Model Reduction Based on Triangle Realization with Pole Retention

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Abstract

Model order reduction is a research direction which has provided interest to many scientists recently. A large number of order reduction algorithms have been introduced in many different approaches among which retaining the important poles of the original system in the reduced root system is a dominant approach with many advantages.

This paper presents a new model in order reduction algorithm for both stable and unstable systems, based on the idea of keeping the dominant poles of the original system in the order reduction process. This algorithm transforms matrix $A$ of the higher-order original system to upper - triangle matrix on which the poles are arranged basically on $H_\infty$ - and $H_2$ - mixed dominant index on the main diagonal of the upper – triangle matrix so that error becomes small and the dominant poles are preserved. The illustration shows the correctness of the model order algorithm.
**I. Introduction**

Modal truncation technique is simple and cheap based on computation \([1]\). It has retained stability and dominant poles of the original system. However, most of the algorithms proposed recently as in \([3]\) \([4]\) \([5]\) had their own disadvantages. In \([5]\), dominant poles were defined, but the algorithm determining the dominant poles showed high computational complexity. In \([4]\), a more simple algorithm was determined but no definition of dominant poles was defined and the reduced system parameters were complex numbers. However, the requirement of the reduction order model problem is that both original and reduced order systems have real number parameters. Therefore, to complete the algorithm in \([4]\), in \([3]\) the algorithm has been modified in order that reduced system parameters are real numbers, but the algorithm is expensive based on the computation.

On the other hand, the general disadvantage of these methods is that the algorithm can reduce asymptotically stable system whereas in fact higher linear system can be unstable. Therefore, in this study, a new reduced order model algorithm based on modal truncation algorithm, Schur decomposition and triangle realiztion is proposed in order to overcome the disadvantages of the above algorithms.

This paper was written in the following structure. A new model order reduction for stable and unstable systems was presented in Section II. In Section III, the effectiveness of the new model order reduction algorithm was illustrated by two examples, namely: reducing the order of higher-order of stable system, and reducing order of higher-order of unstable system. Finally, conclusions were presented in Section V.

**II. New model order reduction algorithm for stable system and unstable system**

**2.1 Problem of order reduction model**

A multiple-inputs multiple-outputs linear system is given with continuous-time constant parameters described in space stated in the following equations:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx 
\end{align*}
\]  

(2.1)

In which, \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^p\), \(y \in \mathbb{R}^q\), \(A \in \mathbb{R}^{nxn}\), \(B \in \mathbb{R}^{nsp}\), \(C \in \mathbb{R}^{qxn}\)

The goal of the order reduction problem with model described by (2.1) is to find models described by systems of equations:

\[
\begin{align*}
\dot{x}_r &= A_r x_r + B_r u \\
y_r &= C_r x_r
\end{align*}
\]  

(2.2)
In which, \( x, u, y \in \mathbb{R}^r, A \in \mathbb{R}^{r \times r}, B_r \in \mathbb{R}^{r \times p}, C_r \in \mathbb{R}^{q \times r} \) and \( r \ll n \), so that the model described by (2.2) can be replaced by the model described in (2.1) to be applied in analysis, design and control system.

### 2.2. New model order reduction algorithm based on triangle realization

**Input:** The original system \((A, B, C)\), which is described by equation (2.1) (unstable system)

**Step 1:** Decompose unstable system into two subsystems, stable subsystem and unstable subsystem.

**Step 1.1:** Transform this system in block diagonal upper Schur form

\[
A_t = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}; B_t = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}; C_t = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}
\]

where \( A_{11} \in \mathbb{R}^{mxm}, A_{12} \in \mathbb{R}^{mx(n-m)}, A_{22} \in \mathbb{R}^{(n-m) \times (n-m)}, B_{11} \in \mathbb{R}^{mxp}, B_{12} \in \mathbb{R}^{(n-m) \times p}, C_{11} \in \mathbb{R}^{q \times m}, C_{12} \in \mathbb{R}^{q \times (n-m)} \)

where \( n \) denotes order of the system, \( m \) denotes the number of stable eigenvalues and \( n - m \) denotes the number of unstable eigenvalues.

