Electromagnetic Wave Diffraction
by Bi-Periodic Thin Conductive Grating

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Abstract

It has been proved that the electromagnetic wave diffraction problem by bi-periodical grating of thin conductive strips can have only one solution. Therefore, only Floquet wave can appear by the diffraction of the plane wave by the grating. The diffraction problem is reduced to regular infinite set of linear algebraic equations by method of integral-summatorial identities.

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1 Introduction

In the modern theory of gratings the electromagnetic waves diffraction problems are formulated as boundary value problems for Maxwell equations (see, for example, [21], [22]). In these books the detailed overview of problem statements and methods of their solving is given.

In the works [8] and [19] the different approaches to the research of scattering wave problems by periodic structures are being discussed. Design problem of periodic structures with given properties were considered in the works of
Asfar, Nayfeh, Bao, Bonnetier, Dobson, Elschner, Schmidt, and several other authors [2], [3], [4], [5], [6], [7].

Simple grating consists of thin conductive screens placed in the same plane. In this case, the boundary diffraction problem is two-sided problem: we should find electromagnetic fields on different sides of a grating satisfying the boundary conditions on the screens and the conjugation conditions out of screens. If the grating is periodic, then the solution of the diffraction problem in each partial area can be found in the form of Floquet waves [9], i.e., as a quasi-periodic solution of Maxwell equations.

From the boundary conditions and conjugation conditions in the grating plane it is easy to pass to dual summatorial equations relative to Floquet coefficients of decomposition of the desired field by harmonics or to equivalent integral equations. It has been established that such problems are ill-posed. By their regularization different approaches can be used. One of the widely used methods is the Riemann-Hilbert problem method [20] (see also [21], Part 2).

In this paper the electromagnetic wave diffraction problem by bi-periodic simple grating is being considered. The over-determined boundary value problem method is used (MODP). The boundary value problem is over-determined problem if at the border of domain area more conditions are given than it is necessary to find a unique solution. Hence, boundary functions in the over-determined problem should satisfy some conditions of solvability. Solvability conditions are used by reducing two-sided boundary value problem to the one-sided problem, by proving the uniqueness of solution and by the regularizing of paired summatorial equations.

Solvability conditions for the over-determined problems for elliptic equations with partial derivatives have been obtained in [10], [11]. The method of the over-determined boundary value problem was used by investigating the electromagnetic waves diffraction problems by conductive thin screens [12], [15], [16] and by investigating elastic waves diffraction problems by heterogeneities of various types [13], [14]. In the work of Anufrieva and Tumakova [1] method of the over-determined boundary value problem was used to study problems on passage of elastic wave through gradient transversely isotropic layer. Boundary value problems, including the over-determined problems, for Maxwell equations in spherical coordinates are considered in the work [18]. Uniqueness of the solution of two-dimensional plane wave diffraction problem by a system of thin conductive screens has been proved in [17].

2 Statement of the diffraction problem

Let the 3D-space be filled with a linear homogeneous and isotropic environment and electromagnetic wave generated by infinitely distant source propagate in
the space. In the plane \( z = 0 \) two-periodic grating is placed consisting of ideally conductive infinitely thin plates (screens). We should find the perturbation of the electromagnetic field by the grating.

Let the electromagnetic field components harmonically depend on time. We will seek for \( z > 0 \) and for \( z < 0 \) solutions \( E^+, H^+ \) and \( E^-, H^- \) of Maxwell equations for complex amplitudes of the electric and magnetic vectors

\[
\text{rot} \ H = -i\omega\varepsilon_0 \varepsilon \ E, \quad \text{rot} \ E = i\omega\mu_0 \mu \ H, \tag{1}
\]

that satisfy the boundary conditions and the conjugation conditions when \( z = 0 \) together with the given complex amplitudes \( E^0, H^0 \) of waves from an external source, as well as conditions at infinity (radiation conditions).

We denote by \( M \) the part of plane \( z = 0 \) filled with screens and by \( N \) we denote the rest part of the plane. Boundary conditions and conjugation conditions for \( z = 0 \) (more precisely, for \( z \to 0^+ \) and \( z \to 0^- \)) are as follows:

\[
E^+_r + E^0_r = 0, \quad E^-_r + E^0_r = 0 \quad \text{on} \quad M, \tag{2}
\]

\[
E^+_r = E^-_r, \quad H^+_r = H^-_r \quad \text{on} \quad N. \tag{3}
\]

Here \( E^\pm_r = (E^\pm_x, E^\pm_y) \), \( H^\pm_r = (H^\pm_x, H^\pm_y) \) are tangential in relation to the plane \( z = 0 \) components of electric and magnetic vectors. Radiation conditions for each half-space we formulate as follows: the solutions of the set of equations (1) can contain only elementary harmonics carrying energy from the plane \( z = 0 \) to infinity or damped in this direction.

