Pointwise Negative Binomial Approximation for Geometric Summands

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Abstract

Stein’s method and the geometric $w$-functions are used to derive a non-uniform bound for the point metric between the distribution of a sum of $n$ independent geometric random variables, each with parameter $p_i = 1 - q_i$, and a negative binomial distribution with parameters $n$ and $p = 1 - q = \frac{n}{n + \sum_{i=1}^{n} \frac{q_i}{p_i}}$. With this bound, it gives a good approximation when all $q_i$ are small or all $q_i$ are close to $q$.

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1 Introduction

Let $X_1, ..., X_n$ be independently distributed geometric random variables, each with probability function $p_{X_i}(k) = p_i q_i^k$ for $k \in \mathbb{N} \cup \{0\}$, and mean $\mu_i = \frac{q_i}{p_i}$ and variance $\sigma_i^2 = \frac{q_i}{p_i^2}$, where $q_i = 1 - p_i$. Let $S_n = \sum_{i=1}^{n} X_i$ and $NB_{n,p}$ denote the negative binomial random variable with parameters $n$ and $p = \frac{n}{n + \sum_{i=1}^{n} \frac{q_i}{p_i}}$. In this case, Vellaisamy and Upadhye [6] used Kerstan’s method to give a uniform bound for the distance between the distributions of $S_n$ and $NB$ as follows:

$$d_A(S_n, NB_{n,p}) \leq \min \left\{ 1.37 \sum_{i=1}^{n} \frac{1}{p_i} \left| 1 - \frac{p_i}{p} \right| \min \left( 2, \sqrt{\frac{2}{nq_e}} \right), 1 \right\},$$  

(1.1)
where \( d_A(S_n, NB_{n,p}) = |P(S_n \in A) - P(NB_{n,p} \in A)| \). Recently, Teerapabolarn [5] used Stein’s method and the geometric \( w \)-functions to give a uniform bound

\[
d_A(S_n, NB_{n,p}) \leq \frac{1 - p^n}{n} \sum_{i=1}^{n} \left| \frac{q_i}{p_i} - \frac{q}{p} \right| \frac{q_ip}{q_ip_i},
\]

which is sharper than that in (1.1). However, for \( A = \{x_0; x_0 \in \mathbb{N} \cup \{0\}\} \) and \( d_{x_0}(S_n, NB_{n,p}) = |P(S_n = x_0) - P(NB_{n,p} = x_0)| \), the result in (1.2) becomes

\[
d_{x_0}(S_n, NB_{n,p}) \leq \frac{1 - p^n}{n} \sum_{i=1}^{n} \left| \frac{q_i}{p_i} - \frac{q}{p} \right| \frac{q_ip}{q_ip_i}
\]

for every \( x_0 \). It is observed that the bound is a uniform constant for the point metric \( d_{x_0}(S_n, NB_{n,p}) \). In this situation, a non-uniform bound with respect to \( x_0 \) is required. In this paper, we focus on deriving a non-uniform bound for \( d_{x_0}(S_n, NB_{n,p}) \) by using Stein’s method and the geometric \( w \)-functions, which are in Section 2. In Section 3, we derive the desired result of this study, and the conclusion of this study is presented in the last section.

2 Method

The following lemma gives the geometric \( w \)-functions, which are directly obtained from [3].

**Lemma 2.1.** For \( 1 \leq i \leq n \), let \( w_i \) be the \( w \)-function associated with the geometric random variable \( X_i \), then we have the following:

\[
w_i(k) = \frac{(1+k)q_i}{p_i \sigma_i^2}, \quad k \in \mathbb{N} \cup \{0\}.
\]

The following relation is an important property for proving the result, which was stated by [2].

\[
Cov(S_n, f(S_n)) = \sum_{i=1}^{n} Cov \left( X_i, f \left( X_i + \sum_{j \neq i} X_j \right) \right)
= \sum_{i=1}^{n} \sigma_i^2 E[w_i(X_i) \Delta f(S_n)],
\]

for any function \( f : \mathbb{N} \cup \{0\} \to \mathbb{R} \) for which \( E|w_i(X_i) \Delta f(S_n)| < \infty \), where \( \Delta f(x) = f(x+1) - f(x) \).