**Step 1.2:** Compute \( S \) came from Lyapunov equation

\[
A_{11}S - SA_{12} + A_{112} = 0
\]

**Step 1.3:** Determine the transition matrix

\[
W = \begin{bmatrix} I_r & S \\ & \ddots & \vdots \\ 0 & \cdots & I_{n-r} \end{bmatrix}
\]

where \( I_m \) and \( I_{n-m} \) are identity matrix of size \( mxm \) and \((n-m) \times (n-m)\)

**Step 1.4:** Compute \((A_d, B_d, C_d) = (W^{-1}AW, W^{-1}B_1, CW)\)

**Step 1.5:** \((A_d, B_d, C_d)\) is partitioned as follows

\[
A_d = \begin{bmatrix} A_{d11} & 0 \\ 0 & A_{d22} \end{bmatrix}; B_d = \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix}; C_d = \begin{bmatrix} C_{d1} & C_{d2} \end{bmatrix}
\]

where \( A_{d11} \in \mathbb{R}^{mxm}, A_{d22} \in \mathbb{R}^{(n-m) \times (n-m)}, B_{d1} \in \mathbb{R}^{mxp}, B_{d2} \in \mathbb{R}^{(n-m) \times p}, C_{d1} \in \mathbb{R}^{q \times m}, C_{d2} \in \mathbb{R}^{q \times (n-m)} \).

#### Stable subsystem \((A_{d11}, B_{d1}, C_{d1})\)

#### Unstable subsystem \((A_{d22}, B_{d2}, C_{d2})\)

**Step 2:** Triangle realization of stable subsystem \((A_{d11}, B_{d1}, C_{d1})\) in which \( A_{d11} \) has triangle form, according to the following steps:

**Step 2.1:** Compute Schur decomposition of \( A_{d1} = U \Delta U^T \), where \( U \) is unitary matrix and \( \Delta \) is upper triangle matrix.
Step 2.2: Compute observability Gramian $Q$ came from Lyapunov equation

$$\Delta Q + Q \Delta + (CU)^T (CU) = 0$$

Step 2.3: Compute Cholesky factorization of observability Gramian $Q$: $Q = R^T R$, where $R$ is upper triangle matrix.

Step 2.4: Compute nonsingular transformation $T = UR^{-1}$

Step 2.5: Compute $(\tilde{A}, \tilde{B}, \tilde{C}) = (T^{-1} A \tau T, T^{-1} B \tau, C \tau T)$

**Step 3.** Re-order the poles on the main diagonal of the upper – triangle matrix $\tilde{A}$ by $H_{\infty}$-, $H_2$- and mixed - dominant index.

**Input:** The triangle realization $(\tilde{A}, \tilde{B}, \tilde{C})$

**Step 3.1:** For each pole $\lambda_i$, $i = 1, \ldots, n$ compute its $H_{\infty}$ - dominant index $R_i = \frac{\| \tilde{C} \tilde{B} \|}{| \text{Re} \lambda_i |}$ (or respectively, its $H_2$ - dominant index $S_i = \sqrt{\text{trace}(\tilde{B}^T \tilde{B})}$, or its mixed - dominant index, $J_i = \max \{ R_i, S_i \}$)

**Step 3.2:** Choose the largest $H_{\infty}$ - dominant index $R_i$ (respectively for $H_2$ - dominant index, mixed - dominant index)

**Step 3.3:** Reorder the pole $\lambda_i$ (and its conjugate $\bar{\lambda}_i$, if it appears) to the first position in the diagonal of $\tilde{A}$ by unitary matrix $U_1$:

$$U_1^T \tilde{A} U_1 = \begin{bmatrix} \lambda_i & * & * & * \\ \bar{\lambda}_i & * & * & * \\ \vdots & \ddots & \ddots & \ddots \\ & & & * \end{bmatrix}$$

**Step 3.4:** Compute new equivalent realization $(U_1^T \tilde{A} U_1, U_1^T \tilde{B}, \tilde{C} U_1)$.

**Step 3.5:** Remove two first rows and columns of $(U_1^T \tilde{A} U_1, U_1^T \tilde{B}, \tilde{C} U_1)$ to obtain a smaller realization $(\tilde{A}, \tilde{B}, \tilde{C})$ with $n - 2$ dimensions.

