

Global Stability Analysis on a Predator-Prey Model with Omnivores

Puji Andayani

Mathematics Department
Brawijaya University
Jl. Veteran Malang 65145, Indonesia

Wuryansari Muharini Kusumawinahyu

Mathematics Department
Brawijaya University
Jl. Veteran Malang 65145, Indonesia

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Abstract

This paper concerns with a stability analysis of a three species predator-prey model, which describes interaction between prey, predator, and omnivore. Some parameter combinations resulting at most five equilibrium points under some sufficient conditions for their existence. It can be shown that all solutions which start from positive initial conditions are bounded. Local stability analysis of the equilibria is performed by using standard method of ODE. The analysis results can be classified into two categories of parameters. For parameters in the first category, there is no positive equilibrium and we rigorously prove that all solution trajectories converge to the boundary equilibria. We also perform some numerical simulations to support the analytical results. For parameters in the second category, we provide some numerical simulations and find that beside global stability, chaotic phenomenon is also captured.

Mathematics Subject Classification: 92D25, 92D40

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1 Introduction

Predator - prey model, which is a Mathematical model describing the growth rate of the population involved in predation process, was originally proposed by Lotka and Volterra and it is well known as Lotka - Volterra model [4]. Actually, interactions in nature involve not only two species, such as herbivores-carnivores which act as prey-predator, but also may involve omnivores as the third species. This complicated process was rarely considered due to its destabilizing effect. Recent elaborate studies of natural food webs have shown that omnivorous is ubiquitous [6].

A three species predator - prey model, where the third species are omnivores, is considered in this paper. This model is constructed by assuming that there are only three species in such an isolated ecosystem. The first species, called as prey, acting as the prey for the second and the third species. The second species, called as predator, only feeds on the first species and can extinct with the absence of prey. The third species, namely omnivores, eat prey and the carcasses of predator. Consequently, the omnivores predation only reduces the prey population but does not affect the predator growth. It is also assumed that the prey population grows logistically and there is competition between omnivores. Based on these assumption, the mathematical model representing those three population density growth rates is governed by a nonlinear ordinary differential equation system, namely

$$\begin{aligned}\dot{x} &= x(1 - y - z - bx), \\ \dot{y} &= y(-c + x), \\ \dot{z} &= z(-e + fx + gy - \beta z).\end{aligned}\tag{1}$$

In this model, $x(t)$, $y(t)$, and $z(t)$ stand for the density of prey, predator, and omnivore populations, respectively. All parameters of model (1) are positive. The death rates of the predator and the omnivore are denoted by c and e , respectively. The parameter f means the rate of omnivore z feeding preys upon x , while the parameter g is the rate of y carcasses predation by z . Parameter b and β are the carrying capacity of the prey and the omnivore, respectively [5].

Interaction among these three species can be observed by learning the populations growth. In order to investigate these growths we perform dynamical analysis involving determination of equilibrium point, its existence condition, and its stability. Both local and global stability analysis are conducted. Numerical simulations are presented to illustrate the analytical result.

2 Equilibria and Boundedness of Solutions

In this section, we provide some sufficient conditions that guarantee the existence of equilibrium points of system (1). Then we show that all solutions of (1) with positive initial values are bounded. There are five equilibrium points of the system (1), namely

$$E_0 = (0, 0, 0), E_1 = (1/b, 0, 0), E_2 = (c, 1 - bc, 0),$$

$$E_3 = \left(\frac{e + \beta}{f + \beta b}, 0, \frac{f - be}{f + \beta b} \right),$$

and

$$E_4 = \left(c, \frac{\beta(1 - bc) - (fc - e)}{g + \beta}, \frac{g(1 - bc) + (fc - e)}{g + \beta} \right).$$

To accommodate biological meaning, the existence conditions for the equilibria require that they are nonnegative. It is obvious that E_0 and E_1 always exist, E_2 exist when $bc < 1$, while E_3 exist when $f > be$. If we pay attention to E_4 , the total densities of predators and omnivores, namely $1 - bc$, has to be positive. Hence, E_4 exist when $bc < 1$ and $0 < fc - e < \beta(1 - bc)$.

