A Food Chain Model Holling Type II
with Distributed Delay

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Abstract

In this research we introduce and study, analytic and numerically, a model of chemostat with distributed delay. This a generalization of the model, with no delay, first treated by Kuang. We give conditions for asymptotic stability, survival of species or extinction of them.

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1. Introduction

We start from the model studied in [5], follow ideas given in [9] and propose the use of distributed delay in order to study a food chain model in the chemostat. Models with distributed delay (that is, integro-differential equations) have been used in biological models for instance in [2], [6], [8] and [9]. We perform local and global analysis of the equilibria on the coordinate axis, a local study for the case of equilibria inside the first octant and some numerical implementations that show how the equilibrium, with positive coordinates (inside first octant), loses stability which allows periodic orbits to appear.

2. The Model and Preliminary Results

In the original paper [5] the chain food model looks like

\begin{align*}
S'(t) &= (S^0 - S(t))D - \frac{m_1 S(t) X(t)}{\gamma_1(a_1 + S(t))}, \\
X'(t) &= X(t) \left( \frac{m_1 S(t)}{a_1 + S(t)} - \frac{m_2 Y(t)}{\gamma_2(a_2 + X(t))} - D_1 \right), \\
Y'(t) &= Y(t) \left( \frac{m_2 X(t)}{a_2 + X(t)} - D_2 \right),
\end{align*}

with \(S(0) = S_0 \geq 0\), \(X(0) = X_0 \geq 0\), \(Y(0) = Y_0 \geq 0\), \(t \geq 0\); \(S(t)\) denotes a primary source of nutrients to population \(X(t)\) (prey) which instead serves as nutrient to population \(Y(t)\) (predator) located at the top of the food chain, \(S^0\) means initial concentration of nutrient, \(\gamma_1\) and \(\gamma_2\) are the growth yield constants of \(X(t)\) and \(Y(t)\) respectively. \(D\) is the washout rate from the chemostat (same as the input rate), \(D_i = D + \varepsilon_i\), \(i = 1, 2\) and \(\varepsilon_1, \varepsilon_2\) being the specific death rates of \(X\) and \(Y\) respectively; \(m_1, m_2\) represent the maximum growth rate for prey and predator population. In the same fashion \(a_1\) and \(a_2\) are saturation rates of the corresponding populations. In case of specific death rates are negligible (which is the case in the chemostat) we may assume \(D_1 = D_2 = D\).

We now measure concentrations, time, \(X\) and \(Y\) in \(S^0\), \(1/D\), \(\gamma_1 S^0\), and \(\gamma_1 \gamma_2 S^0\) units respectively; so equation (1) becomes

\begin{align*}
S'(t) &= 1 - S(t) - \frac{m_1 S(t) X(t)}{a_1 + S(t)}, \\
X'(t) &= X(t) \left( \frac{m_1 S(t)}{a_1 + S(t)} - D_1 - \frac{m_2 Y(t)}{a_2 + X(t)} \right), \\
Y'(t) &= Y(t) \left( \frac{m_2 X(t)}{a_2 + X(t)} - D_2 \right),
\end{align*}
with $S(0) > 0$, $X(0) > 0$, $Y(0) > 0$, $t \geq 0$, $D_i = \frac{\overline{D}_i}{D}$, $i = 1, 2$. The functional responses of the species $X$ and $Y$ are the so called Holling type II (or Michaelis-Menten) form. Notice that if we set $D_1 = D_2 = 1$ system (2) looks like

$$
S'(t) = 1 - S(t) - \frac{m_1S(t)X(t)}{a_1 + S(t)},
$$

$$
X'(t) = X(t) \left( \frac{m_1S(t)}{a_1 + S(t)} - 1 - \frac{m_2Y(t)}{a_2 + X(t)} \right),
$$

$$
Y'(t) = Y(t) \left( \frac{m_2X(t)}{a_2 + X(t)} - 1 \right),
$$

which has been studied, as said before, in [5].

