Non-existence of Screen Homothetic Lightlike Hypersurfaces of an Indefinite Kenmotsu Manifold

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Abstract

We study screen homothetic lightlike hypersurfaces $M$ of an indefinite Kenmotsu manifold $\bar{M}$ with flat transversal connection. We prove that there do not exist screen homothetic lightlike hypersurfaces of indefinite Kenmotsu manifolds with flat transversal connection subject to the conditions: (1) $M$ is locally symmetric, or (2) $M$ is semi-symmetric.

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1 Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. The study of such notion was initiated by Duggal and Bejancu [3] and then studied by many authors (see two books [4, 5]). Now we have lightlike version of a large variety of Riemannian submanifolds.

In this paper, we study lightlike hypersurfaces $M$ of indefinite Kenmotsu manifolds $\bar{M}$ with flat transversal connection, whose shape operator is homothetic to the shape operator of its screen distribution by some constant $\varphi$. The motivation for this geometric restriction comes from the classical geometry of non-degenerate submanifolds for which there are only one type of
shape operator with its one type of respective second fundamental form. The purpose of this paper is to prove that there do not exist screen homothetic lightlike hypersurfaces of indefinite Kenmotsu manifolds with flat transversal connection subject to the conditions: (1) $M$ is locally symmetric, or (2) $M$ is semi-symmetric.

2 Lightlike hypersurfaces

An odd dimensional semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$ is said to be an indefinite Kenmotsu manifold \cite{10, 11} if there exists an almost contact metric structure $(J, \zeta, \theta, \bar{g})$, where $J$ is a $(1,1)$-type tensor field, $\zeta$ is a vector field and $\theta$ is a 1-form such that

\begin{align*}
J^2X &= -X + \theta(X)\zeta, \quad J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\zeta) = 1, \quad (2.1) \\
\theta(X) &= \bar{g}(\zeta, X), \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \theta(X)\theta(Y), \\
\nabla_X\zeta &= -X + \theta(X)\zeta, \quad (2.2) \\
(\nabla_XJ)Y &= -\bar{g}(JX, Y)\zeta + \theta(Y)JX, \quad (2.3)
\end{align*}

for any vector fields $X, Y$ on $\bar{M}$, where $\nabla$ is the Levi-Civita connection of $\bar{M}$.

A hypersurface $(M, g)$ of $\bar{M}$ is called a lightlike hypersurface if the normal bundle $TM^\perp$ of $M$ is a subbundle of the tangent bundle $TM$ of $M$ and coincides with the radical distribution $Rad(TM) = TM \cap TM^\perp$. Then there exists a non-degenerate complementary vector bundle $S(TM)$ of $Rad(TM)$ in $TM$, which is called a screen distribution on $M$, such that

\begin{equation}
TM = Rad(TM) \oplus_{\text{orth}} S(TM), \quad (2.4)
\end{equation}

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of smooth sections of a vector bundle $E$ over $\bar{M}$. For any null section $\xi$ of $Rad(TM)$, there exists a unique null section $N$ of a unique vector bundle $tr(TM)$ \cite{3} in the orthogonal complement $S(TM)^\perp$ of $S(TM)$ in $\bar{M}$ satisfying

\begin{align*}
\bar{g}(\xi, N) &= 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).
\end{align*}

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

\begin{equation}
T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{\text{orth}} S(TM). \quad (2.5)
\end{equation}

We call $tr(TM)$ and $N$ the transversal vector bundle and the null transversal vector field of $M$ with respect to the screen distribution $S(TM)$ respectively.

From now and in the sequel, we denote by $X, Y, Z, U, \cdots$ the vector fields on $M$ unless otherwise specified. Let $P$ be the projection morphism of $TM$
Non-existence of screen homothetic lightlike hypersurfaces