The following lemma guarantee the positivity of the solutions of system (1) when they start from positive initial conditions.

Lemma 2.1. *The space $H_1 = \{(x, 0, 0) \mid x > 0\}$, $H_2 = \{(x, y, 0) \mid x > 0, y > 0\}$, $H_3 = \{(x, 0, z) \mid x > 0, z > 0\}$, and $H_4 = \{(0, y, z) \mid y > 0, z > 0\}$ are invariant under system (1).*

The local as well as global stability of equilibria and their stability conditions in those invariant sets are summarized in the following proposition.

Proposition 2.2. *The space H_1, H_2, H_3 , and H_4 are invariant. Moreover,*

- (i) *on H_1 , the equilibrium E_0 is unstable while E_1 is global asymptotically stable,*
- (ii) *on H_2 , the equilibrium E_2 does not exists and E_1 is global asymptotically stable when $bc > 1$. Otherwise, E_0 and E_1 are saddle and E_2 is global asymptotically stable.*
- (iii) *on H_3 , if $f < be$ then E_3 does not exists and E_1 is global asymptotically stable. Otherwise, E_0 and E_1 are saddle and E_3 is global asymptotically stable.*
- (iv) *on H_4 , the trivial equilibrium E_0 is global asymptotically stable.*

Since the mathematical model is constructed under assumption that preys grow logistically, then any solutions $(x(t), y(t), z(t))$ of (1) has to be bounded. Refers to the work [2], by proofing the following lemma, we show the boundedness of the solution of (1).

Lemma 2.3. *Any solution $(x(t), y(t), z(t))$ of (1) is bounded.*

Proof. Let's consider $u(t), v(t)$ satisfying the following initial value problem

$$\begin{aligned}\frac{du}{dt} &= u(1 - v - bu) \\ \frac{dv}{dt} &= v(-c + u),\end{aligned}\tag{2}$$

where $u(0) = x(0), v(0) = y(0)$. By comparison principle, we have

$$\frac{dx}{dt} \leq \frac{du}{dt} \quad \text{and} \quad \frac{dy}{dt} \leq \frac{dv}{dt}.$$

Moreover, we also have

$$\frac{du}{dt} \leq u(1 - bu).$$

By comparison principle again, it is easy to see that there are some $M > 0$ such that $u(t) \leq M$ for all $t \geq 0$. Take $A > B > 0$, then we have

$$Au'(t) + Bv'(t) = Au(t) - Au(t)v(t) - Abu^2(t) - cBv(t) + Bu(t)v(t)$$

Now set $S(t) = Au(t) + Bv(t)$ and $0 < \varepsilon < c$. It follows that

$$S'(t) + \varepsilon S(t) = A(\varepsilon + 1)u(t) - Au(t)v(t) - Abu^2(t) + (\varepsilon - c)Bv(t) + Bu(t)v(t) \leq A(\varepsilon + 1)M.$$

It implies that $u(t)$ as well as $v(t)$ are bounded and so are $x(t)$ as well as $y(t)$.

Finally, to show that $z(t)$ is bounded, let's take $L > 0$ which satisfies

$$fx(t) + gy(t) \leq L, \quad \text{for all } t \in [0, \infty).$$

According to the third equation of (1), it is clear that $z'(t) < 0$ if $z(t) > (L - e)/\beta$. It follows that $z(t)$ is bounded. The proof is complete. $\square$

3 Local Stability of Equilibria

Based on [5], to investigate the local stability of each equilibrium (x^*, y^*, z^*) , we provide the following Jacobian matrix $J(x^*, y^*, z^*)$.