**Step 3.6:** Do the same procedure from Step 1 to Step 5 for smaller realization $(\tilde{A}, \tilde{B}, \tilde{C})$ and continue this loop until all poles are re-ordered.

**Output:** The equivalent system $(\tilde{A}, \tilde{B}, \tilde{C})$ with the poles that are arranged in descending of $H_{\infty}$-, $H_2$- and mixed-dominant indices on the main diagonal of the upper – triangle matrix $\tilde{A}$

The most important new feature of the algorithm for stable system is the ability to arrange the poles in $H_{\infty}$-, $H_2$- and mixed - dominant indices of decreasing on the
main diagonal of the upper triangular matrix \( \tilde{A} \), which helps to retain dominant poles of the original model in order reduction model.

**Step 4:** Reduce equivalent system \( (\tilde{A}, \tilde{B}, \tilde{C}) \)

**Input step 4:** The equivalent system \( (A, B, C) \)

**Step 4.1:** Choose re-order \( r \) so that \( r \ll n \)

**Step 4.2:** \((\tilde{A}, \tilde{B}, \tilde{C})\) is partitioned as follows

\[
\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \quad \tilde{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}
\]

where \( A_{11} \in \mathbb{R}^{r \times r}, B_i \in \mathbb{R}^{r \times p}, C_i \in \mathbb{R}^{q \times r} \)

**Output step 4:** The reduced order stable subsystem \((A_{11}, B_1, C_1)\).

**Output:** The reduced order unstable system \((A_{d12}, B_{d1}, C_{d1})\)

### III. The illustrative example

#### 3.1. Reducing stable system

The 6th–order stable digital filter that is often selected to evaluate the model order reduction algorithm is given in [6] as follows:

\[
G(z) = \frac{-0.1242z^{5} + 0.1581z^{4} + 0.5273z^{3} + 0.2154z^{2} - 0.0647z + 0.6889}{z^{6} - 1.095z^{5} + 1.299z^{4} - 1.113z^{3} + 1.028z^{2} - 0.6043z + 0.426} \quad (3.1)
\]

Converting discrete-time dynamic system (3.1) to continuous time, we obtain the following model:

\[
G(s) = \frac{-0.1242s^{5} - 0.4629s^{4} - 0.0823s^{3} + 1.504s^{2} + 1.959s + 1.401}{s^{6} + 4.905s^{5} + 10.82s^{4} + 13.13s^{3} + 9.533s^{2} + 3.834s + 0.9407} \quad (3.2)
\]

In performing the reduced order model system (3.2) by new algorithm basic on \( H_{2} \)- dominant index, the results are shown in the following table:

**Table 1. Results of the order reduction of stable system**

<table>
<thead>
<tr>
<th>Order</th>
<th>Reduced order system ( G_{r}(s) )</th>
<th>Error ( |G(s) - G_{r}(s)|<em>{H</em>{2}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( -0.1258s^{4} - 0.51s^{3} + 0.3043s^{2} + 1.413s + 1.698 ) ( s^{5} + 4.903s^{4} + 9.523s^{3} + 8.828s^{2} + 4.008s + 1.151 )</td>
<td>0.0145</td>
</tr>
<tr>
<td>4</td>
<td>( -0.1685s^{3} - 0.1015s^{2} + 0.2835s + 1.052 ) ( s^{4} + 3.318s^{3} + 4.263s^{2} + 2.07s + 0.7264 )</td>
<td>0.0360</td>
</tr>
</tbody>
</table>