From (2.11), we show that

\[ \nabla_X Y = \nabla_X Y + B(X, Y)N, \quad (2.6) \]
\[ \nabla_X N = -A_N X + \tau(X)N; \quad (2.7) \]
\[ \nabla_X P Y = \nabla^*_X P Y + C(X, PY)\xi, \quad (2.8) \]
\[ \nabla_X \xi = -A^*_\xi X - \tau(X)\xi; \quad (2.9) \]

where \( \nabla \) and \( \nabla^* \) are the linear connections on \( TM \) and \( S(TM) \) respectively, \( B \) and \( C \) are the local second fundamental forms on \( TM \) and \( S(TM) \) respectively, \( A_N \) and \( A^*_\xi \) are the shape operators on \( TM \) and \( S(TM) \) respectively and \( \tau \) is a 1-form. As \( \nabla \) is torsion-free, \( \nabla \) is also torsion-free and \( B \) is symmetric on \( TM \). From the fact \( B(X, Y) = g(\nabla_X Y, \xi) \), we show that \( B \) is independent of the choice of a screen distribution \( S(TM) \) and satisfies

\[ B(X, \xi) = 0. \quad (2.10) \]

The above two local second fundamental forms \( B \) and \( C \) of \( M \) and \( S(TM) \) respectively are related to their shape operators by

\[ B(X, Y) = g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \quad (2.11) \]
\[ C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \quad (2.12) \]

From (2.11), we show that \( A^*_\xi \) is \( S(TM) \)-valued self-adjoint such that \( A^*_\xi \xi = 0 \).

Denote by \( \bar{R}, \quad R \) and \( R^* \) the curvature tensors of the connections \( \nabla, \quad \nabla \) and \( \nabla^* \) respectively. Using (2.6)~(2.9), we obtain the Gauss-Codazzi equations:

\[ \bar{R}(X, Y)Z = R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \quad (2.13) \]
\[ + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \]
\[ + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N, \]
\[ \bar{R}(X, Y)N = -\nabla_X (A_N Y) + \nabla_Y (A_N X) + A_N [X, Y] + \tau(X)A_N Y \quad (2.14) \]
\[ - \tau(Y)A_N X + \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X, Y)\}N, \]
\[ R(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A^*_\xi Y - C(Y, PZ)A^*_\xi X \quad (2.15) \]
\[ + \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \]
\[ + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ)\}\xi, \]
\[ R(X, Y)\xi = -\nabla_X (A^*_\xi Y) + \nabla_Y (A^*_\xi X) + A^*_\xi [X, Y] - \tau(X)A^*_\xi Y \quad (2.16) \]
\[ + \tau(Y)A^*_\xi X + \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi. \]

Let \( \nabla^*_X N = \pi(\nabla_X N) \), where \( \pi \) is the projection morphism of \( TM \) on \( tr(TM) \). Then \( \nabla^*_X \) is a linear connection on the transversal vector bundle
The transversal connection $\nabla^\perp$ of $M$ is said to be flat if $R^\perp$ vanishes identically. This definition comes from the definition of flat normal connection in the theory of classical geometry of non-degenerate submanifolds. We have the following result (see [8]).

**Theorem 2.1.** Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then the transversal connection of $M$ is flat if and only if $\tau$ is closed, i.e., $d\tau = 0$, on $M$.

**Note 1.** $d\tau$ is independent of the choice of the null section $\xi$ on $\text{Rad}(TM)$. In fact, if we take $\bar{\xi} = \gamma \xi$, then $\bar{N} = \gamma^{-1} \bar{\xi}$. As $\tau(X) = \bar{g}(\nabla_X N, \xi)$, we have $\tau(X) = \bar{\tau}(X) + X(\ln \gamma)$. If we take the exterior derivative $d$ to this equation, then we get $d\tau = d\bar{\tau}$ [3].

**Note 2.** In case $d\tau = 0$, by the cohomology theory there exist a smooth function $l$ such that $\tau = dl$. Thus $\tau(X) = X(l)$. If we take $\bar{\xi} = \gamma \xi$, then $\tau(X) = \bar{\tau}(X) + X(\ln \gamma)$. Setting $\gamma = \exp(l)$ in this equation, we get $\tau(X) = 0$. We call the pair $\{\xi, N\}$ such that the corresponding 1-form $\tau$ vanishes the canonical null pair of $M$. Although $S(TM)$ is not unique it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/\text{Rad}(TM)$ due to Kupeli [12]. Thus all $S(TM)$ are mutually isomorphic. In the sequel, we deal with only lightlike hypersurfaces $M$ equipped with the canonical null pair $\{\xi, N\}$.