$$J = \begin{pmatrix} 1 - y^* - z^* - bx^* - bx^* & -x^* & -x^* \\ y^* & -c + x^* & 0 \\ fz^* & gz^* & -e + fx^* + gy^* - \beta z^* - \beta z^* \end{pmatrix}$$

For E_0 and E_1 , we have

$$J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -e \end{pmatrix} \quad \text{and} \quad J(E_1) = \begin{pmatrix} -1 & -1/b & -1/b \\ 0 & -c + 1/b & 0 \\ 0 & 0 & -e + f/b \end{pmatrix}.$$

It is clear that E_0 is saddle, while E_1 is asymptotically stable when $f < be$ and $bc > 1$. Jacobian in E_2 is

$$J(E_2) = \begin{pmatrix} -bc & -c & -c \\ 1 - bc & 0 & 0 \\ 0 & 0 & -e + fc + g - gbc \end{pmatrix},$$

which has the following eigenvalues

$$\lambda_1 = g(1 - bc) - (e - fc), \lambda_2 = -\frac{1}{2}(bc - \sqrt{c(b^2c - 4 + 4bc)}), \\ \lambda_3 = -\frac{1}{2}(bc + \sqrt{c(b^2c - 4 + 4bc)}).$$

Since both of λ_2 and λ_3 are negative, local stability of E_2 is determined by λ_1 . Hence E_2 is a stable node when $g(1 - bc) < (e - fc)$.

Local stability of $E_3 = \left(\frac{e + \beta}{f + \beta b}, 0, \frac{f - be}{f + \beta b}\right)$ is determined by investigating the eigenvalues of

$$J(E_3) = \begin{pmatrix} -b\frac{\beta + e}{f + \beta b} & -\frac{\beta + e}{f + \beta b} & -\frac{\beta + e}{f + \beta b} \\ 0 & \frac{\beta - cb\beta + e - fc}{f + \beta b} & 0 \\ -f\frac{be - f}{f + \beta b} & g\frac{be - f}{f + \beta b} & \beta\frac{be - f}{f + \beta b} \end{pmatrix},$$

namely

$$\lambda_1 = -c + \frac{\beta + e}{f + \beta b}, \lambda_2 = -\frac{1}{2}((\beta z_3 + bx_3) + \sqrt{(\beta z_3 + bx_3)^2 - 4fx_3z_3}), \\ \lambda_3 = -\frac{1}{2}((\beta z_3 + bx_3) - \sqrt{(\beta z_3 + bx_3)^2 - 4fx_3z_3}).$$

It is obvious that E_3 is a stable node when $\frac{\beta + e}{f + \beta b} < c$. Finally, the local stability of E_4 is investigated by considering the Jacobian evaluated at E_4 , namely

$$J(E_4) = \begin{pmatrix} -bx_4 & -x_4 & -x_4 \\ y_4 & 0 & 0 \\ fz_4 & gz_4 & -\beta z_4 \end{pmatrix}.$$

The characteristic polynomial $P(\lambda)$ for $J(E_4)$ is

$$P(\lambda) := \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

where

$$\begin{aligned} a_1 &= \beta z_4 + bx_4, \\ a_2 &= bx_4\beta z_4 + z_4fx_4 + y_4x_4, \\ a_3 &= x_4y_4z_4(\beta + g). \end{aligned}$$

It is obvious that $a_1 > 0$ and $a_3 > 0$. If $a_1a_2 > a_3$, then Routh Hurwitz criterion [1] implies that all roots of $P(\lambda)$ have negative real parts, or in other words, E_4 is a stable point. It can be shown that equation

$$a_1a_2 - a_3 = x_4[(\beta z_4 + bx_4)(b\beta z_4 + fz_4) + y_4(bx_4 - gz_4)] \quad (3)$$

is positive if $1 - bc < 0$ and $e - fc > 0$, or $f - be < 0$. This conditions are in contrast to the existence condition of E_4 . It means that E_4 is unstable. Dynamical behavior near E_4 will be demonstrated numerically.

This section is ended by summarizing the existence and stability condition of all of the equilibria in Table 1.