Our main goal is analyze this model in the presence of distributed delays. The recognition of time delays in the growth response of a population that changes in the environment has led to extensive theoretical and experimental studies however, to our knowledge, there has been little emphasis in distributed delays in chemostat models ([3]). The introduction of distributed delay in the above model take us to system

$$
S'(t) = 1 - S(t) - \frac{m_1S(t)X(t)}{a_1 + S(t)},
$$

$$
X'(t) = \frac{m_1S(t)X(t)}{a_1 + S(t)} - X(t) - \frac{m_2X(t)Y(t)}{a_2 + X(t)},
$$

$$
Y'(t) = \int_{-\infty}^{t} \frac{m_2X(\tau)Y(\tau)}{a_2 + X(\tau)}e^{-(t-\tau)(\alpha e^{-\alpha(t-\tau)})}d\tau - Y(t),
$$

with initial conditions $S(0) = S_0 > 0$, $X(0) = \tilde{X}(t) \geq 0$, $Y(0) = \tilde{Y}(t) \geq 0$, $t \leq 0$.

Now we assume that the growth of the predator $Y$ is influenced by the amount of prey in the past. More precisely, we suppose that predator grows up depending on the weight average over the past ([4, 9]) by mean of the the function $Z(t)$ given by

$$
Z(t) = \int_{-\infty}^{t} \frac{m_2X(\tau)Y(\tau)}{a_2 + X(\tau)}e^{-(t-\tau)(\alpha e^{-\alpha(t-\tau)})}d\tau, \alpha > 0.
$$

The linear chain trick is used now, which consists basically in consider equation (5) as a change of variable and generate the following set of differential
Proof. We only have to notice that for any solution $\phi$ of (4), given as before, remains positive on $[0, +\infty)$, then

$Y(t) = 0$ and at the same time $S(t) = 0$, which is impossible.

For Lemma 1.

and (6).

that is,

write down second equation in (4) as

$X(t) = \tilde{X}(0) \exp \left( \int_{0}^{t} \left[ \frac{m_{1}S(\tau)}{a_{1} + S(\tau)} - 1 - \frac{m_{2}Y(\tau)}{a_{2} + X(\tau)} \right] d\tau \right)$.

Finally if we assume existence of $\tau > 0$ such that $Y(\tau) = 0$, define

$\bar{t} = \inf \{ t > 0 : Y(t) = 0 \text{ and } Y(s) > 0 \text{ for } 0 \leq s \leq t \} < \infty$, then $Y(\bar{t}) = 0, Y'(\bar{t}) \leq 0$ and at the same time

$Y'(\bar{t}) = \int_{-\infty}^{\bar{t}} \frac{m_{2}X(\tau)Y(\tau)}{a_{2} + X(\tau)} e^{-\alpha(\tau-t)} e^{-\alpha(\tau-t)} d\tau > 0$

which is impossible. □

Lemma 2. All solutions of (4) are bounded on $[0, +\infty)$. 

The initial conditions for the foregoing system are

$S(0) = S_{0}, X(0) = \tilde{X}(0), Y(0) = \tilde{Y}(0), Z(0) = Z_{0} = \int_{-\infty}^{0} \frac{\alpha m_{2}X(\tau)\tilde{Y}(\tau)}{a_{2} + X(\tau)} e^{(\alpha+1)\tau} d\tau,$

set $\varphi(\phi, t) = (S(t), X(t), Y(t))$ as the solution of (4) with $\phi \in BC_{c}^{3}$ (set of bounded and continuous functions defined on $(-\infty, 0]$ to first octant) given by $\phi(t) = \tilde{S}(t), \tilde{X}(t), \tilde{Y}(t)$, $t \leq 0$; in these conditions $\varphi(\phi, t)$ exists and is unique ([1, 7]). Now we estate and prove some straightforward results for systems (4) and (6).

Lemma 1. For $\phi \in BC_{c}^{3}$, $S(0) \geq 0, \tilde{X}(0) > 0, \tilde{Y}(0) > 0$ corresponding solution $\varphi(\phi, t)$ of (4), given as before, remains positive on $[0, +\infty)$.

Proof. We only have to notice that for any $\xi \geq 0$ such that $S(\xi) = 0,$

$S'(\xi) = 1 - S(\xi) - \frac{m_{1}S(\xi)X(\xi)}{a_{1} + S(\xi)} > 0,$

that is,

$X'(t) = \frac{m_{1}S(t)}{a_{1} + S(t)} - 1 - \frac{m_{2}Y(t)}{a_{2} + X(t)},$
Proposition 6. goes in similar manner.

From last equation in (6) we get \( W(0) = S(0) + X(0) + Y(0) \)

Proof. Let \((S, X, Y)\) be a solution of (4) and \( W = S+X+Y \), then \( W' + W = 1 \), so \( W(t) = 1 + (W(0) - 1) \exp(-t) \), \( W(0) = S(0) + X(0) + Y(0) \).