### 3 Locally symmetric lightlike hypersurfaces

**Definition 1.** A lightlike hypersurface $M$ is screen homothetic[4, 6] if the shape operators $A_N$ and $A^*_{\xi}$ of $M$ and $S(TM)$ respectively are related by $A_N = \varphi A^*_{\xi}$, or equivalently,

$$C(X, PY) = \varphi B(X, Y),$$

where $\varphi$ is a constant on a coordinate neighborhood $U$ in $M$. In particular, if $\varphi = 0$, i.e., $C = 0$ on $U$, then we say that $S(TM)$ is totally geodesic [10] in $M$, and if $\varphi \neq 0$ on $U$, then we say that $M$ is proper screen homothetic [6, 7].

In the sequel, by saying that $M$ is screen homothetic we shall mean not only $M$ is proper screen homothetic but also $S(TM)$ is totally geodesic in $M$.

**Theorem 3.1.** Let $M$ be a locally symmetric lightlike hypersurface of an indefinite Kenmotsu manifold $\bar{M}$. Then $S(TM)$ is not totally geodesic. Moreover, if the transversal connection is flat, then $M$ is not proper screen homothetic.
Proof. From the decomposition (2.5) of $TM$, $\zeta$ is decomposed as follow:

$$\zeta = W + fN, \quad (3.2)$$

where $W$ is a non-vanishing smooth vector field on $M$ and $f = \theta(\xi)$ is a smooth function. Substituting (3.2) into (2.2) and using (2.6) and (2.7), we have

$$\nabla_X W = -X + \theta(X)W + fA_\xi X, \quad (3.3)$$

$$Xf + f\tau(X) + B(X, W) = f\theta(X). \quad (3.4)$$

Substituting (3.4) into $[X,Y]f = X(Yf) - Y(Xf)$, we have

$$(\nabla_X B)(Y,W) - (\nabla_Y B)(X,W) + \tau(X)B(Y,W) - \tau(Y)B(X,W) \quad (3.5)$$

$$+ f\{B(Y,A_\xi X) - B(X,A_\xi Y) + 2d\tau(X,Y)\} = 2f d\theta(X,Y).$$

Using (2.13), (2.14) and (3.2), the equation (3.5) reduce to

$$\bar{g}(\bar{R}(X,Y)\zeta, \xi) = 2f d\theta(X,Y). \quad (3.6)$$

Substituting (3.3) into $R(X,Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W$ and using (2.13), (2.14), (3.2) $\sim$ (3.6) and the fact that $\nabla$ is torsion-free, we have

$$\bar{R}(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta. \quad (3.7)$$

Taking the scalar product with $\zeta$ to (3.7) and using the facts that $\bar{g}(\bar{R}(X,Y)\zeta, \zeta) = 0$ and $\theta(X) = \bar{g}(X, \zeta)$, we have $d\theta = 0$, i.e., the 1-form $\theta$ is closed on $TM$.

Călin [1] proved that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assume in this case. Taking $Y = \zeta$ to (2.6) and using (2.2), we get

$$\nabla_X \zeta = -X + \theta(X)\zeta, \quad B(X, \zeta) = 0.$$

Taking the scalar product with $N$ to the first equation and using (2.8), we have

$$C(X, \zeta) = -\eta(X),$$

where $\eta$ is a 1-form such that $\eta(X) = \bar{g}(X, N)$.

Assume that $M$ is screen homothetic. Using the last two equations, we get

$$-\eta(X) = C(X, \zeta) = \varphi B(X, \zeta) = 0.$$

It is a contradiction as $\eta(\xi) = 1$. Thus $\zeta$ is not tangent to $M$ and $f \neq 0$. Let $e = \theta(N)$. Assume that $e = 0$. Applying $\nabla_X$ to $\bar{g}(\zeta, N) = 0$ and using (2.2) and (2.7), we have $\bar{g}(A_\xi X, \zeta) = -\eta(X)$. Replacing $X$ by $\xi$ to this and using (3.1) and the fact that $A_\xi^2 \xi = 0$, we have $0 = \varphi\bar{g}(A_\xi^2 \xi, \zeta) = -\eta(\xi) = -1$. It is a contradiction. Thus $e$ is non-vanishing function.
Case 1. Assume that $S(TM)$ is totally geodesic in $M$, i.e., $\varphi = 0$. Substituting (3.2) into (3.7) and using (2.13), (2.14), (3.5) and the fact that $d\theta = 0$, we have