Table 1. Existence and stabilities condition of equilibria.

Equil.	Existence Cond.	Stability Cond.
E_0	-	always saddle
E_1	-	$bc > 1$ and $f < be$
E_2	$bc < 1$	$g(1 - bc) < e - fc$
E_3	$f > be$	$\frac{\beta + e}{f + b\beta} < c$
E_4	$bc < 1$ and $0 < fc - e < \beta(1 - bc)$	$bc > 1$ and $fc < e$

4 Global Stability of Boundary Equilibria

By following the works of [2] and [3], we study the global stability of the boundary equilibria. Let $(x(t), y(t), z(t))$ is the solution of the system (1) with the positive initial value $(x(0), y(0), z(0))$. When $z = 0$ the system (1) becomes

$$\begin{aligned} x'(t) &= x(t)(1 - y(t) - bx(t)), \\ y'(t) &= y(t)(-c + x(t)). \end{aligned} \quad (4)$$

The following propositions summarized the global stability of the boundary equilibria on H_2 and H_3 region.

Proposition 4.1.

- (1) If $bc > 1$ then the boundary equilibrium E_1 is globally stable on H_2 .
- (2) If $bc < 1$ then the boundary equilibrium E_2 is globally stable on H_2 .

Proposition 4.2.

- (1) If $f > be$ then the boundary equilibrium E_3 is globally stable on H_3 .
- (2) If $f < be$ then the boundary equilibrium E_1 is globally stable on H_3 .

Furthermore, we study the global stability of system (1) in R_3^+ . First, we study the asymptotic behavior of $y(t)$ as $t \rightarrow \infty$.

Lemma 4.3. *If $bc > 1$, then $\lim_{t \rightarrow \infty} y(t) = 0$.*

Proof. Let $\varepsilon \in (0, |1/b - c|)$ there exists a time T_1 , such that $x(t) \leq 1/b + \varepsilon$, for $t \geq T_1$. Then we have $-c + 1/b + \varepsilon < 0$.

$$y'(t) \leq y(t)(-c + 1/b + \varepsilon), \quad \text{for } t \geq T_1$$

Suppose $v(t)$ be the solution of the following equations

$$\begin{cases} v'(t) \leq v(t)(-c + 1/b + \varepsilon), & \text{for } t \geq T_1 \\ v(T_1) = y(T_1). \end{cases} \quad (5)$$

By comparison principle we have

$$y(t) \leq v(T_1)e^{(-c+1/b+\varepsilon)(t-T_1)}, y(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

Thus we have

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

□

The following Theorem (5.6) implies that the local stability of boundary equilibrium $E_1 = (1/b, 0, 0)$ of system (1) indicates its global stability.

Theorem 4.4. *If $bc > 1$ and $f < be$, then $E_1 = (1/b, 0, 0)$ is globally stable in R_+^3 .*

Proof. Based on Lemma (4.3) we know that $\lim_{t \rightarrow \infty} y(t) = 0$. For all $\varepsilon > 0$, where $\varepsilon \in (0, |\frac{e-f}{f+g}|)$, there exists T_2 such that $y(t) \leq \varepsilon$, for $t \geq T_2$. Then we have

$$-e + fx(t) \leq -e + f/b + f\varepsilon, \quad \text{for } t \geq T_1.$$

Take $T = \max\{T_1, T_2\}$, so that we find the following inequality

$$z'(t) \leq z(t)(-e + f/b + (f+g)\varepsilon - \beta z(t)) \leq z(t)(-e + f/b + (f+g)\varepsilon), \quad \text{for } t \geq T.$$

Suppose $w(t)$ be the solution of the following equations

$$\begin{cases} w'(t) \leq w(t)(-e + f/b + (f+g)\varepsilon), & \text{for } t \geq T \\ w(T) = z(T). \end{cases} \quad (6)$$

