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Spanning Trees on Decorated Centered Cubic Lattices

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Abstract

In this paper we compute the number of spanning trees on the following decorated centred cubic lattices; base- centred cubic, side- centered cubic and edge- centred cubic lattices. For these lattices we also determine the asymptotic growth constant.

Keywords: Spanning trees, Asymptotic growth constant, Decorated centred cubic lattices

1 Introduction

The problem of the enumeration of the number of spanning trees on the network is considered by Kirchhoff in his analysis of electric circuits [6]. Kirchhoff showed that the spanning trees problem is related to the problem of computing the two-node resistance of a resistor electrical network. The number of spanning trees is an important measure of reliability of a network and useful for designing electrical circuits. Spanning tree is of interest in statistical physics. It is also closely concerned with the q-state Potts model [5, 15].

There are two approaches for calculating the number of spanning trees are the matrix tree theorem (Laplacian matrix) [9, 10, 11] and the Tutte polynomial [1, 4, 7, 13]. The enumeration of spanning trees and the computation of their asymptotic growth constants on uniform lattices or graphs were studied extensively, see for instance [2, 3, 8, 12, 14].

In this paper we will use the matrix tree theorem to determine the number of spanning trees and the thermodynamic limit (asymptotic growth constant) for the following decorated cubic centered lattices: base-centered cubic, side-centered cubic and edge-centered cubic lattice.

2 Definitions and method (A brief formulation)

In this section, we briefly present basic definitions, expressions and the general method (matrix tree theorem) that we use[8] in this work.

Consider a lattice L that is a uniform (periodic) tiling of d-dimensional space and is decomposable into a hypercubic array of $N_1 \times N_2 \times \cdot\cdot\cdot \times N_d$ unit cells, each containing s sites labeled by $1,2,\ldots,s$ so that the number of sites in the lattice is $n = sN_1N_2 \cdot\cdot\cdot N_d$. The unit cell can be specified by the coordinate $\boldsymbol{n} = \{n_1, n_2, \ldots, n_d\}$, where $n_i = 0, 1, 2, \ldots, N_i - 1$. The connection between the sites of the unit cells $\boldsymbol{n}$ and $\boldsymbol{n}'$ can be described by an adjacency matrix $\mathbf{A}(\boldsymbol{n}, \boldsymbol{n}')$ which is $s \times$ s matrix and defined by

$$A_{\alpha\beta}(\mathbf{n},\mathbf{n}') = \begin{cases} 1 & \text{if site } \alpha \text{ in cell } \mathbf{n} \text{ and site } \beta \text{ in cell } \mathbf{n}' \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

By imposing periodic boundary conditions, the translational symmetry is $A(\boldsymbol{n}, \boldsymbol{n}') = A(\boldsymbol{n} - \boldsymbol{n}')$ and therefore $A(\boldsymbol{n}) = a(n_1, n_2, \ldots, n_d)$. The degree matrix D_s for a unit cell is a $s \times$ s diagonal matrix whose elements

$$D_{\alpha\beta} = \kappa_\alpha \delta_{\alpha\beta} \qquad (2)$$

where κ_α is the degree or coordination number of site α and $\delta_{\alpha\beta}$ is the Kronecker delta function defined as $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$. The Laplacian matrix of the lattice is defined by

$$L(\mathbf{\Theta}) = D_s - \sum_n A(n) e^{i\mathbf{n}.\mathbf{\Theta}} \qquad (3$$

where $\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_d)$ is the d-dimensional vector. The very important theorem for counting the number of spanning trees $N_{ST}(L)$ in graph theory is given by [1, 7]

$$N_{\mathrm{ST}}(L)=\frac{1}{n}\prod_{i=1}^{n-1}\lambda_i \qquad (4$$

where λ_i are the non-zero eigenvalues of the Laplacian matrix **L** of the lattice.

It is known that a determinant of the Laplacian matrix is equal to the product of its eigenvalues, so that $N_{ST}(L)$ can be written as [8]

$$N_{ST}\left(L\right)=\frac{\lambda_1\lambda_2...\lambda_{s-1}}{sN_1N_2...N_d}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2,....,\ell_d\neq 0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}..\prod_{\ell_d=0}^{N_d-1}\det\mathbf{L}\left(\theta_1=\frac{2\pi\ell_1}{N_1},\theta_2=\frac{2\pi\ell_2}{N_2},..,\theta_d=\frac{2\pi\ell_d}{N_d}\right) \qquad (5$$

where $\lambda_1,\lambda_2,..,\lambda_{s-1}$ are the non-zero eigenvalues of $\mathbf{L}(0,0,0)$.