$$R(X,Y)W = \theta(X)Y - \theta(Y)X. \quad (3.8)$$

Applying $\nabla_X$ to $\theta(Y) = g(Y,\zeta)$ and using (2.2) and (2.6), we have

$$(\nabla_X\theta)(Y) = eB(X,Y) - g(X,Y) + \theta(X)\theta(Y). \quad (3.9)$$

Applying $\nabla_Z$ to (3.8) and using (3.3), (3.9) and the fact that $\nabla_ZR = 0$, we have

$$R(X,Y)Z = \{g(X,Z) - eB(X,Z)\}Y - \{g(Y,Z) - eB(Y,Z)\}X. \quad (3.10)$$

Replacing $Z$ by $W$ to (3.10) and then, comparing this result with (3.8) and using the fact that $\theta(X) = g(X,W) + f\eta(X)$, we have

$$\{f\eta(X) + eB(X,W)\}Y = \{f\eta(Y) + eB(Y,W)\}X.$$

Replacing $Y$ by $\xi$ to this and using the fact that $X = PX + \eta(X)\xi$, we have

$$fPX = eB(X,W)\xi, \quad \forall X \in \Gamma(TM).$$

The left term of this equation belongs to $S(TM)$ and the right term belongs to $TM^\perp$. Thus $fPX = 0$ and $eB(X,W) = 0$ for all $X \in \Gamma(TM)$. From the first equation of this results, we have $f = 0$. It is a contradiction as $f \neq 0$. Thus $S(TM)$ is not totally geodesic in $M$.

Case 2. Assume that $M$ is screen homothetic. Substituting (3.2) into (3.7) with $d\theta = 0$ and then, taking the scalar product with $N$, we have

$$g(R(X,Y)W, N) = \theta(X)\eta(Y) - \theta(Y)\eta(X). \quad (3.11)$$

As the transversal connection is flat, i.e., $d\tau = 0$, we have $\tau = 0$ by Note 2. Substituting $W = PW + e\xi$ to (3.11) and using (2.11), (2.15), (2.16) and (3.1), we have

$$(\nabla_XC)(Y, PW) - (\nabla_YC)(X, PW) = \theta(X)\eta(Y) - \theta(Y)\eta(X). \quad (3.12)$$

Applying $\nabla_X$ to $C(Y, PW) = \varphi B(Y, W)$ and using (2.8)$\sim$(2.11), we have

$$(\nabla_XC)(Y, PW) = \varphi(\nabla_XB)(Y, W).$$

Substituting this equation into (3.12) and using (3.5) with $\tau = 0$, we have

$$\theta(X)\eta(Y) - \theta(Y)\eta(X) = 0, \quad \forall X, Y \in \Gamma(TM).$$
Replacing $Y$ by $\xi$ to this, we have $g(X,W) = 0$. This implies $W = e\xi$. Consequently the structure vector field $\zeta$ of $M$ is decomposed as

$$\zeta = e\xi + fN.$$  \hspace{1cm} (3.13)

From the fact that $\bar{g}(\zeta, \zeta) = 1$ and (3.13), we get $2ef = 1$. Applying $\nabla_X$ to (3.13) and using (2.2), (2.7), (2.9) and the fact that $\eta(X) = 2e\theta(X)$, we have

$$-PX - e\theta(X)\xi + f\theta(X)N = -(e + \varphi f)A^*\xi X + X[e]\xi + X[f]N.$$  

Taking the scalar product with $\xi$ and $N$ to this result by turns, we get

$$X[f] = f\theta(X), \quad X[e] = -e\theta(X), \quad (e + \varphi f)A^*\xi X = PX. \hspace{1cm} (3.14)$$

If $e + \varphi f = 0$, then we obtain $PX = 0$ for all $X \in \Gamma(TM)$. It is an impossible result if $S(TM) \neq \{0\}$. Thus the function $\alpha = (e + \varphi f)^{-1}$ is a non-vanishing one and we get

$$A^*\xi X = \alpha PX, \quad B(X, Y) = \alpha g(X, Y). \hspace{1cm} (3.15)$$

Applying $\nabla_X$ to $\alpha = (e + \varphi f)^{-1}$ and $\alpha e$, and then, using (3.14)\textsubscript{1,2}, we have

$$X[\alpha] = \alpha(2\alpha e - 1)\theta(X), \quad X[\alpha e] = 2\alpha e(\alpha e - 1)\theta(X). \hspace{1cm} (3.16)$$