Bode plots of the original system, reduced order systems results are shown Fig.1
Fig. 1. Bode plots of the original system, reduced order systems

Remarks of the results: From the result of order reduction, evaluation of the reduced order error and bode plots showed that:
- The error between the original system and the 5th-order system is smaller than the error between the original system and the 4th-order system (0.0145 < 0.0360).
- Bode plots of the 5th-order system almost coincides with bode plots of the original system. Bode plots of 4th-order system has small deviations with bode plots of the original system in the high frequency (frequency > 5.4 rad/s)
Therefore, the 5th, 4th-order system can be used to replace the original system. The above results prove that the new algorithm can reduce high order stable system.

3.2. Reducing unstable system

A high order unstable system as follows:

\[ R(s) = \frac{H(s)}{D(s)} \]

\[ H(s) = -2.23.10^{-7}s^{30} - 4.67.10^{-4}s^{29} - 0.266s^{28} - 22.96s^{27} - 1006s^{26} - 2.853.10^4s^{25} 
- 5.837.10^5s^{24} - 9.144.10^6s^{23} - 1.139.10^7s^{22} - 1.158.10^8s^{21} - 9.776.10^9s^{20} 
- 6.949.10^9s^{19} - 4.199.10^{11}s^{18} - 2.172.10^{12}s^{17} - 9.663.10^{13}s^{16} - 3.71.10^{13}s^{15} 
- 1.231.10^{14}s^{14} - 3.53.10^{14}s^{13} - 8.74.10^{14}s^{12} - 1.862.10^{15}s^{11} - 3.398.10^{15}s^{10} 
- 5.276.10^{15}s^9 - 6.903.10^{15}s^8 - 7.511.10^{15}s^7 - 6.676.10^{15}s^6 - 4.721.10^{15}s^5 
- 2.556.10^{15}s^4 - 9.953.10^{14}s^3 - 2.482.10^{14}s^2 - 2.977.10^{13}s - 0.00439 \]
Model reduction based on triangle realization with pole retention

\[ D(s) = 4.971 \times 10^{-14} s^{30} + 2.032 \times 10^{-10} s^{29} + 2.663 \times 10^{-7} s^{28} + 1.221 \times 10^{-4} s^{27} + 9.72 \times 10^{-3} s^{26} \]
\[ + 0.3918 s^{25} + 10.14 s^{24} + 187.1 s^{23} + 2612 s^{22} + 2.862 \times 10^4 s^{21} + 2.523 \times 10^5 s^{20} \]
\[ + 1.82 \times 10^6 s^{19} + 1.088 \times 10^7 s^{18} + 5.428 \times 10^7 s^{17} + 2.273 \times 10^8 s^{16} + 8.005 \times 10^8 s^{15} \]
\[ + 2.372 \times 10^9 s^{14} + 5.91 \times 10^9 s^{13} + 1.225 \times 10^{10} s^{12} + 2.107 \times 10^{10} s^{11} + 2.962 \times 10^{10} s^{10} \]
\[ + 3.341 \times 10^{10} s^9 + 2.941 \times 10^{10} s^8 + 1.931 \times 10^9 s^7 + 8.743 \times 10^8 s^6 + 2.286 \times 10^8 s^5 \]
\[ + 1.519 \times 10^7 s^4 - 5.226 \times 10^7 s^3 + 3.610^{-6} s^2 + 5.32 \times 10^{-22} s \]

Converting unstable system (3.2) to minimal realization and pole-zero cancellation, we obtain the following result:

