\Delta^{-1}_{\alpha(\ell_3)}u\left(\hat{\ell}_3(k-r_2\ell_2-r_1\ell_1+\ell_3)\right). \qquad (18)$$

Continuing this process we get the proof of the theorem.

Corollary 3.3 *Taking $n=2$, $u(k)=k^2$ in the Theorem 3.2, we have*

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\alpha^{r_2+r_1}(k-r_2\ell_2-r_1\ell_1)^2+\Big\{\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\alpha^{[\frac{k-r_1\ell_1}{\ell_2}]+r_1+1}$$
$$\Big(\frac{(\hat{\ell}_2(k-r_1\ell_1+\ell_2))^2}{(1-\alpha)}-\frac{2\ell_2(\hat{\ell}_2(k-r_1\ell_1+\ell_2))}{(1-\alpha)^2}+\frac{{\ell_2}^2(1+\alpha)}{(1-\alpha)^3}\Big)\Big\}$$
$$=\Big(\frac{(k+\ell_2+\ell_1)^2}{(1-\alpha)^2}-\frac{2(\ell_1+\ell_2)(k+\ell_2+\ell_1)}{(1-\alpha)^3}+\frac{(1+\alpha)({\ell_1}^2+{\ell_2}^2)+2\ell_1\ell_1}{(1-\alpha)^4}\Big)$$
$$-\alpha^{[\frac{k}{\ell_1}]+1}\Big(\frac{(\ell_2+\hat{\ell}_1(k))^2}{(1-\alpha)^2}-\frac{2(\ell_1+\ell_2)(\ell_2+\hat{\ell}_1(k))}{(1-\alpha)^3}+\frac{(1+\alpha)({\ell_1}^2+{\ell_2}^2)+2\ell_1\ell_2}{(1-\alpha)^4}\Big). \qquad (19)$$

Proof. Since $\Delta_{\alpha(\ell_2)}k^0=(k+\ell_2)^0-\alpha k^0=(1-\alpha)(1)$, we have

$$\Delta^{-1}_{\alpha(\ell_2)}(1)=\frac{1}{1-\alpha}. \qquad (20)$$

From $\Delta_{\alpha(\ell_2)}k=(k+\ell_2)-\alpha k=k(1-\alpha)+\ell_2(1)$ and (20), we get

$$\Delta^{-1}_{\alpha(\ell_2)}k=\frac{k}{1-\alpha}-\frac{\ell_2}{(1-\alpha)^2}. \qquad (21)$$

Now $\Delta_{\alpha(\ell_2)}k^2=(k+\ell_2)^2-\alpha k^2=k^2(1-\alpha)+2\ell_2k+\ell_2^2$ yields
$\Delta^{-1}_{\alpha(\ell_2)}k^2=\dfrac{k^2}{(1-\alpha)}-\dfrac{2\ell_2}{(1-\alpha)}\Delta^{-1}_{\alpha(\ell_2)}k-\dfrac{\ell_2^2}{(1-\alpha)}\Delta^{-1}_{\alpha(\ell_2)}(1)$
and hence by (20) and (21), we find

$$\Delta^{-1}_{\alpha(\ell_2)}k^2=\frac{k^2}{(1-\alpha)}-\frac{2\ell_2k}{(1-\alpha)^2}+\frac{\ell_2^2(1+\alpha)}{(1-\alpha)^3}. \qquad (22)$$

Taking $\Delta^{-1}_{\alpha(\ell_1)}$ on both sides of (22) and applying (20) and(21) for ℓ_1, we arrive

$$\Delta^{-1}_{\alpha(\ell_1)}\Delta^{-1}_{\alpha(\ell_2)}k^2 = \frac{k^2}{(1-\alpha)^2} - \frac{2(\ell_1+\ell_2)k}{(1-\alpha)^3} + \frac{(1+\alpha)(\ell_1^2+\ell_2^2)+2\ell_1\ell_2}{(1-\alpha)^4}. \qquad (23)$$

Now the proof follows by applying (22) and(23) in the Theorem (3.2).

Following example is an verification of corollary (3.3).