Applying $\nabla_X$ to $\theta(Y) = g(Y, \zeta)$ and using (2.2), (2.6) and (3.15), we have

$$\nabla_X \theta(Y) = (\alpha e - 1)g(X, Y) + \theta(X)\theta(Y). \hspace{1cm} (3.17)$$

Substituting (3.2) into (3.7) with $d\theta = 0$ and using (2.10), (2.13), (2.14), (3.5) and the facts that $W = e\xi$, $A_N = \varphi A^*\xi$ and $\alpha = (e + f\varphi)^{-1}$, we have

$$R(X, Y)\xi = \alpha \{\theta(X)Y - \theta(Y)X\}, \quad \forall X, Y \in \Gamma(TM). \hspace{1cm} (3.18)$$

From (2.9), (3.14)\textsubscript{3} and the facts that $X = PX + \eta(X)\xi$ and $2ef = 1$, we have

$$\nabla_X \xi = -\alpha X + 2\alpha e \theta(X)\xi. \hspace{1cm} (3.19)$$

Applying $\nabla_Z$ to (3.18) and using (3.16)~(3.19), we have

$$(\nabla_Z R)(X, Y)\xi = \alpha R(X, Y)Z + \alpha(\alpha e - 1)\{g(X, Z)Y - g(Y, Z)X\}. \hspace{1cm} (3.20)$$

Assume that $M$ be locally symmetric. The equation (3.20) is reduced to

$$R(X, Y)Z = (\alpha e - 1)\{g(Y, Z)X - g(X, Z)Y\}. \hspace{1cm}$$

Taking $Z = \xi$ to this and then, comparing this result with (3.18), we have

$$\alpha \{\theta(X)Y - \theta(Y)X\} = 0.$$  

Replacing $Y$ by $\xi$ to this and using the facts that $X = PX + \eta(X)\xi$ and $\theta(X) = f\eta(X)$, we have $\alpha PX = 0$ for all $X \in \Gamma(TM)$. Thus we have $\alpha = 0$. It is a contradiction as $\alpha \neq 0$. Thus $M$ is not proper screen homothetic.
4 Semi-symmetric lightlike hypersurfaces

**Definition 2.** A lightlike hypersurface $M$ is called *semi-symmetric* [5, 10] if

$$R(X,Y)R = 0, \quad \forall X, Y \in \Gamma(TM).$$

**Theorem 4.1.** Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $\bar{M}$ with flat transversal connection. If $M$ is semi-symmetric and $\dim M > 2$, then neither $M$ is proper screen homothetic nor $S(TM)$ is totally geodesic in $M$.

**Proof.** It is well-known that, for any lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$, $J(Rad(TM))$ and $J(tr(TM))$ are vector subbundles of $S(TM)$ of rank 1 [8, 9]. Assume that $M$ is screen homothetic. Then we can use the equations (3.1)∼(3.7), (3.11)∼(3.20) in Section 3. Applying $J$ to (3.13) and using $J\zeta = 0$ and $ef = 1$, we get $J\xi = -2f^2JN$. Thus $J(Rad(TM)) \cap J(tr(TM)) \neq \{0\}$. From this result we have

$$J(Rad(TM)) = J(tr(TM)).$$

Using this result, the tangent bundle $TM$ of $M$ is decomposed as follow:

$$TM = Rad(TM) \oplus_{\text{orth}} \{ J(Rad(TM)) \oplus_{\text{orth}} H \}, \quad (4.1)$$

where $H$ is a non-degenerate and almost complex distribution on $S(TM)$ with respect to $J$. Consider a timelike vector field $V$ and its 1-form $v$ defined by

$$V = -f^{-1}J\xi = e^{-1}JN, \quad v(X) = -g(X,V), \quad (4.2)$$

for all $X \in \Gamma(TM)$. Applying $J$ to (4.2) and using the fact that $2ef = 1$, we get

$$JV = e\xi - fN. \quad (4.3)$$

Denote by $Q$ the projection morphism of $TM$ on $H$ with respect to the decomposition (4.1). Then any vector field $X$ on $M$ is expressed as follow:

$$X = QX + v(X)V + \eta(X)\xi.$$

Applying $J$ to this and using (4.3) and the fact that $\theta(X) = f\eta(X)$, we have