\[ R_m(s) = \frac{H_m(s)}{D_m(s)} \quad (3.4) \]
\[ H_m(s) = -4.485 \times 10^6 s^{27} - 4.231 \times 10^6 s^{26} - 1.912 \times 10^6 s^{25} - 5.51 \times 10^{11} s^{23} - 1.138 \times 10^{13} s^{23} \]
\[ - 1.794 \times 10^9 s^{22} - 2.245 \times 10^9 s^{22} - 2.29 \times 10^8 s^{20} - 1.939 \times 10^7 s^{19} - 1.381 \times 10^6 s^{16} \]
\[ - 8.359 \times 10^7 s^7 - 4.33 \times 10^6 s^6 - 1.929 \times 10^6 s^5 - 7.413 \times 10^5 s^4 - 2.462 \times 10^5 s^3 \]
\[ - 7.066 \times 10^2 s^2 - 1.75 \times 10^3 s^2 - 3.732 \times 10^2 s - 6.814 \times 10^2 s^2 - 1.058 \times 10^2 s^3 \]
\[ - 1.385 \times 10^3 s^3 - 1.508 \times 10^3 s^3 - 1.341 \times 10^2 s^3 - 9.487 \times 10^2 s^3 - 5.138 \times 10^2 s^3 \]
\[ - 2.001 \times 10^2 s^2 - 4.99 \times 10^2 s - 5.987 \times 10^2 \]
\[ D_m(s) = s^7 + 2.088 s^6 + 1.803 \times 10^5 s^2 + 7.485 \times 10^5 s^4 + 1.966 \times 10^8 s^{23} + 3.66 \times 10^8 s^{22} \]
\[ + 5.14 \times 10^7 s^2 + 5.657 \times 10^6 s^2 + 5.003 \times 10^5 s^2 + 3.618 \times 10^5 s^2 + 2.166 \times 10^4 s^1 \]
\[ + 1.083 \times 10^4 s^1 + 4.539 \times 10^3 s^1 + 1.601 \times 10^3 s^1 + 4.748 \times 10^2 s^1 + 1.182 \times 10^2 s^1 \]
\[ + 2.456 \times 10^1 s^1 + 4.226 \times 10^1 s^1 + 5.944 \times 10^1 s^1 + 6.709 \times 10^1 s^1 + 5.908 \times 10^1 s^1 \]
\[ + 3.881 \times 10^1 s^1 + 1.758 \times 10^1 s^1 + 4.598 \times 10^1 s^1 + 3.058 \times 10^1 s^1 - 1.051 \times 10^1 s^1 \]

Performing reduced order model unstable system (3.4) by new algorithm based on \( H_2 \) - dominant index, the results are shown in the following:

**Step 1:** The unstable system is decomposed into two subsystems, stable subsystem and unstable subsystem as follows.

**Stable subsystem:** \( R_{stable}(s) = \frac{A(s)}{B(s)} \)

with

\[ A(s) = -4.485 \times 10^6 s^2 - 2.435 \times 10^5 s^2 - 1.904 \times 10^6 s^2 - 5.422 \times 10^5 s^2 - 1.098 \times 10^5 s^2 \]
\[ - 1.683 \times 10^4 s^9 - 2.029 \times 10^4 s^9 - 1.974 \times 10^4 s^9 - 1.576 \times 10^4 s^9 - 1.046 \times 10^4 s^9 \]
\[ - 5.827 \times 10^3 s^9 - 2.737 \times 10^3 s^9 - 1.088 \times 10^2 s^9 - 3.666 \times 10^2 s^9 - 1.046 \times 10^2 s^9 \]
\[ - 2.518 \times 10^1 s^9 - 5.092 \times 10^1 s^9 - 8.575 \times 10^1 s^9 - 1.188 \times 10^2 s^9 - 1.331 \times 10^2 s^9 \]
\[ - 1.175 \times 10^2 s^2 - 7.877 \times 10^2 s^2 - 7.366 \times 10^2 s^2 - 1.144 \times 10^2 s^1 - 1.659 \times 10^2 \]
$B(s) = s^{24} + 2088s^{23} + 1.806.10^4s^{22} + 7.504.10^6s^{21} + 1.973.10^8s^{20} + 3.681.10^9s^{19} + 5.178.10^{10}s^{18} + 5.71.10^{11}s^{17} + 5.062.10^{12}s^{16} + 3.671.10^{13}s^{15} + 2.204.10^{14}s^{14} + 1.105.10^{15}s^{13} + 4.653.10^{15}s^{12} + 1.649.10^{16}s^{11} + 4.918.10^{16}s^{10} + 1.233.10^{17}s^9 + 2.583.10^{17}s^8 + 4.492.10^{17}s^7 + 6.408.10^{17}s^6 + 7.371.10^{17}s^5 + 6.669.10^{17}s^4 + 4.57.10^{17}s^3 + 2.23.10^{17}s^2 + 6.9.10^6s + 1.018.10^{16}$