Let the periodic grating of screens have a period \( p \) in \( x \) axis direction and period \( q \) in \( y \) axis direction. Designations \( M \) and \( N \) will be used also for two parts of the rectangle of periods \([0, p] \times [0, q]\). Part \( M \) is filled by screens and part \( N \) supplements the first part to the whole rectangle.

We say that the function \( f(x, y, z) \) is quasi-periodic (by variables \( x \) and \( y \)), if

\[
f(x, y, z) = e^{i\alpha x + i\beta y} f_0(x, y, z), \tag{4}
\]

where \( \alpha \) and \( \beta \) are some numbers (Floquet parameters) and \( f_0(x, y, z) \) is periodic function of variables \( x \) and \( y \). Coefficients of decomposition of the function \( f_0(x, y, z) \) into Fourier series by exponents

\[
f_0(x, y, z) = \sum \sum f_{m,n}(z) e_{m,n}(x, y), \quad e_{m,n}(x, y) = e^{i\frac{2\pi}{p}m x + i\frac{2\pi}{q}n y} \tag{5}
\]

are usually called as Floquet coefficients. Quasi-periodic solutions of Maxwell equations are also called Floquet waves.

Let us prove that the solution of the diffraction problem of Floquet wave with parameters \( \alpha, \beta, p, q \) by the two-periodic grating can be Floquet wave only with the same parameters.
3 Auxiliary boundary value problems

Let us specify the structure of solutions of Maxwell equations (1) and the form of radiation conditions for the upper and lower half-spaces.

All unknown functions should be continuously differential when $z \neq 0$ and their limits should exist when $z \to 0 \pm 0$ almost everywhere on the plane $(x, y)$, except possibly some lines of zero measure (in our case, these lines are boundaries of domains $\mathcal{M}$ and $\mathcal{N}$).

To use the technique of integral Fourier transformation by the variables $x$ and $y$, we assume further that for each $z \neq 0$ values of solutions of a set of equations (1) are locally integrable functions of slow growth of variables $x$ and $y$. We will consider these functions as distributions and therefore, the limit transit on a plane $z = 0$ with $z \to 0 \pm 0$ will be understood in the sense of the theory of distributions. When we carry out a Fourier transformation by the variables $(x, y) \to (\xi, \eta)$ for Fourier transforms and for Fourier pre-images, we will use the same notations: for example, $f(\xi, \eta)$ is a Fourier transform of function (distribution) $f(x, y)$.

We apply the Fourier transformation by the variables $x$ and $y$ to the Maxwell equations (1). The equations for Fourier transforms

$$
-i \eta H_z - \frac{\partial H_y}{\partial z} = -i \omega \varepsilon_0 E_x, \quad -i \eta E_z - \frac{\partial E_y}{\partial z} = i \omega \mu_0 \mu H_x, \\
\frac{\partial H_x}{\partial z} + i \xi H_z = -i \omega \varepsilon_0 E_y, \quad \frac{\partial E_x}{\partial z} + i \xi E_z = i \omega \mu_0 \mu H_y, \\
-i \xi H_y + i \eta H_x = -i \omega \varepsilon_0 E_z, \quad -i \xi E_y + i \eta E_x = i \omega \mu_0 \mu H_z 
$$

(6)
do not contain derivatives of functions $E_z$ and $H_z$ (here only the derivatives by variable $z$ remain). We exclude these functions and get a pair of vector equations

$$
i \omega \varepsilon_0 \varepsilon E'_\tau = -H_\tau \mathbf{R}(\xi, \eta), \quad i \omega \mu_0 \mu H'_\tau = E_\tau \mathbf{R}(\xi, \eta). \quad (7)$$