$$JX = FX - \theta(X)V + ev(X)\xi - f\nu(X)N, \quad (4.4)$$

where $F$ is a tensor field of type $(1,1)$ globally defined on $M$ by

$$FX = JQX, \quad \forall X \in \Gamma(TM).$$
Applying $\tilde{\nabla}_X$ to $fV = -J\xi$ and $v(Y) = -g(Y, V)$ by turns and using (2.1), (2.3), (2.6), (2.9), (2.11), (3.14), (4.2) to (4.4) and the fact that $F\xi = 0$, we get

$$\nabla_X V = e(\rho - 1)v(X)\xi + \rho FX, \quad \rho = 2ae - 1, \quad (4.5)$$

$$\nabla_X v(Y) = -\rho g(FX, Y) + (\rho + 1)v(X)\theta(Y). \quad (4.6)$$

As $\rho = 2ae - 1$, we get $ae = (\rho + 1)/2$. Thus the equation (3.17) reduce to

$$\nabla_X\theta(Y) = \frac{1}{2}(\rho - 1)g(X, Y) + \theta(X)\theta(Y). \quad (4.7)$$

Applying $\tilde{\nabla}_Y$ to (4.4) and using (2.3), (2.6) to (2.9), (3.13) to (3.15) and (4.2) to (4.7), we get

$$\nabla_X F = (\rho + 1)\theta(Y)FX + \rho v(Y)PX + \rho g(X, Y)V \quad (4.8)$$

$$+ e(\rho - 1)g(FX, Y)\xi, \quad \forall X, Y \in \Gamma(TM).$$

Substituting (4.5) into the right term of $R(X, Y)V = \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X, Y]} V$ and using the fact that $\nabla$ is torsion-free, we have

$$R(X, Y)V = e\{(X\rho)v(Y) - (Y\rho)v(X)
+ (\rho^2 + \rho - 2)[v(X)\theta(Y) - v(Y)\theta(X)]\}\xi
+ (X\rho)FY - (Y\rho)FX + \rho(\rho + 1)[\theta(Y)FX - \theta(X)FY]
+ \frac{1}{2}(\rho^2 + 1)[v(Y)PX - v(X)PY]. \quad (4.9)$$

Assume that $M$ is semi-symmetric, i.e., $R(U, Z)R = 0$. In case $M$ is proper screen homothetic. Applying $\nabla_U$ to (3.20) and using (3.2) and (3.20), we have

$$(\nabla_U \nabla_Z R)(X, Y)\xi = \alpha(\nabla_U R)(X, Y)Z + \alpha(\nabla_Z R)(X, Y)U$$

$$- \alpha\theta(U)R(X, Y)Z + \alpha R(X, Y)\nabla_U Z$$

$$+ \alpha(2ae - 1)(ae - 1)\theta(U)\{g(X, Z)Y - g(Y, Z)X\}$$

$$+ \alpha(ae - 1)\{2ae\theta(Z)[g(U, X)Y - g(U, Y)X]$$

$$+ 2ae g(U, Z)\theta(X)Y - \theta(Y)X]$$

$$+ g(X, \nabla_U Z)Y - g(Y, \nabla_U Z)X\}.$$

From the last equation and (3.20), we have

$$\theta(Z)\{R(X, Y)U + (ae - 1)[g(X, U)Y - g(Y, U)X]\}$$

$$= \theta(U)\{R(X, Y)Z + (ae - 1)[g(X, Z)Y - g(Y, Z)X]\}.$$
In case $S(TM)$ is totally geodesic, $\varphi = 0$ and $\alpha e = 1$. Thus $R$ is given by

$$R(X, Y)Z = 2\theta(Z)\{\theta(X)Y - \theta(Y)X\}. \quad (4.11)$$

**Case 1.** Let $M$ be proper screen homothetic. Replacing $Z$ by $V$ to (4.10) and then, comparing the $S(TM)$-components of this result and (4.9), we get

$$(\alpha e - 1)\{v(X)PY - v(Y)PX\}$$

$$= (X\rho)FY - (Y\rho)FX + \rho(\rho + 1)\{\theta(Y)FX - \theta(X)FY\}$$

$$+ \frac{1}{2}(\rho^2 + 1)\{v(Y)PX - v(X)PY\}, \quad \forall X, Y \in \Gamma(TM).$$