Unstable subsystem: $R_{\text{unstable}}(s) = \frac{-5.838.10^4s^2 - 9.163.10^4s - 5.881.10^4}{s^3 - 0.1032s^2 + 1.185.10^{-17}s - 1.513.10^{-18}}$

**Step 2-4:** Performing reduced order model system follows step 2 – 4 in new algorithm basic on $H_2$ - dominant index, the results shown in the following table:

**Table 2. Results of the order reduction of unstable system**

<table>
<thead>
<tr>
<th>Order</th>
<th>Reduced order system $R_i(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$-4.485.10^8s^4 - 6.804.10^7s^3 - 4.123.10^7s^2 - 1.235.10^6s^1 - 1.816.10^5s - 1.09.10^4$ $s^5 + 2009s^4 + 1.833.10^7s^3 - 1913s^2 + 2.165.10^{13}s - 2.804.10^{14}$</td>
</tr>
<tr>
<td>4</td>
<td>$-4.485.10^8s^4 - 2.65.10^7s^3 - 1.141.10^7s^2 - 1.833.10^6s - 1.176.10^4$ $s^5 + 2000s^4 - 206.5s^3 + 2.369.10^{14}s - 3.026.10^{15}$</td>
</tr>
</tbody>
</table>

Step response and bode plots of the original system, reduced order systems results are shown Fig. 2.

![Step Response](image1.png)  
![Bode Diagram](image2.png)

**Fig. 2. Step response (a) and bode plots (b) of the original system, reduced order systems**

**Remarks of the results:** Step response and bode plots of the 4th, 5th - order system almost coincides with step and bode plots of the original system. Therefore, the 5th, 4th - order system can be used to replace the original system. The above results show that the high order unstable system can be reduced by the new algorithm.

To compare the effectiveness of the new order model reduction algorithm with the balanced truncation, the authors perform enough order reduction of the
unstable system based on balanced truncation algorithm [2] (using `balancmr` function in Matlab - Simulink), results were obtained as shown in the following table:

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Reduced order system $R_i(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balancmr</td>
<td>$\frac{-4.485.10^5 s^4 - 5.351.10^5 s^3 + 7.513.10^5 s^2 + 2.822.10^7 s + 1.307.10^7}{s^4 + 2000 s^3 - 206.5 s^2 + 1.258.10^{10} s - 4.767.10^{12}}$</td>
</tr>
</tbody>
</table>

Step response and bode plots of the original system, reduced order systems results are shown Fig.3.

![Step Response](image1)

![Bode Diagram](image2)

(a) Step Response  
(b) Bode Diagram

**Fig. 3. Step response (a) and bode plots (b) of the original system, reduced order systems**

**Remarks of the results:** Step response and bode plots of the 4th-order system based on new algorithm almost coincides with step and bode plots of the original system. Step and bode plots of the 4th-order system based on the balanced truncation has large deviations with step and bode plots of the original system. Therefore, the 4th-order system based on the balanced truncation cannot be used to replace the original system.

**IV. Conclusions**

This paper introduced in detail the new model order reduction algorithm for stable and unstable systems. The most important new feature of the algorithm for stable system is the ability to arrange the poles in $H_\infty$, $H_2$, and mixed-dominant indices in decreasing the main diagonal of the upper triangular matrix $A$ and the ability to retain the dominant poles of the original model in order reduction model. The effectiveness of the new model order reduction algorithm is illustrated by reducing order of higher-order stable system and unstable system. The simulation results showed the correctness of the proposed algorithm.
References


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