On Heaviside Step Function with a Bulge Function by Using Laplace Transform

P. Haarsa\(^1\) and S. Pothat\(^2\)

\(^1\)Department of Mathematics, Srinakharinwirot
Bangkok 10110, Thailand
\(^2\)Wad Ban-Koh School, Bandara, Amphoe Pichai
Uttaradit 53220, Thailand

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Abstract

In this paper, we study the nonhomogeneous second order differential equation of the Heaviside step function with a bulge function. The Laplace transform, inverse Laplace transform and Power series expansion are employed to obtain the solution.

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1 Introduction

The Laplace transform can be employed in not only solving the linear ordinary differential equations with constant coefficient but also can be used with ordinary differential equations with variable coefficients. In addition, the Laplace transforms of derivatives have been studied in numerous approaches to solve the ODEs. Ig. Cho and Hj. Kim [3] showed that the Laplace transform of derivative can be expressed by an infinite series or Heaviside function. T. Lee and H. Kim [4] found the representation of energy equation by Laplace transform. In this paper, we applied the Laplace transform to the nonhomogeneous second order differential equation with a bulge function involved the Heaviside step function.
2 Preliminary Notes

We begin our study by contributing out the Laplace transform, Heaviside step function and the Power series expansion which can be used in this study.

**Definition 2.1.** The Laplace Transform [1]. Given a function \( f(t) \) defined for all \( t \geq 0 \), the Laplace transform of \( f \) is the function \( F \) defined as follow:

\[
F(s) = L \{ f(t) \} = \int_0^\infty e^{-st} f(t) dt.
\]  

(1)

for all values of \( s \) for which the improper integral converges

The nonhomogeneous differential equation with constant coefficients [2]. An equation of the form

\[
a_n \frac{d^ny}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = f(x).
\]  

(2)

is called the higher order nonhomogeneous linear differential equations. In this paper, we study the nonhomogeneous second order differential equation with a bulge function in the form \( y'' + \omega^2 y = e^{-\frac{(t-\ell)^2}{2}} \). The Laplace transform of the first and second derivatives are expressed respectively by \( L \{ y' \} = sF(s) - y(0) \) and \( L \{ y'' \} = s^2 F(s) - sy(0) - y'(0) \).

**Lemma 2.2.** The Laplace transform of the bulge function \( e^{-\frac{(t-\ell)^2}{2}} \) is expressed by

\[
L \left\{ e^{-\frac{(t-\ell)^2}{2}} \right\} = e^{-\frac{\ell^2}{2}} \left[ \frac{1}{s} + \frac{-1 + \ell^2}{s^3} + \frac{l(s^2 - 3 + \ell^2)}{s^4} \right].
\]  

(3)

**Proof.** The Power series expansion \( e^x \) is of the form

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]  

(4)

Hence, by substituting equation (4) with \( x = -\frac{(t-\ell)^2}{2} \), we derive

\[
e^{-\frac{(t-\ell)^2}{2}} = e^{-\frac{\ell^2}{2}} + e^{-\frac{\ell^2}{2}} t \ell + e^{-\frac{\ell^2}{2}} \left( -\frac{1}{2} + \frac{\ell^2}{2} \right) t^2 + e^{-\frac{\ell^2}{2}} \left( -\frac{1}{2} + \frac{\ell^3}{6} \right) t^3.
\]  

(5)

By taking the Laplace transform to equation (5) and using the fact that the Laplace transform is linear, we obtain

\[
L \left\{ e^{-\frac{(t-\ell)^2}{2}} \right\} = e^{-\frac{\ell^2}{2}} \left[ \frac{1}{s} + \frac{-1 + \ell^2}{s^3} + \frac{l(s^2 - 3 + \ell^2)}{s^4} \right].
\]  

(6)
Heaviside step function of a bulge function of a piecewise continuous function \( f(t) = \left\{ \begin{array}{ll} e^{-\frac{(t-l)^2}{2}}/a ; & 0 < t < \xi_1 \\ t > \xi_1 \end{array} \right. \) is expressed by

\[
 f(t) = e^{-\frac{(t-l)^2}{2}} + au(t-\xi_1) - e^{-\frac{(t-l)^2}{2}}u(t-\xi_1). \tag{7}
\]

where \( a,\xi_1 \) are constants.

Lemma 2.3. The Laplace transform of \( e^{-\frac{(t-l)^2}{2}}u(t-\xi_1) \) is expressed by

\[
 L \left\{ e^{-\frac{(t-l)^2}{2}}u(t-\xi_1) \right\} = \frac{Ae^{-\xi_1 s}}{s} + A\xi_1 s + \frac{B\xi_1^2}{s} + \frac{C\xi_1^3}{s} \tag{8}
\]

where \( A = e^{-\frac{l^2}{2}}, B = \left( -\frac{1}{2} + \frac{l^2}{2} \right), \) and \( C = \left( -\frac{1}{2} + \frac{l^3}{6} \right) \).