Lemma 3. If \( \Theta \) is the \( \omega \)-limit set corresponding to a solution of system (6) on \([0, +\infty)\), then

\[
\Theta \subseteq \Delta_* = \{(S, X, Y, Z) : S + X + Y + \frac{Z}{\alpha} = 1, S \geq 0, X \geq 0, Y \geq 0, Z \geq 0\}.
\]

Proof. Because the \( \omega \)-limit set is the smallest positive invariant set of system (6), then any other positive invariant set associated to this equation must contain it, therefore we only need to show that \( \Delta_* \) is invariant with respect to solutions of (6).

Let \((S^*(0), X^*(0), Y^*(0), Z^*(0)) \in \Delta_* \) be initial data of solution \((S^*(t), X^*(t), Y^*(t), Z^*(t))\) for \( t \geq 0 \), define \( M = 1 - (S^*(t) + X^*(t) + Y^*(t) + \frac{Z^*(t)}{\alpha}) \), notice that \( M' = -M(t) \), \( M(0) = S^*(0) + X^*(0) + Y^*(0) + \frac{Z^*(0)}{\alpha} \); but then \( M(t) = M(0) \exp(-t), t \geq 0 \).

Corollary 4. Any attractor of solutions of (6) is in \( \Delta_* \).

At this stage we define new parameters \( \mu_1, \mu_2, \mu_\alpha \) in order to simplify calculations,

\[
\mu_1 = \frac{a_1}{m_1 - 1}, \quad \mu_2 = \frac{a_2}{m_2 - 1} \quad \text{and} \quad \mu_\alpha = \frac{a_2}{\alpha + \frac{m_2}{\alpha} - 1}.
\]

We finish this section establishing a couple of results that characterize existence of species involved.

Proposition 5. If \( m_1 \leq 1 \) or \( \mu_1 \geq 1 \), species \( X \) dies out in (6).

Proof. Since any solution starting in \( \Delta_* \) remains in it, we put \( S = 1 - (X + Y + \frac{Z}{\alpha}) \), so if \( m_1 \leq 1 \), \( X(0) > 0 \), and first equation in (6) now looks like

\[
X'(t) = \frac{m_1(1 - (X(t) + Y(t) + \frac{Z(t)}{\alpha}))}{a_1 + 1 - (X(t) + Y(t) + \frac{Z(t)}{\alpha})} - \frac{m_2 Y(t)}{a_2 + X(t)} < \frac{m_1}{a_1 + 1} X(t) < 0,
\]

where the hypothesis has been used to obtain last inequality. The case \( \mu_1 \geq 1 \) goes in similar manner.

Proposition 6. For \( m_2 \leq \frac{\alpha + 1}{\alpha} \), species \( Y \) and \( Z \) in (6) disappear.

Proof. From last equation in (6) we get \( \frac{Z'(t)}{\alpha + 1} < Y(t) - Z(t) = -Y'(t) \), or \( \frac{Z'(t)}{\alpha + 1} + Y'(t) < 0 \). In other words, \( \frac{Z(t)}{\alpha + 1} + Y(t) \) is decreasing function and because \( Z \) and \( Y \) are always positive conclusion follows.

Remark 7. According to last two propositions necessary conditions for surviving of species \( X, Y, \) and \( Z \) are

\[
m_1 > 1, \quad m_2 > \frac{\alpha + 1}{\alpha} \quad \text{and} \quad 0 < \mu_1 < 1.
\]
3. Local Analysis in the Border of First Octant

We begin this section by taking a closer look to manifold \( \Delta^* \), noticing that because each trajectory tends to its \( \omega \)-limit set is enough to reduce the study of solutions to \( \Delta^* \), and in this set we may reduce (6) by taking

\[
S = 1 - \left( X + Y + \frac{Z}{\alpha} \right).
\]

Therefore, on \( \Delta^* \), (6) becomes, after dropping \( t \) in variables \( X, Y \) and \( Z \),

\[
\begin{align*}
X'(t) &= \left[ \frac{m_1 \left( 1 - X(t) - Y(t) - \frac{Z(t)}{\alpha} \right)}{a_1 + 1 - X(t) - Y(t) - \frac{Z(t)}{\alpha}} - 1 - \frac{m_2 Y(t)}{a_2 + X(t)} \right] X(t), \\
Y'(t) &= Z(t) - Y(t), \\
Z'(t) &= \frac{\alpha m_2 X(t) Y(t)}{a_2 + X(t)} - (\alpha + 1)Z(t).
\end{align*}
\] (10)