The number of spanning trees $N_{ST}(L)$ grows asymptotically as $\exp(nz_L)$ in the thermodynamic limit, $n\to\infty$, where z_L is called the asymptotic growth constant for spanning trees in the thermodynamics limit on graphs or lattices and given by [8]

$$z_L=\frac{1}{s}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}....\int_{-\pi}^{\pi}\frac{d\theta_d}{2\pi}\log\left[\det(\mathbf{L}(\theta_1,...\theta_d))\right] \qquad (6)$$

It was shown in [3] that the asymptotic growth constant on the homeomorphic expansion of k-regular lattices with p lattice points inserted on each edge is given by

$$z_{\text{homeomorphic expansion}}=\frac{(\frac{k}{2}-1)\log(p+1)+z_{\textit{regular lattice}}}{\frac{k}{2}p+1} \qquad (7)$$

where k is the coordination number for a regular lattice.

3 Decorated centered cubic lattices, *d=3*

In this section, the number of spanning trees and the asymptotic growth constants on the decorated centered cubic lattices are calculated.

3.1 Base-centered cubic lattice

The base-centered cubic lattice is a simple cubic lattice with extra vertices at the centers of the horizontal faces of the cube as shown in Fig.1.The unit cell is a cube containing two sites $s=2$ numbered 1 and 2.

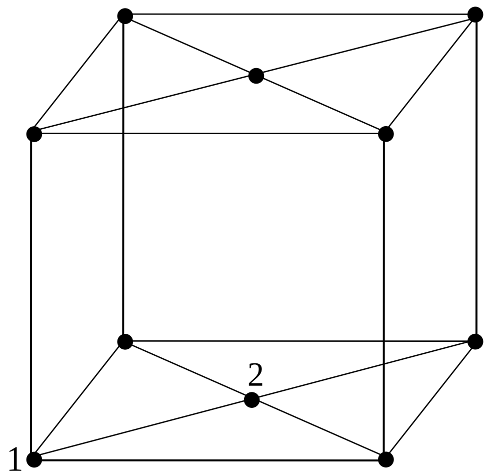

Fig.1. The base-centered cubic lattice.

The degree matrix and adjacency matrices are

$$D_2 = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}, \quad A(0,0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A(1,0,0) = A(-1,0,0)^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$
$$A(0,1,0) = A(0,-1,0)^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A(0,0,1) = A(0,0,-1)^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$A(1,1,0) = A(-1,-1,0)^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{8}$$

where $\mathbf{A}^T$ is the transpose of the matrix **A**.
Therefore, using Eq.(3) the Laplacian matrix can be written as

$$\begin{aligned} \mathbf{L}(\theta_1,\theta_2,\theta_3) = {} & \mathrm{D}_2 - A(0,0,0) - A(1,0,0)\,\mathrm{e}^{i\theta_1} - A(-1,0,0)\,\mathrm{e}^{-i\theta_1} - A(0,1,0)\,\mathrm{e}^{i\theta_2} \\ & - A(0,-1,0)\,\mathrm{e}^{-i\theta_2} - A(0,0,1)\,\mathrm{e}^{i\theta_3} - A(0,0,-1)\,\mathrm{e}^{-i\theta_3} - A(1,1,0)\,\mathrm{e}^{i(\theta_1+\theta_2)} \\ & - A(-1,-1,0)\,\mathrm{e}^{-i(\theta_1+\theta_2)} \end{aligned} \tag{9a}$$

Substituting Eq.(8) into the above equation yields

$$\mathbf{L}(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} 10 - 2\cos\theta_1 - 2\cos\theta_2 - 2\cos\theta_3 & -(1+e^{i\theta_1})(1+e^{i\theta_2}) \\ -(1+e^{-i\theta_1})(1+e^{-i\theta_2}) & 4 \end{pmatrix} \tag{9b}$$

and hence the determinant is

$$\det(\mathbf{L}(\theta_1,\theta_2,\theta_3)) = 36 - 12\cos\theta_1 - 12\cos\theta_2 - 8\cos\theta_3 - 4\cos\theta_1\cos\theta_2 \tag{10}$$

The non-zero eigenvalues of **L** (0, 0, 0) is $\lambda_1 = 8$ .Hence, the number of spanning trees N_{ST} given by Eq. (5) becomes

$$N_{ST}(L_{base\ cubic\ lattice}) = \frac{8}{2N_1N_2N_3} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2,\ldots,\ell_d \neq 0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \prod_{\ell_3=0}^{N_3-1} \left(36 - 12\cos\frac{2\pi\ell_1}{N_1} - 12\cos\frac{2\pi\ell_2}{N_2} - 8\cos\frac{2\pi\ell_3}{N_3} - 4\cos\frac{2\pi\ell_1}{N_1}\cos\frac{2\pi\ell_2}{N_2} \right) \quad (11)$$