Substituting $\rho = 2ae - 1$ and $X[\rho] = 4ae(\alpha e - 1)\theta(X)$ to this, we have

$$\alpha(2ae - 1)\{v(X)PY - v(Y)PX\} + 2ae\{\theta(X)FY - \theta(Y)FX\} = 0. \quad (4.12)$$

Replacing $Y$ by $\xi$ to this and using the facts that $v(\xi) = 0$ and $P\xi = FX = 0$ for all $X \in \Gamma(TM)$. Substituting (4.4) into the equation $\bar{g}(JX, JY) = g(X, Y) - \theta(X)\theta(Y)$ for any $X, Y \in \Gamma(TM)$, we have $g(FX, FY) = g(X, Y) + v(X)v(Y)$. As $\alpha \neq 0$, we get $\alpha\{g(X, Y) + v(X)v(Y)\} = 0$. This implies $g(X, Y) = 0$ for any $X, Y \in \Gamma(H)$. It is a contradiction as $\dim M > 2$. Thus $M$ is not proper screen homothetic.

**Case 2.** Let $S(TM)$ be totally geodesic in $M$. Then $\alpha e = 1$ and (4.12) is reduced to

$$v(X)PY - v(Y)PX + 2\{\theta(X)FY - \theta(Y)FX\} = 0.$$

Replacing $Y$ by $V$ to this and using the facts that $\theta(V) = 0$ and $FV = 0$, we have

$$PX = v(X)V, \quad \forall X \in \Gamma(TM).$$

It is a contradiction as $\dim M > 2$. Thus $S(TM)$ is not totally geodesic.

An odd dimensional semi-Riemannian manifold $(\tilde{M}, \bar{g})$ is called an *indefinite Sasakian manifold* [5, 8] if there exists an almost contact metric structure $(\phi, \mathcal{U}, u, \bar{g})$, where $\phi$ is a $(1, 1)$-type tensor field, $\mathcal{U}$ is a vector field and $u$ is a 1-form such that

$$\phi^2X = -X + u(X)\mathcal{U}, \quad \phi\mathcal{U} = 0, \quad u \circ \phi = 0, \quad u(\mathcal{U}) = 1,$$

$$\bar{g}(\mathcal{U}, \mathcal{U}) = \epsilon, \quad \bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon u(X)u(Y),$$

$$u(X) = \epsilon\bar{g}(X, \mathcal{U}), \quad du(X, Y) = \bar{g}(\phi X, Y), \quad \epsilon = \pm 1,$$

$$\tilde{\nabla}_X\mathcal{U} = -\epsilon\phi X, \quad (\tilde{\nabla}_X\phi)Y = \bar{g}(X, Y)\mathcal{U} - \epsilon u(Y)X,$$

for any $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. 
The following result is a very marvellous one in the common sense.

**Theorem 4.2.** Let \( M \) be a lightlike hypersurface of an indefinite Kenmotsu manifold \( \bar{M} \) with flat transversal connection. If \( S(TM) \) is totally geodesic in \( M \), then the leaf \( M^* \) of \( S(TM) \) is an indefinite Sasakian manifold with an almost contact metric structure \((F,V,v,g)\) such that \( \epsilon = \bar{g}(V,V) = -1 \).

**Proof.** Assume that \( S(TM) \) is totally geodesic in \( M \). Then we have \( \alpha \epsilon = 1 \) and \( \rho = 1 \). Take \( X, Y \in \Gamma(S(TM)) \), then (4.5) and (4.8) reduce to

\[
\nabla_X V = FX, \quad (\nabla_X F)Y = g(X,Y)V + v(Y)X,
\]

respectively. By direct calculations, we show that \((F,V,v,g)\) is an almost contact metric structure on \( M^* \). Thus \( M^* \) is an indefinite Sasakian manifold.