Using comparison principle we have $z(t) \leq w(t)$ for $t \geq T$,

$$z(t) \leq w(T)e^{(-e+f/b+(f+g)\varepsilon)(t-T)}.$$

Thus we find,

$$\lim_{t \rightarrow \infty} z(t) = 0.$$

Now for all $\varepsilon > 0$, there exists $T_2 > 0$ such that $|y(t)| < \varepsilon, |z(t)| < \varepsilon$, for all $t \geq T_2$. We have inequality

$$x'(t) \geq x(t)(1 - bx(t) - 2\varepsilon), \quad t \geq T_2.$$

Suppose $u(t)$ be the solution of the following system

$$\begin{cases} u'(t) = u(t)(1 - 2\varepsilon - bu(t)) \\ u(T_2) = x(T_2). \end{cases} \quad (7)$$

By comparison principle again we have $u(t) \leq x(t)$, while $\lim_{t \rightarrow \infty} u(t) \geq \frac{1-2\varepsilon}{b}$. Let $M > 0$, there exists $x(t) \geq 1/b - 2\varepsilon/b$, for $t \geq M$. Take $L = \max\{T_1, M\}$,

$$|x(t) - 1/b| < 2\varepsilon/b, \quad \text{for } t \geq L.$$

It is mean that

$$\lim_{t \rightarrow \infty} x(t) = 1/b.$$

□

Theorem 4.5. *If $f > be, bc > 1$, and $\beta(1 - bc) < (fc - e)$, then boundary equilibrium $E_3 = (\frac{e+\beta}{f+\beta b}, 0, \frac{f-be}{f+\beta b})$ is globally stable in R_+^3 .*

Proof. Let $\mathbf{p} := (x(0), y(0), z(0)) \in R_+^3$ be fixed. By Lemma 2.5 we have $\lim_{t \rightarrow \infty} y(t) = 0$. It follows that there exists some $\mathbf{q} = (q_1, 0, q_3) \in \omega(\mathbf{p})$. Note that $\omega(\mathbf{p})$ is invariant and closed. Now we focus on the following system

$$\begin{cases} \frac{du}{dt} = u(1 - v - w - bu), \\ \frac{dv}{dt} = v(-c + u), \\ \frac{dw}{dt} = w(-e + fu + gv - \beta w), \\ (u(0), v(0), w(0)) = \mathbf{q}. \end{cases} \quad (8)$$

By part (1) of Proposition 4.2, we have E_{B3} is globally stable on x - z plane. Since $\mathbf{q}$ belongs to x - z plane, it follows that

$$\lim_{t \rightarrow \infty} (u(t), v(t), w(t)) = E_{B3}. \quad (9)$$

On the other hand, since $\omega(\mathbf{p})$ is invariant and $\mathbf{q} \in \omega(\mathbf{p})$, it is clear that

$$(u(t), v(t), w(t)) \in \omega(\mathbf{p}) \quad \text{for } t > 0. \quad (10)$$

By (9), (10) and the fact that $\omega(\mathbf{p})$ is closed, one can easily see that $E_{B3} \in \omega(\mathbf{p})$. Recall that E_{B3} is locally stable on R_+^3 and then the fact $E_{B3} \in \omega(\mathbf{p})$ implies that $(x(t), y(t), z(t))$ converges to E_{B3} . $\square$

5 Numerical Simulation

In this section, some numerical simulations are presented to demonstrate the chaotic and global stability phenomena. To demonstrate the existence of chaotic phenomenon near E_4 , take a set of parameter values $b = 0.2, c = 0.4, e = 0.1, f = 0.3, g = 0.5$, and $\beta = 0.4$. The numerical result in figure 1, shows the orbit of two solutions which start from their positive initial condition. We can see that the trajectories do not tend to E_4 but a chaotic phenomena look like happens. The second and third simulations illustrates the global sta-