Besides, and as we know, \( S(t) > 0 \) if \( S(0) > 0 \) thus we restrict the study of system (10) to the region

\[
\Gamma = \left\{ (X,Y,Z)/X \geq 0, Y \geq 0, Z \geq 0, X + Y + \frac{Z}{\alpha} \leq 1 \right\}
\]

and the following straightforward result shows up;

**Lemma 8.** Solutions of (10) are such that \( \lim_{t \to \infty} Z(t) = 0 \) if and only if \( \lim_{t \to \infty} Y(t) = 0 \).

We notice now that equilibria of (10) are \( E_0 = (0,0,0) \) (origin) and \( E_{\mu_1} = (1 - \mu_1,0,0) \) located at the border of first octant; moreover, conditions for non existence or existence of unique equilibrium inside first octant will be given later on.

**Theorem 9.** If \( 0 < \mu_1 < 1 \) then \( E_0 \) is a saddle point.

**Proof.** We only need to look at the signs of eigenvalues of Jacobian matrix associated to (10) at the point \( E_0, \hat{J}(E_0) \). Indeed,

\[
\hat{J}(E_0) = \begin{bmatrix}
\frac{m_1}{a_1+1} & -1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -(\alpha + 1) & 1
\end{bmatrix},
\]

but \( \frac{m_1}{a_1+1} - 1 > 0 \) under the the given hypothesis. \( \Box \)

**Theorem 10.** Let us assume \( m_1 > 1, m_2 > \frac{\alpha+1}{\alpha}, 0 < \mu_1 < 1, \) and \( 0 < \mu_2 < 1. \) Then

1. For \( \mu_1 + \mu_2 \geq 1 \) the equilibrium point \( E_{\mu_1} \) is locally asymptotically stable.
2. For \( \mu_1 + \mu_2 < 1 \) the following two cases show up;
   a. \( E_{\mu_1} \) is locally asymptotically stable if \( \alpha < \frac{a_2+1-\mu_1}{(m_2-1)(1-\mu_1-\mu_2)}. \)
(b) $E_{\mu_1}$ is unstable if $\alpha > \frac{a_2+1-\mu_1}{(m_2-1)(1-\mu_1-\mu_2)}$.

**Proof.** If we denote Jacobian matrix associated to $E_{\mu_1}$ as $J(E_{\mu_1})$ and $P(\lambda) = \det(\lambda I - J(E_{\mu_1}))$ its corresponding characteristic polynomial, then

$$P(\lambda) = \left(\lambda + (1 - \mu_1)\frac{a_1 m_1}{(a_1 + \mu_1)^2}\right) \left(\lambda^2 + (\alpha + 2)\lambda + \alpha \left(1 - \frac{m_2(1 - \mu_1)}{1 + a_2 - \mu_1}\right) + 1\right),$$

therefore $P(\lambda) = 0$ if and only if $\lambda = -(1 - \mu_1)\frac{a_1 m_1}{(a_1 + \mu_1)^2} < 0$ or

$$\lambda^2 + (\alpha + 2)\lambda + \alpha \left(1 - \frac{m_2(1 - \mu_1)}{1 + a_2 - \mu_1}\right) + 1 = 0. \quad (12)$$

In order to show (1) we notice that $\mu_1 + \mu_2 \geq 1$ implies $1 - \frac{m_2(1 - \mu_1)}{a_2 + (1 - \mu_1)} > 0$ and being $\alpha > 0$, it follows that all roots of $P(\lambda)$ have negative real part.