As an example, if $N_1 = N_2 = N_3 = 2$ then, number of spanning trees of the base-centered cubic lattice is

$$N_{ST}(L_{base\ cubic\ lattice}) = \frac{1}{2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2,\ldots,\ell_d \neq 0)}}^{1} \prod_{\ell_2=0}^{1} \prod_{\ell_3=0}^{1} \left(36 - 12\cos\pi\ell_1 - 12\cos\pi\ell_2 - 8\cos\pi\ell_3 - 4\cos\pi\ell_1\cos\pi\ell_2 \right) \quad (12)$$

$$= 2\,113\,929\,216.$$

Using Eq. (6), the asymptotic growth constant for spanning trees is given by

$$z_{\text{base cubic lattice}} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \log\left(36 - 12\cos\theta_1 - 12\cos\theta_2 - 8\cos\theta_3 - 4\cos\theta_1\cos\theta_2 \right) \quad (13a)$$

The numerical computation of (13) yields the value

$$z_{base\ cubic\ lattice} = 1.738692709\ldots \quad (13b)$$

3.2 Side-centered cubic lattice

The side–centered cubic lattice is a simple cubic lattice with additional vertices at the centers of vertical faces as shown in Fig.2. Each unit cell consisting of three vertices $s = 3$ labeled by 1, 2 and 3. Therefore, the degree matrix and the adjacency matrices are

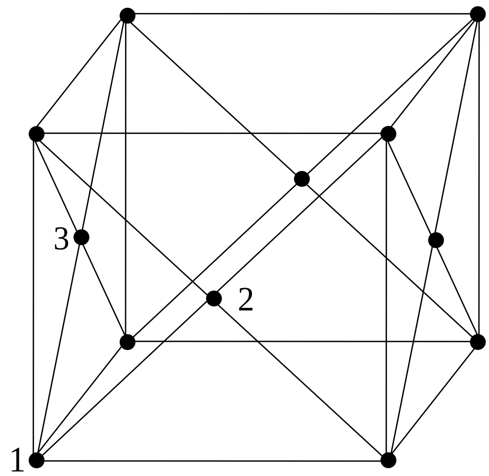

Fig.2. The side-centered cubic lattice.

$$D_3 = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A(0,0,0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(1,0,0) = A(-1,0,0)^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A(0,1,0) = A(0,-1,0)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(0,0,1) = A(0,0,-1)^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A(1,0,1) = A(-1,0,-1)^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(0,1,1) = A(0,-1,-1)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{14}$$

and one has

$$\mathbf{L}(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} 14 - 2\cos\theta_1 - 2\cos\theta_2 - 2\cos\theta_3 & -(1+e^{i\theta_1})(1+e^{i\theta_3}) & -(1+e^{i\theta_2})(1+e^{i\theta_3}) \\ -(1+e^{-i\theta_1})(1+e^{-i\theta_3}) & 4 & 0 \\ -(1+e^{-i\theta_2})(1+e^{-i\theta_3}) & 0 & 4 \end{pmatrix}, \tag{15}$$

with

$$\det \mathbf{L}(\theta_1,\theta_2,\theta_3) = 16(12 - 3\cos\theta_1 - 3\cos\theta_2 - 4\cos\theta_3 - \cos\theta_1\cos\theta_3 - \cos\theta_2\cos\theta_3), \quad (16)$$

The non-zero eigenvalues of **L** (0, 0 ,0) are $\lambda_1 = 12$, $\lambda_2 = 4$. Thus the number of spanning trees of the side-centered cubic lattice is

$$N_{ST}(L_{side\ cubic\ lattice}) = \frac{48}{3N_1N_2N_3}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2,\ell_3\neq 0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\prod_{\ell_3=0}^{N_3-1}\left(192 - 48\cos\frac{2\pi\ell_1}{N_1} - 48\cos\frac{2\pi\ell_2}{N_2} - 64\cos\frac{2\pi\ell_3}{N_3} - 16\cos\frac{2\pi\ell_1}{N_1}\cos\frac{2\pi\ell_3}{N_3} - 16\cos\frac{2\pi\ell_2}{N_2}\cos\frac{2\pi\ell_3}{N_3}\right)$$

(17a)

As an example, we compute the number of spanning trees of a finite side-centered cubic lattice with $N_1 = N_2 = N_3 = 2$.

$$N_{ST}(L_{side\ cubic\ lattice}) = 2\prod_{\ell_1=0}^{1}\prod_{\ell_2=0}^{1}\prod_{\ell_3=0}^{1}\times$$
$$\left(192 - 48\cos\pi\ell_1 - 48\cos\pi\ell_2 - 64\cos\pi\ell_3 - 16\cos\pi\ell_1\cos\pi\ell_3 - 16\cos\pi\ell_2\cos\pi\ell_3\right)$$
$$= 230\,897\,441\,832\,960$$