## 5 Lightlike hypersurfaces of a space form

**Definition 3.** An indefinite Kenmotsu manifold \( \bar{M} \) is called an indefinite Kenmotsu space form, denoted by \( \bar{M}(\bar{c}) \), if it has the constant \( J \)-sectional curvature \( \bar{c} \) [11]. The curvature tensor \( \bar{R} \) of this space form \( \bar{M}(\bar{c}) \) is given by

\[
4\bar{R}(X,Y)Z = (\bar{c} - 3)\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}
+ (\bar{c} + 1)\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta
+ \bar{g}(JY,Z)JX + \bar{g}(JZ,X)JY - 2\bar{g}(JX,Y)JZ\}, \quad \forall X, Y, Z \in \Gamma(TM).
\]

It is well known [11] that if an indefinite Kenmotsu manifold \( \bar{M} \) is a space form, then it is Einstein and \( \bar{c} = -1 \), i.e., \( \bar{R} \) is given by

\[
\bar{R}(X,Y)Z = \bar{g}(X,Z)Y - \bar{g}(Y,Z)X, \quad \forall X, Y, Z \in \Gamma(TM). \quad (5.1)
\]

**Theorem 5.1.** Let \( M \) be a lightlike hypersurface of an indefinite Kenmotsu space form \( \bar{M}(\bar{c}) \). Then neither \( M \) is proper screen homothetic nor \( S(TM) \) is totally geodesic in \( M \).

**Proof.** Assume that \( M \) is screen homothetic. Taking the scalar product with \( \xi \) to (2.13) and (5.1) and then, comparing the resulting two equations, we have

\[
(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) = 0. \quad (5.2)
\]

Comparing (3.5) with \( d\theta = 0 \) and (5.2) with \( Z = W \) and using (3.1) and the fact that \( f \neq 0 \), we have \( d\tau = 0 \). By Theorem 2.1, we show that the transversal connection is flat. Thus we can use (3.1)~(3.7) and (3.11)~(3.20) in Section 3 and (4.1)~(4.9) in Section 4.
**Case 1.** If $M$ is proper screen homothetic, then, replacing $Z$ by $V$ to (2.13) and using (5.1) and (5.2), for any $X, Y \in \Gamma(TM)$, we have

$$R(X,Y)V = (1 - \alpha^2 \varphi)\{v(Y)PX - v(X)PY\} + \{v(Y)\eta(X) - v(X)\eta(Y)\}\xi.$$

Comparing the $\text{Rad}(TM)$-components of this equation and (4.9), we obtain

$$(X\rho)v(Y) - (Y\rho)v(X) + (\rho^2 + \rho - 2)\{v(X)\theta(Y) - v(Y)\theta(X)\}$$

$$= 2\{\theta(X)v(Y) - \theta(Y)v(X)\}, \quad \forall X, Y \in \Gamma(TM).$$

Substituting $\rho = 2\alpha e - 1$ and $X[\rho] = 4\alpha e (\alpha e - 1)\theta(X)$ to this, we have

$$\alpha e\{\theta(X)v(Y) - \theta(Y)v(X)\} = 0, \quad \forall X, Y \in \Gamma(TM).$$

Replacing $Y$ by $\xi$ to this equation and using $v(\xi) = 0, \theta(\xi) = f$ and $2ef = 1$, we have $\alpha v(X) = 0$ for all $X \in \Gamma(TM)$. Taking $X = V$, we get $\alpha = 0$. It is a contradiction as $\alpha \neq 0$. Thus $M$ is not proper screen homothetic.

**Case 2.** If $S(TM)$ is totally geodesic in $M$, then we show that $\alpha e = \rho = 1$. As the transversal connection is flat, (4.9) is reduced to

$$R(X,Y)V = v(Y)PX - v(X)PY + 2\{\theta(Y)FX - \theta(X)FY\}.$$

Replacing $Z$ by $V$ to (2.13) and using (5.1) and (5.2), we have

$$R(X,Y)V = v(Y)PX - v(X)PY + \{v(Y)\eta(X) - v(X)\eta(Y)\}\xi,$$

for all $X, Y \in \Gamma(TM)$. Comparing the last two equations, we obtain

$$2\{\theta(Y)FX - \theta(X)FY\} = \{v(Y)\eta(X) - v(X)\eta(Y)\}\xi,$$

for all $X, Y \in \Gamma(TM)$. Taking the scalar product with $N$ to this, we have

$$v(Y)\eta(X) - v(X)\eta(Y) = 0, \quad \forall X, Y \in \Gamma(TM).$$

Taking $Y = \xi$ to this equation, we get $v(X) = 0$ for any $X \in \Gamma(TM)$. It is a contradiction as $v(V) = 1$. Thus $S(TM)$ is not totally geodesic in $M$.

**References**


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