Example 3.4 *Put* $k = 10, \ell_1 = 3, \ell_2 = 4$ *and* $\alpha = 3$ *in equation (19).*

$$\sum_{r_1=0}^{3}\sum_{r_2=0}^{[\frac{10-3r_1}{4}]} 3^{(r_2+r_1)}(10-4r_2-3r_1)^2$$
$$+\Big\{\sum_{r_1=0}^{3} 3^{[\frac{10-3r_1}{4}]+r_1+1}\Big(\frac{(\hat{4}(10-3r_1+4))^2}{-2} - \frac{2\times\hat{4}(10-3r_1+4)}{4} - 8\Big)\Big\}$$
$$= \Big(\frac{(17)^2}{4} + \frac{14(17)}{8} + \frac{100+24}{16}\Big) - 3^4\Big(\frac{5^2}{4} + \frac{14(5)}{8} + \frac{100+24}{16}\Big).$$

Corollary 3.5 *For* $k \in \big[\sum_{i=1}^{n}\ell_i, \infty\big)$, *we have*

$$\sum_{i=1}^{n}\left\{\sum_{(r\ell)_{1\to i}}^{[k]} \Delta^{-1}_{\ell_{[i+1\to n]}} u\left(\hat{\ell}_{i+1}(k - \sum_{t=1}^{i} r_t\ell_t + \ell_{i+1}) + \sum_{t=i+2}^{n}\ell_t\right)\right\}$$
$$= \Delta^{-1}_{\ell_{[1\to n]}}u(k + \sum_{t=1}^{n}\ell_t) - \Delta^{-1}_{\ell_{[1\to n]}}u\Big(\hat{\ell}_1(k+\ell_1) + \sum_{t=2}^{n}\ell_t\Big) \qquad (24)$$

where $\hat{\ell}_{n+1} = 0$ *and* $\hat{\ell}_{n+1}(k) = k$, $\sum_{t=m}^{n}\ell_t = 0$,*when* $m > n$.

Proof. The proof follows by taking $\alpha = 1$ in the Theorem 3.2.

Corollary 3.6 *From (6), taking* $n = 2$ *and* $u(k) = 2^k$ *in the Corollary 3.5, we have*

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]} 2^{k-r_2\ell_2-r_1\ell_1} + \sum_{r_1=0}^{[\frac{k}{\ell_1}]}\frac{2^{\hat{\ell_2}(k-r_1\ell_1+\ell_2)}}{(2^{\ell_2}-1)} = \frac{2^{k+\ell_1+\ell_2} - 2^{\hat{\ell_1}(k)+\ell_2}}{(2^{\ell_1}-1)(2^{\ell_2}-1)}. \qquad (25)$$

The following example is an verification of corollary 3.6.

Example 3.7 *Taking* $k = 11, \ell_1 = 2$ *and* $\ell_2 = 3$ *in (25), we have*

$$\sum_{r_1=0}^{5}\sum_{r_2=0}^{[\frac{11-2r_1}{3}]} 2^{11-2r_1-3r_2} + \sum_{r_1=0}^{5}\frac{2^{\hat{3}(11-2r_1+3)}}{(2^3-1)} = \frac{2^{16}-2^4}{(2^2-1)(2^3-1)}.$$

Theorem 3.8 *Let $\ell > 0$ and $k \in [\ell, \infty)$. Then*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+n-1}{n-1} \alpha^r u(k-r\ell) + \alpha^{([\frac{k}{\ell}]+1)} \sum_{r=1}^{n-1} \frac{([\frac{k}{\ell}]+r)^{(r)}}{r!} \Delta_{\alpha(\ell)}^{-(n-r)} u(\hat{\ell}(k)+(n-1-r)\ell)$$

$$= \Delta_{\alpha(\ell)}^{-n} u(k+n\ell) - \alpha^{([\frac{k}{\ell}]+1)} \Delta_{\alpha(\ell)}^{-n} u(\hat{\ell}(k)+(n-1)\ell). \quad (26)$$

Proof. From (8), we have

$$u(k) + \alpha u(k-\ell) + \cdots + \alpha^{[\frac{k}{\ell}]} u(\hat{\ell}(k)) = \Delta_{\alpha(\ell)}^{-1} u(k+\ell) - \alpha^{[\frac{k}{\ell}]+1} \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)). \quad (27)$$

Since $\hat{\ell}(k) = \hat{\ell}(k-r\ell)$ for $r = 1, 2, ...[\frac{k}{\ell}]$, replacing k by $k-\ell$, $k-2\ell$,...$\hat{\ell}(k)$, in (27) and multiplying the corresponding expressions by α, α^2, α^3, ..., $\alpha^{[\frac{k}{\ell}]}$ respectively and then adding all the resultant expressions, we arrive