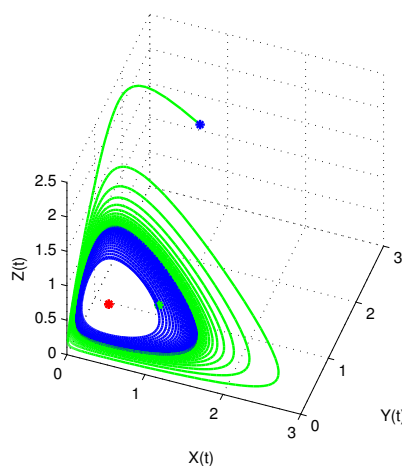


Figure 1: Chaotic phenomena near E_4

bility phenomena. Here, we choose $b = 1.5, c = 0.7, e = 0.3, f = 0.4, g = 0.4$, and $\beta = 0.1$. Figure 2 shows that the solution of system 1 tends to $E_1 = (2/3, 0, 0)$ as t tends to infinity. It means that ultimately the population of

prey are persist, whereas the populations of predator and omnivore are extinct. Hence, this set of parameter values can be used to predict when the interaction will be won by preys. Another global stability occurs when $f - be > 0$ and

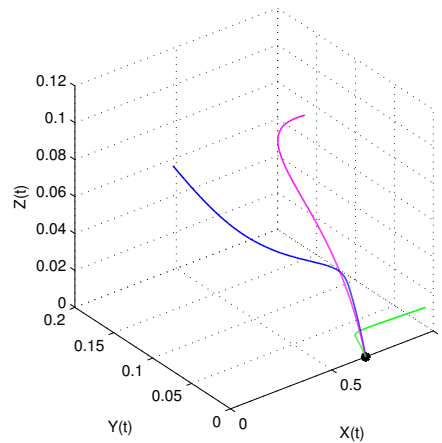


Figure 2: Global stability of boundary equilibrium $E_1 = (1/b, 0, 0)$

$1 - bc < 0$. Let's take $b = 1, e = 0.02, f = 1.3, c = 0.5, g = 0.02$, and $\beta = 0.1$ to illustrate this situation. Refer to Figure 3, we can see that under these parameter the orbit of solutions tend to equilibrium $E_3 = (0.0857, 0, 0.9143)$. In this case, preys and predators are extinct, whereas omnivore persist under these chosen parameters.

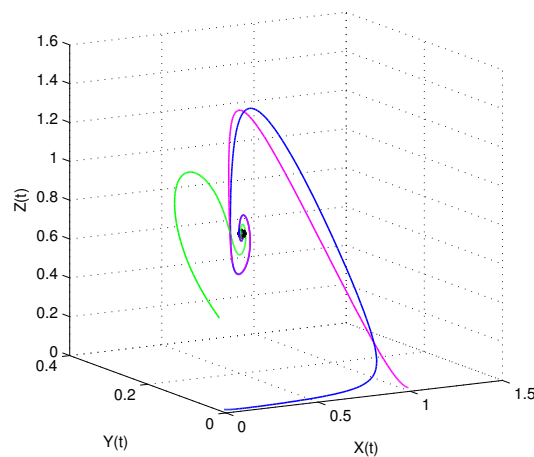


Figure 3: Global stability of boundary equilibrium $E_3 = (0.0857, 0, 0.9143)$

6 Conclusions

The existence of non-negative equilibria of the system (1) has been investigated. A complete classification of the equilibria, with respect to the various parameters based on their existence conditions, is provided. The boundedness of solution is also proved. Different from [4], we provide a complete exploration and classification for the local stability conditions of each equilibrium. In [4], global stability was considered in each plane separately. In this paper we employ a different approach to determine the sufficient condition that guarantee the global stability. The interesting result for interior equilibrium is that the existence conditions contradict with the stability condition. Hence, the trajectories of the system will not tend to interior equilibrium. Chaotic phenomena has been considered in [4] by investigating the trajectories using period doubling and trapping region analysis for some specific parameters. By choosing a set of parameters different from [4], our numerical result also captures the chaotic behavior.

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