Proof. From equation (4) and the Heaviside step function, we have

\[
 e^{-\frac{(t-l)^2}{2}}u(t-\xi_1) = e^{-\frac{l^2}{2}}u(t-\xi_1) + e^{-\frac{l^2}{2}}ltu(t-\xi_1) + e^{-\frac{l^2}{2}}t^2u(t-\xi_1) + e^{-\frac{l^2}{2}}t^3u(t-\xi_1). \tag{9}
\]

Therefore, by taking the Laplace transform to equation (9), we obtain

\[
 L \left\{ e^{-\frac{(t-l)^2}{2}}u(t-\xi_1) \right\} = AL \{u(t-\xi_1)\} + AL \{tu(t-\xi_1)\} \\
 + ABL \{t^2u(t-\xi_1)\} + ACL \{t^3u(t-\xi_1)\} \\
 = \frac{Ae^{-\xi_1 s}}{s} + A\xi_1 s + \frac{B\xi_1^2}{s} + \frac{C\xi_1^3}{s} \tag{= K}. \tag{10}
\]
3 Main Result

Lemma 3.1. The Laplace transform of Heaviside step function of a bulge function of a piecewise continuous function \( f(t) = \begin{cases} e^{-\frac{(t-\xi)^2}{a}} & ; 0 < t < \xi_1 \\ a & ; t > \xi_1 \end{cases} \) can be expressed by

\[
M + \frac{ae^{-\xi_1 s}}{s} + K. \tag{11}
\]

where \( a, \xi_1 \) are constants.

Proof. By taking the Laplace transform to equation (7) and lemma 2.2 and 2.3, we have

\[
L \{ f(t) \} = L \left\{ e^{-\frac{(t-\xi)^2}{a}} \right\} + L \left\{ a - e^{-\frac{(t-\xi)^2}{a}} \right\} u(t - \xi_1)
\]

\[
= L \left\{ e^{-\frac{(t-\xi)^2}{a}} \right\} + L \{ au(t - \xi_1) \} - L \left\{ e^{-\frac{(t-\xi)^2}{a}} u(t - \xi_1) \right\}
\]

\[
= e^{-\frac{l^2}{2}} \left[ \frac{1}{s} - 1 + \frac{l^2}{s^3} + \frac{l(s^2 - 3 + l^2)}{s^4} \right] + \frac{ae^{-\xi_1 s}}{s} \\
- A e^{-\xi_1 s} \left[ \frac{1}{s^2} + \frac{\xi_1}{s} \right] - Ae^{-\xi_1 s} \left[ \frac{2}{s^3} + \frac{2\xi_1}{s^2} + \frac{\xi_1^2}{s} \right] \\
- AC e^{-\xi_1 s} \left[ \frac{6}{s^4} + \frac{6\xi_1}{s^3} + \frac{3\xi_1^2}{s^2} + \frac{\xi_1}{s} \right]
\]

\[
= M + \frac{ae^{-\xi_1 s}}{s} + K \tag{12}
\]

where \( M = e^{-\frac{l^2}{2}} \left[ \frac{1}{s} + \frac{1 + l^2}{s^3} + \frac{l(s^2 - 3 + l^2)}{s^4} \right] \) and \( K = A e^{-\xi_1 s} + Ae^{-\xi_1 s} \left[ \frac{1}{s^2} + \frac{\xi_1}{s} \right] + \frac{2}{s^3} + \frac{2\xi_1}{s^2} + \frac{\xi_1^2}{s} \). \[Q.E.D.\]

Lemma 3.2. The solution of the nonhomogeneous differential equation with constant coefficients \( y'' + \omega^2 y = f(t) = \begin{cases} e^{-\frac{(t-\xi)^2}{a}} & ; 0 < t < \xi_1 \\ a & ; t > \xi_1 \end{cases} \) where \( y(0) = \omega_0, y'(0) = \omega_1 \) is expressed by

\[
y(t) = \frac{\omega_0}{\omega} \cos \omega t + \frac{\omega_1}{\omega} \sin \omega t + e^{-\frac{l^2}{2}} \left[ 1 - \frac{l^2}{2} + \frac{l^2 t^2}{2} \right. \\
+ lt - \frac{lt^3}{3} + \frac{l^3 t^3}{6} \left] + a \times Heaviside(t - \xi_1) + L^{-1} \{ K \}. \tag{13}
\]

where \( a, l, \omega, \omega_0 \) and \( \omega_1 \) are constants.
Proof. By taking the Laplace transform to the nonhomogeneous differential equation with constant coefficients and the Heaviside step function and lemma 3.1, we obtain

\[ s^2 L \{y''(t)\} - s\omega_0 - \omega_1 + \omega^2 L \{y(t)\} = L \{\text{Heaviside step function of } f(t)\} . \]

Or

\[ L \{y(t)\} = \frac{s\omega_0}{s^2 + \omega^2} + \frac{\omega_1}{s^2 + \omega^2} + M + \frac{ae^{-\xi_1 s}}{s} + K. \quad (14) \]

The inverse Laplace transform can be used to equation (14) to obtain the solution of the nonhomogeneous differential equation with the Heaviside step function as

\[ y(t) = \omega_0 \cos \omega t + \frac{\omega_1}{\omega} \sin \omega t + e^{-t^2/2} \left[ 1 - \frac{t^2}{2} + \frac{t^4}{2} \right] + a \times \text{Heaviside}(t - \xi_1) + L^{-1} \{K\} . \quad (15) \]

4 Conclusion

In this paper, we study the nonhomogeneous second order differential equation of the Heaviside step function with a bulge function which is denoted by \( f(t) = e^{-(t-l)^2} \) where \( l \) is a positive constant. Ig. Cho and Hj. Kim [3] studied the Laplace transform of derivative expressed by Heaviside function. For this research, we found the method to solve the nonhomogeneous second order differential equation with a bulge function involved the Heaviside step function. The Laplace transform, the inverse Laplace transform and the Power series expansion were employed in this method.

References


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