Matrix

$$
\mathbf{R}(\xi, \eta) = \begin{pmatrix}
\xi \eta & \eta^2 - k^2 \\
k^2 - \xi^2 & -\xi \eta 
\end{pmatrix}, \quad k^2 = \omega^2 \mu_0 \mu \varepsilon_0 \varepsilon,
$$

has the following property: if you square it then you get the diagonal matrix with elements $-k^2(k^2 - \xi^2 - \eta^2)$ on the diagonal. Therefore the tangential components of electric and of magnetic vectors should satisfy the equations

$$
E''_\tau + (k^2 - \xi^2 - \eta^2) E_\tau = 0, \quad H''_\tau + (k^2 - \xi^2 - \eta^2) H_\tau = 0.
$$

Let, for example,

$$
E_\tau(\xi, \eta, z) = a(\xi, \eta) e^{i \gamma(\xi, \eta) z} + b(\xi, \eta) e^{-i \gamma(\xi, \eta) z},
$$
where
\[ \gamma(\xi, \eta) = \sqrt{k^2 - \xi^2 - \eta^2}, \quad \text{Re}\gamma(\xi, \eta) \geq 0 \quad \text{or} \quad \text{Im}\gamma(\xi, \eta) \geq 0, \]
and \(a(\xi, \eta), b(\xi, \eta)\) are arbitrary vector-functions. Then
\[ H_\tau(\xi, \eta, z) = \frac{1}{\omega \mu_0 \mu} \gamma(\xi, \eta) [ -a(\xi, \eta) e^{i\gamma(\xi, \eta)z} + b(\xi, \eta) e^{-i\gamma(\xi, \eta)z} ] R(\xi, \eta). \]

For \(b(\xi, \eta) = 0\) we have positively oriented solutions, and for \(a(\xi, \eta) = 0\) we have negatively oriented solutions. The direction in which energy is transferred through the plane \(z = 0\), is defined by the normal component of Poynting vector.

We will seek solutions of Maxwell equations (1), satisfying the conditions
\[ E_\tau(x, y, 0) = e(x, y), \quad H_\tau(x, y, 0) = h(x, y). \quad (8) \]

**Theorem 3.1** Vector-functions \(e(x, y), h(x, y)\) are traces of positively oriented solution of equations (1) if and only if their Fourier transforms satisfy the condition
\[ \omega \epsilon_0 \epsilon \gamma(\xi, \eta) e(\xi, \eta) = h(\xi, \eta) R(\xi, \eta). \quad (9) \]
Vector-functions \(e(x, y), h(x, y)\) are traces of negatively oriented solution of equations (1) if and only if their Fourier transforms satisfy the condition
\[ \omega \epsilon_0 \epsilon \gamma(\xi, \eta) e(\xi, \eta) = -h(\xi, \eta) R(\xi, \eta). \quad (10) \]

Proof. If the solution of equations (1) is positively oriented, then \(E'_\tau = i\gamma(\xi, \eta) E_\tau\). Therefore by the first equation (7)
\[ \omega \epsilon_0 \epsilon \gamma(\xi, \eta) E_\tau = -H_\tau R(\xi, \eta). \]

We pass to the limit when \(z \to 0\) and get equality (9). On the other hand, by the conditions (8) independently from each other complex amplitudes of the electric and magnetic vectors are found. If condition (9) is fulfilled, they will satisfy the equations (7).

Condition (10) is obtained similarly.

We note that the conditions (9) and (10) are necessary and sufficient conditions for the solvability of over-determined boundary value problem (1), (8) in a class of oriented solutions. If we seek non-oriented solutions, this problem is not over-determined problem.

Let us show now that the diffraction problem of the electromagnetic wave by a system of screens (not necessarily periodic), placed in the plane \(z = 0\) can be reduced to boundary value problems for Maxwell equations only in one of half-spaces.
Theorem 3.2 The diffraction problem (1)-(3) is equivalent to the following boundary value problems: either you should seek in the upper half-space positively oriented solution of the set of equations (1) which satisfies the boundary conditions

$$E^+_\tau(x, y, 0) = -E^0_\tau(x, y, 0) \quad \text{on} \quad \mathcal{M}, \quad H^+_\tau(x, y, 0) = 0 \quad \text{on} \quad \mathcal{N}, \quad (11)$$

or you should seek in the lower half-space negatively oriented solution of the set of equations (1) which satisfies the boundary conditions

$$E^-_\tau(x, y, 0) = -E^0_\tau(x, y, 0) \quad \text{on} \quad \mathcal{M}, \quad H^-_\tau(x, y, 0) = 0 \quad \text{on} \quad \mathcal{N}. \quad (12)$$

Proof. We will seek a solution of the diffraction problem as a pair of two solutions of boundary value problems: in the upper half-space the functions $e^+(x, y)$, $h^+(x, y)$ are used in the boundary conditions (8) and in the lower half-space the functions $e^-(x, y)$, $h^-(x, y)$ are used in these boundary conditions.

