Isolate Domination in the Join and Corona of Graphs

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Abstract
A subset \( S \subseteq V(G) \) is called an isolate set if the subgraph induced by \( S \) has an isolated vertex. This set \( S \) is called an isolate dominating set if it is both isolate and dominating. The minimum cardinality of an isolate dominating set is called the isolate domination number and is denoted by \( \gamma_0(G) \). In this paper, we look at another aspect of the isolate dominating set and characterized the lower and upper bounds of the isolate domination number and those graphs resulting from some binary operations such as join and corona.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a simple graph (a graph without loops or multiple edges), where \( V(G) \) is the vertex set and \( E(G) \) is the edge set of \( G \), respectively. A set \( S \subseteq V(G) \) is called a dominating set of \( G \) if for each \( u \in V(G) \setminus S \), there exists \( v \in S \) such that \( uv \in E(G) \). A subset \( S \subseteq V(G) \) is called an isolate set if the subgraph induced by \( S \) has an isolated vertex. This set \( S \) is called an isolate dominating set if it is both isolate and dominating. The minimum cardinality of an isolate dominating set is called the isolate domination number.
and is denoted by $\gamma_0(G)$. An isolate dominating set $S$ of $G$ with $|S| = \gamma_0(G)$ is called a $\gamma_0$-set.

Domination becomes the most fascinating topic in graph theory due to its application in networking. Haynes et. al. in [4] listed at least 75 different types of domination related parameters. The concept of isolate domination was introduced by Hamid and Balamurugan [5] in 2013 and investigated some properties relating to the maximum degree of a graph. In [7], they established a chain connecting various isolate parameters with the existing domination and independent domination. Moreover, in [6], they characterized a unicyclic graphs on which the order equals the sum of the isolate domination number and its maximum degree.

Two vertices $u, v \in V(G)$ are said to be adjacent if $uv \in E(G)$. A dominating set $S \subseteq V(G)$ is said to be an independent dominating set of $G$ if every two vertices $u, v \in S$, $uv \notin E(G)$. The minimum cardinality of the independent dominating set is called an independent domination number and is denoted by $i(G)$. For any graph theoretic concept not defined in this paper, please see [1, 2, 3].

2 Preliminary Results

Theorem 2.1 Let $G$ be a graph of order $n \geq 1$. Then $\gamma_0(G) = 1$ if and only if $\Delta(G) = n - 1$.

Proof: Suppose $\gamma_0(G) = 1$ and let $S = \{x\}$ be the minimum isolate dominating set of $G$. If $G$ is a trivial graph, then we are done. Suppose $G$ is not trivial. Clearly, each vertex in $\langle V(G) \setminus S \rangle$ is adjacent to $x$. Since $|V(G)| = n$, $deg_G(x) = n - 1$. Hence, $\Delta(G) = n - 1$. The converse is clear. □

The following Corollary follows directly from Theorem 2.1.

Corollary 2.2 Let $K_n$, $S_{n-1}$, and $W_{n-1}$ be complete graph, star graph, and wheel graph of $n \geq 2$ vertices, respectively. Then

$$\gamma_0(K_n) = \gamma_0(S_{n-1}) = \gamma_0(W_{n-1}) = 1.$$  

Theorem 2.3 Let $G$ be a non-trivial graph. Then $\gamma_0(G) = 2$ if and only if $i(G) = 2$.

Proof: The proof is obvious. □

Remark 2.4 If $\Delta(G) = 0$, then $\gamma(G) = n$ for any graph $G$ of order $n$.

Theorem 2.5 Let $G$ be a non-trivial graph. Then $\gamma_0(G) = n - 1$ if and only if $G$ has exactly two vertices of degree 1 and $\Delta(G) = 1$. 
Proof: Suppose $\gamma_0(G) = n - 1$ and let $S = V(G) \setminus \{x\}$ be the minimum isolate dominating set of $G$. Suppose further that $\Delta(G) \neq 1$ or $G$ has no exactly two vertices of degree 1 (i.e., has more than two vertices of degree one). Consider the following cases:

**Case 1: $\Delta(G) \neq 1$**

Let $u \in V(G)$. If $\deg_G(u) = 0$, then $S$ is not a dominating set by Remark 2.4 which is a contradiction. Now, suppose $\deg_G(u) \geq 2$. Then there exist at least two neighbours of $u$, say $x_1, x_2, ..., x - k$, where $2 \leq k \leq n - 1$. This implies that $u \in S$ is an isolated vertex in $\langle S \rangle$. Thus, $V(G) \setminus \{x_1, x_2\}$ is an isolate dominating set and this is a contradiction.