For Stein’s method in the negative binomial approximation, it can be applied for every \( n \in \mathbb{N} \) and \( 0 < p = 1 - q < 1 \), for every \( x_0 \in \mathbb{N} \cup \{0\} \) and
bounded real-valued function \( f = f(x_0) : \mathbb{N} \cup \{0\} \to \mathbb{R} \) defined as in Brown and Phillips [1]. So, Stein’s equation for these conditions is as follows:

\[
P(S_n = x_0) - P(NB_{n,p} = x_0) = E[q(n + S_n)f(S_n + 1) - S_nf(S_n)]. \tag{2.3}
\]

For \( x_0 \in \mathbb{N} \cup \{0\} \) and \( x \in \mathbb{N} \), [4] showed that

\[
\sup_{x \geq 1} |\Delta f(x)| \leq \delta(x_0) = \begin{cases} \frac{1-p^n}{nq} & \text{if } x_0 = 0, 1, \\ \min \left\{ \frac{1}{x_0}, \frac{1-p^n}{(n+x_0-1)q} \right\} & \text{if } x_0 > 1. \end{cases} \tag{2.4}
\]

3 Result

The following theorem presents a non-uniform bound on the pointwise negative binomial approximation to the distribution of \( S_n \).

**Theorem 3.1.** For \( x_0 \in \mathbb{N} \cup \{0\} \), then we have the following:

\[
d_{x_0}(S_n, NB_{n,p}) \leq \delta(x_0) \sum_{i=1}^{n} \frac{q_i}{p_i} - q \frac{q_i p}{p_i}, \tag{3.1}
\]

where \( \delta(x_0) \) is defined in (2.4).

**Proof.** From (2.3), it follows that

\[
d_{x_0}(S_n, NB_{n,p}) = \left| E[nqf(S_n + 1) + qS_n\Delta f(S_n) - pS_nf(S_n)] \right|
\]

\[
= p \left| \frac{nq}{p} E[\Delta f(S_n)] + \frac{q}{p} E[S_n \Delta f(S_n)] - Cov(S_n, f(S_n)) \right|.
\]

Using (2.2) and lemma 2.1, we have

\[
d_{x_0}(S_n, NB_{n,p}) = p \left| \sum_{i=1}^{n} \left\{ E \left[ \left( \mu_i + \frac{q}{p} X_i \right) \Delta f(S_n) \right] - \sigma_i^2 E[w_i(X_i)\Delta f(S_n)] \right\} \right|
\]

\[
\leq p \sum_{i=1}^{n} E \left\{ \left| \frac{q_i}{p_i} + \frac{q}{p} X_i - \sigma_i^2 w_i(X_i) \right| \Delta f(S_n) \right\}
\]

\[
\leq \sup_{x \geq 1} |\Delta f(x)| p \sum_{i=1}^{n} E \left\{ \left| \frac{q_i}{p_i} + \frac{q}{p} X_i - (1 + X_i) \frac{q_i}{p_i} \right| \right\}
\]

\[
\leq \delta(x_0) \sum_{i=1}^{n} \left| \frac{q_i}{p_i} - \frac{q}{p} \right| \frac{q_i p}{p_i}.
\]

Hence, (3.1) is obtained. \( \square \)
4 Conclusion

In this study, a non-uniform bound for the point metric between the distribution of a sum of independent geometric random variables and a negative binomial distribution with parameters $n$ and $p = \frac{n}{n+\sum_{i=1}^{n} \frac{q_i}{p}}$ was derived by Stein’s method and the geometric $w$-functions. With this bound, it gives a good approximation when all $q_i$ are small or all $q_i$ are close to $q$. In addition, the bound in this study is sharper than that presented in (1.3).

References


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