Under hypothesis in (2) it is easy to notice that for $\alpha < \frac{a_2+1-\mu_1}{(m_2-1)(1-\mu_1-\mu_2)}$, expression $\alpha(1 - \frac{m_2(1 - \mu_1)}{a_2 + (1 - \mu_1)} + 1)$ is positive which takes us to conclude that all roots of polynomial $P(\lambda)$ have negative real part implying local asymptotically stability of $E_{\mu_1}$. Besides, if $\alpha > \frac{a_2+1-\mu_1}{(m_2-1)(1-\mu_1-\mu_2)}$, then $\alpha(1 - \frac{m_2(1 - \mu_1)}{a_2 + (1 - \mu_1)} + 1) > 0$, so it must exist a couple of roots whose product is negative and consequently $E_{\mu_1}$ is unstable. \qed

**Theorem 11.** If $\mu_1 + \mu_\alpha < 1$, then $E_{\mu_1}$ is unstable.

**Proof.** Notice first that $\mu_1 + \mu_\alpha < 1$ implies $1 - \frac{m_2(1 - \mu_1)}{a_2 + (1 - \mu_1)} > 0$; therefore $\alpha(1 - \frac{m_2(1 - \mu_1)}{a_2 + (1 - \mu_1)} + 1) > a_2 + 1 - \mu_1$ but $(1 - \mu_1)(m_2 - 1) - a_2 = (m_2 - 1)(1 - \mu_1 - \mu_2)$, thus $\alpha > \frac{a_2+1-\mu_1}{(m_2-1)(1-\mu_1-\mu_2)}$ and conclusion follows from Theorem 10. \qed

**Theorem 12.** For $\mu_1$ and $\mu_\alpha$ given by (8),

(1) If $\mu_1 + \mu_\alpha \geq 1$, then there is not equilibrium point in the interior of first octant.

(2) If $\mu_1 + \mu_\alpha < 1$, then there exists, for any $\alpha$ fixed, unique equilibrium point $E_c = (X_c, Y_c, Z_c)$ in the interior of first octant.

**Proof.** If $E_c = (X_c, Y_c, Z_c)$ is an equilibrium point in the interior of first octant, in particular it has to lay inside $\Gamma$, then $Z_c = Y_c$ and $X_c = \mu_\alpha > 0$, moreover $Y_c$ satisfies the relation

$$(m_1 - 1)(1 - \mu_1 - \mu_\alpha - \frac{\alpha + 1}{\alpha} Y_c) = \frac{\alpha + 1}{\alpha} \frac{Y_c}{\mu_\alpha} (1 - a_1 - \mu_\alpha - \frac{\alpha + 1}{\alpha} Y_c). \quad (13)$$

Now if $\mu_1 + \mu_\alpha \geq 1$ and because $\mu_\alpha + \frac{\alpha + 1}{\alpha} Y_c < 1$ inside $\Gamma$, it is clear that right hand and left hand side in (13) have opposite sign which is impossible to occur, thus first part is obtained. For second part we only have to become aware that there is only one $Y_c$, with $0 < Y_c < \frac{\alpha}{\alpha + 1}$ satisfying (13), and $X_c$ as we already know. \qed

**Corollary 13.** If $\mu_1 + \mu_\alpha \geq 1$, then species $Y$ and $Z$ die out.
We finish this section by giving conditions under which equilibrium point $E_c = (X_c, Y_c, Z_c)$ with positive coordinates, inside $\Gamma$ ($Z_c = Y_c$ inside $\Gamma$), is asymptotically stable. In this case corresponding jacobian matrix of (10) at $E_c$, $\hat{J}(E_c)$, is given by

$$
\hat{J}(E_c) = \begin{bmatrix}
a - b & -\left(\frac{\alpha+1}{\alpha} + b\right) & -\frac{b}{\alpha} \\
0 & -1 & 1 \\
c & \alpha + 1 & -(\alpha + 1)
\end{bmatrix},
$$

$$
a = \frac{m_2 \mu \alpha Y_c}{(a_2 + \mu \alpha)^2}, \quad b = \frac{a_1 m_1 \mu \alpha}{(a_1 + 1 - \mu - \frac{\alpha+1}{\alpha} Y_c)^2}, \quad c = \frac{\alpha a_2 m_2 Y_c}{(a_2 + \mu \alpha)^2},
$$

and characteristic polynomial

$$P(\lambda) = \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0,$$

$$p_2 = (b - a) + (2 + \alpha), \quad p_1 = (\alpha + 2)(b - a) + \frac{cb}{\alpha}, \quad p_0 = c \frac{\alpha + 1}{\alpha} (b + 1). \quad (14)$$

**Theorem 14.** Let us suppose $\mu_1 + \mu_\alpha \geq 1$ and

$$\frac{Y_c}{m_2 \mu_\alpha^2} < \left(\frac{\alpha}{\alpha + 1}\right)^2 \frac{a_1 m_1}{(a_1 + 1 - \mu - \frac{\alpha+1}{\alpha} Y_c)^2}. \quad (15)$$

If

$$\frac{1}{\mu_\alpha} < \frac{1}{a_1 m_1 \alpha + 1 (a_1 + 1 - \mu - \frac{\alpha+1}{\alpha} Y_c)^2},$$

then $E_c$, given as before, is locally asymptotically stable.