(17b)

From Eq. (6) the asymptotic growth constant for spanning trees is given by

$$z_{\text{side cubic lattice}} = \frac{1}{3}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_3}{2\pi}\times$$
$$\log\{16(12 - 3\cos\theta_1 - 3\cos\theta_2 - 4\cos\theta_3 - \cos\theta_1\cos\theta_3 - \cos\theta_2\cos\theta_3)\}$$

(18)

and the numerical evaluation gives $z_{\text{side cubic lattice}} = 1.7211738959\ldots$

3.3 Edge-centered cubic lattice

The edge-centered cubic lattice is shown in Fig. 3, which is a homeomorphic expansion of simple cubic lattice with one vertex (site) inserted on each edge-midpoint. The unit cell is a cube containing four sites $s = 4$ numbered 1, 2, 3 and 4. The degree matrix and the adjacency matrices are

$$D_4 = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, A(0,0,0) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (19a)$$

$$A(1,0,0) = A(-1,0,0)^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{19b}$$

$$A(0,1,0) = A(0,-1,0)^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{19c}$$

$$A(0,0,1) = A(0,0,-1)^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{19d}$$

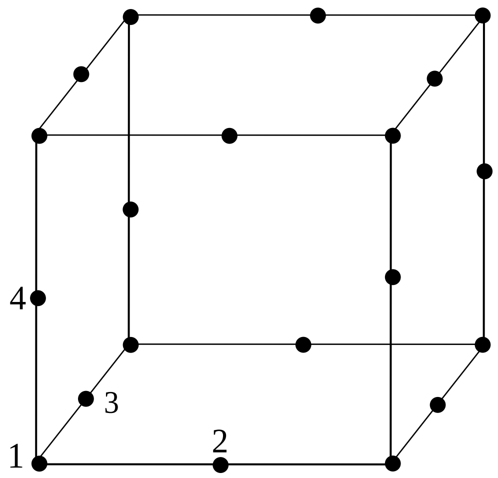

Fig.2. The side-centered cubic lattice.

Also the Laplacian matrix is given by

$$\mathbf{L}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} 6 & -1-e^{-i\theta_1} & -1-e^{-i\theta_2} & -1-e^{-i\theta_3} \\ -1-e^{i\theta_1} & 2 & 0 & 0 \\ -1-e^{i\theta_2} & 0 & 2 & 0 \\ -1-e^{i\theta_3} & 0 & 0 & 2 \end{bmatrix}, \tag{20}$$

with $\det[\mathbf{L}(\theta_1,\theta_2,\theta_3)] = 24 - 8\cos\theta_1 - 8\cos\theta_2 - 8\cos\theta_3$. The Laplacian matrix

L(0,0,0) has the eigenvalues of $\lambda_1 = 8, \quad \lambda_2 = \lambda_3 = 2$. Thus the number of spanning trees of the edge-centered cubic lattice is

$$N_{ST}(L_{edge\ cubic\ lattice}) = \frac{32}{4N_1N_2N_3} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2,\ell_3 \neq 0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \prod_{\ell_3=0}^{N_3-1} \left(24 - 8\cos\frac{2\pi\ell_1}{N_1} - 8\cos\frac{2\pi\ell_2}{N_2} - 8\cos\frac{2\pi\ell_3}{N_3} \right) \tag{21}$$

As an example, we compute the number of spanning trees of a finite edge-centered cubic lattice with $N_1 = N_2 = N_3 = 2$.

$$N_{ST}(L_{edge\ cubic\ lattice}) = \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2,\ell_3 \neq 0)}}^{1} \prod_{\ell_2=0}^{1} \prod_{\ell_3=0}^{1} \left(24 - 8\cos\pi\ell_1 - 8\cos\pi\ell_2 - 8\cos\pi\ell_3 \right) = 503316480. \tag{22}$$

The asymptotic growth constant is

$$z_{\text{edge cubic lattice}} = \frac{1}{4} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \log(24 - 8\cos\theta_1 - 8\cos\theta_2 - 8\cos\theta_3) \tag{23}$$

The numerical evaluation of (23) gives $z_{\text{edge cubic lattice}} = 0.76492094....$. Since the edge-centered cubic lattice is the homeomorphic expansion of the simple cubic lattice ($k = 6$) with $p = 1$ vertex inserted on each edge, one can use Eq. (7) to calculate the asymptotic growth constant of the edge-centered cubic lattice from that of the simple cubic lattice [8,12], we have

$$z_{\text{edge cubic lattice}} = \frac{2\log 2 + z_{simple\ cubic\ lattice}}{4} = 0.765111 \tag{24}$$

One can note that inserting vertices on the edges of the simple cubic lattice reduce its asymptotic growth constant.

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