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+1}{1} \alpha^r u(k-r\ell) = \Delta_{\alpha(\ell)}^{-2} u(k+2\ell) - \alpha^{[\frac{k}{\ell}]+1} \Big\{ \Delta_{\alpha(\ell)}^{-2} u(\hat{\ell}(k)+\ell)$$

$$+ \left(\left[\frac{k}{\ell}\right] + 1 \right) \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)) \Big\}. \quad (28)$$

Applying the process mentioned above to (28), we find that

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+2}{2} \alpha^r u(k-r\ell) = \Delta_{\alpha(\ell)}^{-3} u(k+3\ell) - \alpha^{[\frac{k}{\ell}]+1} \Big\{ \Delta_{\alpha(\ell)}^{-3} u(\hat{\ell}(k)+2\ell)$$

$$- \frac{([\frac{k}{\ell}]+1)_1^{(1)}}{1!} \Delta_{\alpha(\ell)}^{-2} u(\hat{\ell}(k)+\ell) - \frac{([\frac{k}{\ell}]+2)_1^{(2)}}{2!} \Delta_{\alpha(\ell)}^{-1} u(\hat{\ell}(k)) \Big\}. \quad (29)$$

Proceeding like this we get the proof of the theorem.

Corollary 3.9 *From (7), taking $n = 4$ and $u(k) = 2^k$ in (26), we have*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+3}{3} \alpha^r 2^{k-r\ell} + \alpha^{[\frac{k}{\ell}]+1} \Big\{ \frac{([\frac{k}{\ell}]+1) 2^{\hat{\ell}(k)+2\ell}}{(2^\ell-\alpha)^3} + \frac{([\frac{k}{\ell}]+2)_1^{(2)} 2^{\hat{\ell}(k)+\ell}}{2!(2^\ell-\alpha)^2} +$$

$$\frac{([\frac{k}{\ell}]+3)_1^{(3)} 2^{\hat{\ell}(k)}}{3!(2^\ell-\alpha)} \Big\} = \frac{2^{k+4\ell} - \alpha^{[\frac{k}{\ell}]+1} 2^{\hat{\ell}(k)+3\ell}}{(2^\ell-\alpha)^4}. \quad (30)$$

Following is an illustration of Corollary 3.9.

Example 3.10 *Taking $k = 7$, $\ell = 2$, $\alpha = 3$ in (30), we have*

$$\sum_{r=0}^{3} \binom{r+3}{3} 3^r 2^{7-2r} + 3^4 \left\{ \frac{4 \times 2^5}{(2^2-3)^3} + \frac{10 \times 2^3}{(2^2-3)^2} + \frac{20 \times 2}{(2^2-3)} \right\} = \frac{2^{15} - 3^4 \times 2^7}{(2^2-3)^4}.$$

Theorem 3.11 *Let $\ell > 0$ and $k \in [\ell, \infty)$. Then*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+n-1}{n-1} u(k - r\ell) + \sum_{r=1}^{n-1} \frac{([\frac{k}{\ell}] + r)^{(r)}}{r!} \Delta_\ell^{-(n-r)} u(\hat{\ell}(k) + (n-1-r)\ell)$$

$$= \Delta_\ell^{-n} u(k + n\ell) - \Delta_\ell^{-n} u(\hat{\ell}(k) + (n-1)\ell). \quad (31)$$

Proof. The proof of (31) follows by taking $\alpha = 1$ in the Theorem 3.8.

Corollary 3.12 *From (5), taking $n = 3$, $u(k) = k_\ell^{(t)}$ in (31), we have*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+2}{2} (k - r\ell)_\ell^{(t)} + \frac{([\frac{k}{\ell}] + 1)(\hat{\ell}(k) + \ell)_\ell^{(t+2)}}{1!\ell^2 (t+2)^{(2)}} + \frac{([\frac{k}{\ell}] + 2)_1^{(2)} (\hat{\ell}(k))_\ell^{(t+1)}}{2!\ell(t+1)}$$

$$= \frac{(k + 3\ell)_\ell^{(t+3)} - (\hat{\ell}(k) + 2\ell)_\ell^{(t+3)}}{\ell^3 (t+3)^{(3)}}. \quad (32)$$

The following example is an illustration of Corollary 3.12.