**Case 2: $G$ has more than two vertices of degree one**

Let $u_1, u_2, ..., u_k \in V(G)$ such that $\deg_G(u_1) = \deg_G(u_2) = ... = \deg_G(u_k) = 1$, where $3 \leq k \leq n$. If two of these vertices (say $u_1$ and $u_2$) are neighbours, then there exists another vertex $z \in V(G)$ such that $u_3z \in E(G)$. Since $\deg_G(u_1) = \deg_G(u_2) = 1$ and $u_3z \in E(G)$, $V(G) \setminus \{u_1, u_3\}$ is an isolate dominating set of $G$. This is a contradiction to the fact that $S$ is the minimum isolate dominating set of $G$.

For the converse, let $u, v \in V(G)$ such that $\deg_G(u) = \deg_G(v) = 1$. Since $\Delta(G) = 1$, other vertices of $G$ has degree 0, i.e., it is an isolated vertex. This implies that $V(G) \setminus \{u\}$ is the minimum isolate dominating set of $G$. Hence, $\gamma_0(G) = n - 1$. □

**Theorem 2.6** Let $G$ be a graph of $n$ vertices. Then $\gamma_0(G) = n$ if and only if $G = \overline{K}_n$.

**Proof:** Suppose that $\gamma_0(G) = n$. Suppose further that $G \neq \overline{K}_n$. Then there exists $u \in V(G)$ such that $\deg_G(u) \geq 1$. This implies that $u$ has at least one neighbour. By taking all the neighbours of $u$ outside the isolate dominating set, we obtain $\gamma_0(G) \leq n - 1$. This is a contradiction.

The converse is obvious. □

**Corollary 2.7** For any graph $G$, $1 \leq \gamma_0(G) \leq n$.

**Proof:** Follow from Theorems 2.1 and 2.6. □

### 3 Join of Graphs

Let $A$ and $B$ be sets which are not necessarily disjoint. The *disjoint union* of $A$ and $B$, denoted by $A \sqcup B$, is the set obtained by taking the union of $A$ and $B$
treat each element in \( A \) as distinct from each element in \( B \). The union \( G_1 \cup G_2 \) of graphs \( G_1 \) and \( G_2 \) with disjoint vertex-sets \( V(G_1) \) and \( V(G_2) \), respectively, is the graph \( G \) where \( V(G) = V(G_1) \cup V(G_2) \) and \( E(G) = E(G_1) \cup E(G_2) \).

Note that the induced subgraph of a subset \( S \subseteq V(G + H) \) is always connected whenever both \( S \cap V(G) \) and \( S \cap V(H) \) are non-empty. Thus, we have the following lemma.

**Lemma 3.1** Let \( G \) and \( H \) be any non-trivial graphs, and let \( S \subseteq V(G + H) \).
If \( S \cap V(G) \neq \emptyset \) and \( S \cap V(H) \neq \emptyset \), then the isolate dominating set of \( G + H \) does not exist.

**Theorem 3.2** Let \( G \) and \( H \) be any non-trivial graphs. Then \( S \subseteq V(G + H) \)
is an isolate dominating set of \( G + H \) if and only if either \( S \subseteq V(G) \) is an isolate dominating set of \( G \) or \( S \subseteq V(H) \) is an isolate dominating set of \( H \).

**Proof**: Suppose \( S \subseteq V(G + H) \) is an isolate dominating set of \( G + H \). Then by Lemma 3.1, either \( S \subseteq V(G) \) or \( S \subseteq V(H) \). Since \( S \) is an isolate dominating set of \( G + H \), it follows that \( S \) is either an isolate dominating set of \( G \) or \( H \).

The converse is clear. □

**Corollary 3.3** Let \( G \) and \( H \) be any non-trivial graphs. Then

\[
\gamma_0(G + H) = \min\{\gamma_0(G), \gamma_0(H)\}.
\]

**Proof**: Suppose \( \gamma_0(G) \leq \gamma_0(H) \). Let \( S \subseteq V(G) \) be a minimum isolate dominating set of \( G \). Then by Theorem 3.2, \( S \) is an isolate dominating set of \( G + H \). Thus, \( \gamma_0(G + H) \leq \gamma_0(G) \). Next, let \( S^* \) be an isolate dominating set of \( G + H \) such that \( |S^*| < |S| \). Then by Theorem 3.2, \( S^* \) is an isolate dominating set of \( G \). This is a contradiction to the fact that \( |S| = \gamma_0(G) \). Thus, \( \gamma_0(G + H) \geq \gamma_0(G) \). Hence, we have the equality \( \gamma_0(G + H) = \gamma_0(G) \). Similarly, using similar argument we can show that \( \gamma_0(G + H) \geq \gamma_0(G) \), whenever \( \gamma_0(G) \geq \gamma_0(H) \). □