**Proof.** Theorem 12 guarantees existence and uniqueness of $E_c$, its local asymptotic stability will be obtained by using Routh Hurwitz’s criterion together with the given hypothesis. Actually, $E_c$ is locally asymptotically stable if and only if $p_0 > 0$, $p_1 > 0$, $p_2 > 0$ and $p_2 p_1 - p_0 > 0$. But $p_0 > 0$; both $p_1$ and $p_2$ are positive if and only if $b - a > 0$. Besides

$$b - a = -\mu_\alpha \left[\frac{Y_c}{m_2 \mu_\alpha^2} - \left(\frac{\alpha}{\alpha + 1}\right)^2 \frac{a_1 m_1}{(a_1 + 1 - \mu - \frac{\alpha+1}{\alpha} Y_c)^2}\right] > 0.$$

In the same token, being $b - a > 0$; $p_2 p_1 - p_0 > 0$ if and only if $b - (\alpha + 1) > 0$ and

$$b - (\alpha + 1) = \mu_\alpha \left[\frac{a_1 m_1}{(a_1 + 1 - \mu - \frac{\alpha+1}{\alpha} Y_c)^2} - \frac{\alpha + 1}{\mu_\alpha}\right] > 0$$

as desired. Notice we have used all inequalities in the hypothesis. \qed
4. Global Analysis in the Border of First Octant

We begin by giving conditions under which \( E_0 \) is a global attractor, actually the following result, which is a direct consequence of Propositions 5 and 6, comes out,

**Theorem 15.** If \( \mu_1 > 1 \) and \( m_2 \leq \frac{\alpha+1}{\alpha} \), then all solutions of (10) satisfy

\[
\lim_{t \to \infty} (X(t), Y(t), Z(t)) = (0, 0, 0).
\]

In the same fashion for \( E_{\mu_1} \) in first octant, and in absence of \( E_c \), is the attractor.

**Theorem 16.** If \( \mu_1 < 1 \) and \( \mu_1 + \mu_\alpha \geq 1 \), then all solutions of (10) satisfy

\[
\lim_{t \to \infty} (X(t), Y(t), Z(t)) = (1 - \mu_1, 0, 0).
\]

**Proof.** The global stability will be obtained by showing that for any solution of (10), \((X(t), Y(t), Z(t))\), satisfies \( \lim_{t \to \infty} Y(t) = \lim_{t \to \infty} Z(t) = 0 \) and then \( \lim_{t \to \infty} X(t) = 1 - \mu_1 \). Actually, by hypothesis \( 1 - \mu_1 < \mu_\alpha \), and from first equation in (10),

\[
X'(t) < \left( \frac{m_1(1 - X(t))}{a_1 + 1 - X(t)} - 1 \right) X(t)
\]

\[
= \left( \frac{m_1 - 1 - a_1}{a_1} X(t) - (m_1 - 1)X^2(t) \right)
\]

\[
= \frac{1}{\mu_1} \left( 1 - \mu_1 - X(t) \right) X(t),
\]

therefore

\[
X'(t) < \frac{1 - \mu_1}{\mu_1} X(t) - \frac{1}{\mu_1} X^2(t),
\]

but \( X(t) > 0 \) for all \( t \geq 0 \); so we set \( v(t) = X^{-1}(t) ; t \geq 0 \), thus (16) now looks like

\[
v'(t) + \frac{1 - \mu_1}{\mu_1} v(t) > \frac{1}{\mu}, \quad (17)
\]

and from here \( \lim_{t \to +\infty} v(t) > \frac{1}{1 - \mu_1} \) or better, \( \lim_{t \to \infty} X(t) < 1 - \mu_1 \); but then there is \( p > 0 \) such that \( \lim_{t \to \infty} X(t) = p \). Because \( X(t) < 1 - \mu_1 < \mu_\alpha \), there exists \( T > 0 \) such that \( X(t) < \frac{a_2}{\alpha + m_2}, t \geq T \), thus \( \left( \frac{\alpha}{\alpha + 1} m_2 - 1 \right) X(t) - a_2 < 0 \), \( t \geq T \). On the other hand, from second and third equation in (10),