Example 3.13 *Taking $t = 4$, $k = 1501$, $\ell = 2$ in (32) we have*

$$\sum_{r=0}^{750} \binom{r+2}{2} (1501 - 2r)_2^{(4)} + \frac{751(2)_2^{(6)}}{120} + \frac{(752)_1^{(2)} (1)_2^{(5)}}{20} = \frac{(1507)_2^{(7)} - (5)_2^{(7)}}{1680}.$$

Corollary 3.14 *If k is an integer multiple of ℓ and $u(0) = 0$, $\Delta_\ell^{-1} u(0) = 0$, $\Delta_\ell^{-2} u(\ell) = 0$,$\Delta_\ell^{-3} u(2\ell) = 0$, ..., $\Delta_\ell^{-n} u((n-1)\ell) = 0$, then*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+n-1}{n-1} u(k - r\ell) = \Delta_\ell^{-n} u(k + n\ell). \quad (33)$$

Proof. Since k is a integer multiple of ℓ, we have $\hat{\ell}(k) = 0 = \hat{\ell}(k - r\ell)$ for $r = 1,\ 2,\ ...,\ [\frac{k}{\ell}]$ and hence the proof follows from (31).

Example 3.15 *From (4) and (5), taking $n = 3$ and $u(k) = k^t$ in (33), we have*

$$\sum_{r=0}^{[\frac{k}{\ell}]} \binom{r+2}{2} (k - rl)^t = \sum_{q=1}^{t} S_q^t \ell^{t-q} \frac{(k + 3\ell)_\ell^{(q+3)}}{\ell^3 (q+3)^{(3)}}. \quad (34)$$

Since $S_1^3 = 1$, $S_2^3 = 3$, $S_3^3 = 1$, when $t = 3$, $k = 500$, $\ell = 4$ in (34), we get

$$\sum_{r=0}^{125} \binom{r+2}{2} (500 - 4r)^3 = \frac{1 \times 4^2 (512)_4^{(4)}}{64(4)^{(3)}} + \frac{3 \times 4 (512)_4^{(5)}}{64(5)^{(3)}} + \frac{1 \times (512)_4^{(6)}}{64(6)^{(3)}}.$$

Conclusion: All the formulae and the theorems are very useful when k is very large and ℓ_i's are small.

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References

[1] N. R. O. Bastos, R. A. C. Ferreira, and D. F. M. Torres, *Discrete-Time Fractional Variational Problems, Signal Processing*, vol. 91, no. 3, pp. 513-524, 2011. http://dx.doi.org/10.1016/j.sigpro.2010.05.001

[2] R. A. C. Ferreira and D. F. M. Torres, *Fractional h-difference equations arising from the calculus of variations, Applicable Analysis and Discrete Mathematics*, 5(1) (2011), 110-121. http://dx.doi.org/10.2298/aadm110131002f

[3] Jerzy Popenda and Blazej Szmanda, *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, XVII(1), (1984), 153 - 164.

[4] M. Maria Susai Manuel, G. Britto Antony Xavier and E.Thandapani, *Theory of Generalized Difference Operator and Its Applications*, Far East Journal of Mathematical Sciences, 20(2) (2006), 163 - 171.

[5] M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier, *Solutions and Applications of Certain Class of α-Difference Equations*, International Journal of Applied Mathematics, 24(6) (2011), 943-954.

[6] M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier, *Theory of Generalized α-Difference Operator and its Applications in Number Theory*, Advances in Differential Equations and Control Processes, 9(2) (2012), 141-155.

[7] M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier, *Generalized $\alpha-$Bernoulli Polynomials through Weighted Pochhammer Symbols*, Far East Journal of Mathematical Sciences, Vol. 4, No. 1, 2013, 23-29.

[8] K. S. Miller and B. Ross, *Fractional Difference Calculus in Univalent Functions*, Horwood, Chichester, UK, 139-152,1989.

[9] M. M. Susai Manuel, G. B. A. Xavier, V. Chandrasekar and R. Pugalarasu, *Theory and application of the Generalized Difference Operator of the n^{th} kind(Part I)*, Demonstratio Mathematica,vol.45, no.1, pp.95-106, 2012.

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