### 4 Corona of Graphs

The *corona* of two graphs \( G \) and \( H \), denoted by \( G \circ H \), is the graph obtained by taking one copy of \( G \) of order \( n \) and \( n \) copies of \( H \), and then joining the \( i \)-th vertex of \( G \) to every vertex in the \( i \)-th copy of \( H \). For every \( v \in V(G) \), we denote by \( H^v \) the copy of \( H \) whose vertices are joined or attached to the vertex \( v \). For each \( v \in V(G) \), we denote \( T^v \subseteq V(H^v) \) and \( D^v \subseteq V(\{\{v\}\} + H^v) \). The next result is a characterization of an isolate dominating set for a corona of two graphs.
**Theorem 4.1** Let $G$ and $H$ be any graphs. Then a nonempty set $C \subseteq V(G\circ H)$ is an isolate dominating set of $G \circ H$ if and only if there exists $v \in V(G)$ such that either of the following hold:

(i.) $T^v$ is an isolate dominating set of $H^v$ and

$$C = \left( \bigcup_{u \in V(G) \setminus \{v\}} D^u \right) \cup T^v;$$

(ii.) $\deg_G(v) = 0$ and

$$C = \left( \bigcup_{u \in V(G) \setminus \{v\}} D^u \right) \cup \{v\},$$

where $T^u$ is a dominating set of $H^u$ whenever $u \notin D^u$

**Proof**: Suppose $C \subseteq V(G\circ H)$ is an isolate dominating set of $G \circ H$. Then there exists at least one isolated vertex in $G \circ H$, say $x$. Consider the following cases:

**Case 1**: $x \notin V(G)$

Then $x \in V(H^v)$ for some $v \in V(G)$. Let $T^v \subseteq V(H^v)$. Since $x \in C$ is an isolated vertex, $v \notin C$. This implies that $T^v$ is a dominating set of $\langle \{v\} \rangle + H^v$ and $x \in T^v$. Thus, $T^v$ is an isolate dominating set of $H^v$. Since $C$ is a dominating set of $G \circ H$, $D^u$ is a dominating set for each $u \in V(G) \setminus \{v\}$. If $u \notin D^u$, then $D^u = T^u$. This implies that $T^u$ is a dominating set of $H^u$. Moreover,

$$C = \left( \bigcup_{u \in V(G) \setminus \{v\}} D^u \right) \cup T^v.$$ 

**Case 2**: $x \in V(G)$

Then $\deg_G(x) = 0$. This follows that $\{x\}$ is a dominating set of $\langle \{x\} \rangle + H^x$. Similarly, if $u \notin D^u$, then $D^u = T^u$ for each $u \in V(G) \setminus \{x\}$. This implies that $T^u$ is a dominating set of $H^u$. Moreover,

$$C = \left( \bigcup_{u \in V(G) \setminus \{v\}} D^u \right) \cup \{v\}.$$ 

The converse is clear. □

**Corollary 4.2** Let $G$ and $H$ be any graphs of order $m \geq 1$ and $n \geq 1$, respectively. Then

$$\gamma_0(G \circ H) = \begin{cases} m & \text{if } \delta(G) = 0 \\ m - 1 + \gamma_0(H) & \text{if } \delta(G) \neq 0 \end{cases}$$
Proof: Consider the following cases:

Case 1: $\delta(G) = 0$

Let $v \in V(G)$ such that $\text{deg}_G(u) = 0$. Choose $D^u = \{u\}$ for each $u \in V(G) \setminus \{v\}$. Then

$$C = \left( \bigcup_{u \in V(G) \setminus \{v\}} D^u \right) \cup \{v\} = V(G)$$

is an isolate dominating set of $G \circ H$ by Theorem 4.1. Hence,

$$\gamma_0(G \circ H) \leq |V(G)| = m.$$ 

Next let $C^*$ be a minimum isolate dominating set of $G \circ H$. Then

$$C^* = \left( \bigcup_{u \in V(G) \setminus \{v\}} \{u\} \right) \cup \{v\} = V(G).$$

Hence, $\gamma_0(G \circ H) \geq |V(G)| = m$. Thus, $\gamma_0(G \circ H) = m$.

Case 2: $\delta(G) \neq 0$

Similarly, let $v \in V(G)$ and choose $D^u = \{u\}$ for each $u \in V(G) \setminus \{v\}$. Clearly $v \notin C$. By Theorem 4.1, $T^v$ is an isolate dominating set of $\langle \{v\} \rangle + H^v$ and

$$C = \left( \bigcup_{u \in V(G) \setminus \{v\}} \{u\} \right) \cup T^v.$$ 

Hence, $\gamma_0(G \circ H) \leq |V(G)| - 1 + \gamma_0(H) = |V(G)| - 1 + |T^v|$. Next let $C^*$ be a minimum isolate dominating set of $G \circ H$. Then

$$C^* = \left( \bigcup_{u \in V(G) \setminus \{v\}} \{u\} \right) \cup T^v.$$ 

Then $|V(G)| - 1 + |T^v| = |V(G)| - 1 + \gamma_0(H) \leq \gamma_0(G \circ H)$. Thus,

$$\gamma_0(G \circ H) = |V(G)| - 1 + \gamma_0(H). \Box$$

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References


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