\[
\frac{1}{\alpha + 1} Z'(t) + Y'(t) = \frac{Y(t)}{a_2 + X(t)} \left( \frac{\alpha}{\alpha + 1} m_2 - 1 \right) X(t) - a_2 < 0.
\]
Now if \( \frac{1}{\alpha+1} Z(t) + Y(t) \) tends, as \( t \) goes to infinity, to a value \( c > 0 \), by Lemma 8, both \( Z(t) \) and \( Y(t) \) tend to values \( c_1 > 0 \) and \( c_2 > 0 \) respectively, which implies that whole solution approaches to \( (p, c_1, c_2) \) as \( t \to +\infty \) and by hypothesis this is impossible to happen, unless \( c_1 = c_2 = 0 \) but then \( (X(t), Y(t), Z(t)) \to (p, 0, 0) \) for \( t \to +\infty \). Al these mean that the \( \omega \)-limit set of any solution of system (10), with the given hypothesis, is contained in \( \Theta = [0, 1 - \mu_1] \times \{0\} \times \{0\} \), but the only \( \omega \)-limit points in \( \Theta \) are \( (0, 0, 0) \) and \( (1 - \mu_1, 0, 0) \) indeed, as known, \( (0, 0, 0) \) is a saddle point and \( x \) axis is an orbit which goes in opposite direction to this point, therefore it does not belong to \( \Theta \); the only one left in \( \Theta \) is then \( (1 - \mu_1, 0, 0) \), and this is exactly \( \lim_{t \to +\infty} X(t) = 1 - \mu_1 \). □

5. Periodic Orbits and Numerical Implementation

In this Section we perform some numerical implementations in order to show behavior of orbits in the invariant region when time goes by; with same initial data, values of \( \alpha \) are varied which causes appearance of possible period orbits as we shall show below. One of the basic problems in delayed models is indeed the search of periodic solutions, which in our model into consideration may show up when inequality (15) is reversed causing that in equilibrium point \( E_c \) appears a Hopf bifurcation. This is one of the goal we reach by means of the numerical simulation. Along the whole simulation we chose the following constants

\[
m_1 = 8.5, \quad m_2 = 6, \quad a_1 = 0.6, \quad a_2 = 0.6.
\]

In the same token, and for each picture below, we pick the initial data as

\[
X(0) = 0.1, \quad Y(0) = 0.7, \quad Z(0) = 0.8.
\]

The first picture, Figure 1, corresponds to above given values and \( \alpha = 1.02 \), and shows behavior of the corresponding orbit when \( t \) increases. It is worth to notice that orbits remain bounded as predicted by the theoretical results.

Another fact we may get from different implementations and \( \alpha \in [1.02, 1.03] \) is that orbits approach to \( E_c = (0.29561, 0.2274, 0.22764) \), picture above illustrates this situation.

Figure 2 corresponds to \( \alpha = 3.9 \), now \( E_c = (0.15892, 0.32445, 0.32445) \) and it is possible to observe how orbits get away from \( E_c \) and period orbits start to show up.

In next and last picture, Figure 3, \( \alpha = 4.5, \quad E_c = (0.15349, 0.32939, 0.32939) \), again orbits get away from \( E_c \) and possible period orbits start to show up.

Finally we would like to mention that introduction of distributed delay may cause a destabilizing effect in the model inducing oscillations and, very often, periodic orbits appear which is a very rare situation in model where delay is absent. At the same time from the results in this research, analytic and numeric, we may infer that introduction of distributed delay in the present food chain model opens the way to get desired good and precise results from the biologic point of view with regard to the coexistence of three species in the ecosystem since in the case of ”big delays” coexistence appears in the form
of one point with positive coordinates while for ”small delays” it shows up by means of periodic orbits. With respect to the equilibria in this model we must say that global stability of $E_{\mu_1}$ implies that predator will be expel from chemostat regardless of initial density levels of both prey and predator, also $E_c$ appears at the moment $E_{\mu_1}$ looses stability.

**Figure 1.** Orbit of system (10), $\alpha = 1.02$.

**Figure 2.** Orbit of system (10), $\alpha = 3.9$. 
Figure 3. Orbit of system (10), $\alpha = 4.5$.  

References


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