In the case when the environments in the upper and in the lower half-spaces are identical, it follows by conditions (2) and (3) that equality $e^+(x, y) = e^-(x, y)$ should be fulfilled both on $\mathcal{M}$ and on $\mathcal{N}$. Fourier transforms of these vector-functions are equal also: $e^+(\xi, \eta) = e^-(\xi, \eta)$. Then by equalities (9) and (10), we have $h^+(\xi, \eta) = -h^-(\xi, \eta)$ and $h^+(x, y) = -h^-(x, y)$. But $h^+(x, y) = h^-(x, y)$ on $\mathcal{N}$. Therefore, $h^+(x, y) = h^-(x, y) = 0$ on $\mathcal{N}$ and conditions $e^+(x, y) = -E^0_\tau(x, y, 0), \quad e^-(x, y) = -E^0_\tau(x, y, 0)$ remain on $\mathcal{M}$. •

We note that if a plane $z = 0$ is the media interface with different properties, then solution of the diffraction problem should be sought in the form of a sum consisting of the solution of the problem of reflection and refraction of wave from an external source on this boundary and of additional perturbation of electromagnetic field by the screens. In this case, the diffraction problem can be reduced to the unilateral boundary value problems also.

4 The uniqueness theorem and its corollaries

The statement on the uniqueness of the solution of the electromagnetic wave diffraction problem by the system of screens placed in the plane $z = 0$ is easy to get as a corollary of theorem on the uniqueness of the solutions of the mixed boundary value problem for Maxwell equations in the half-space.

Theorem 4.1 The set of equations (1) can have in the upper half-space only one oriented solution that satisfies the conditions

$$E^+(x, y, 0) = e(x, y) \quad \text{on} \quad \mathcal{M}, \quad H^+(x, y, 0) = h(x, y) \quad \text{on} \quad \mathcal{N}. \quad (13)$$
Proof. We suppose that boundary value problem has two solutions. Traces of the difference of these solutions on the plane \( z = 0 \) are such that

\[
e(x, y) = 0 \quad \text{on} \quad \mathcal{M} \quad \text{and} \quad h(x, y) = 0 \quad \text{on} \quad \mathcal{N}.
\]

We consider the expression

\[
S(x, y) = \left[ -\frac{\partial^2 h_x(x, y)}{\partial x \partial y} + (k^2 + \frac{\partial^2}{\partial x^2}) h_y(x, y) \right] e^*_x(x, y) \\
+ \left[ -\left( k^2 + \frac{\partial^2}{\partial y^2} \right) h_x(x, y) + \frac{\partial^2 h_y(x, y)}{\partial x \partial y} \right] e^*_y(x, y),
\]

here \( * \) is a sign of the complex conjugation. By constructing, \( S(x, y) = 0 \) both on \( \mathcal{M} \) and on \( \mathcal{N} \). Therefore

\[
\iint S(x, y) \, dx \, dy = 0.
\]

Let us pass in this equality to Fourier transforms of individual multipliers by Parseval’s formula. We get

\[
\iint \left\{ \left[ \xi \eta h_x(\xi, \eta) + (k^2 - \xi^2) h_y(\xi, \eta) \right] e^*_x(\xi, \eta) \\
+ \left[ (\eta^2 - k^2) h_x(\xi, \eta) - \xi \eta h_y(\xi, \eta) \right] e^*_y(\xi, \eta) \right\} d\xi \, d\eta = 0.
\]

It is convenient to write down the expression in the braces as scalar product of two vector functions:

\[
\iint (h(\xi, \eta) R(\xi, \eta), e(\xi, \eta)) \, dx \, dy = 0.
\]

But from the condition (9) (or from the condition (10)) it follows that

\[
\omega \varepsilon_0 \varepsilon \iint \gamma(\xi, \eta) (e(\xi, \eta), e(\xi, \eta)) \, dx \, dy = 0.
\]

As the real and imaginary parts of the function \( \gamma(\xi, \eta) \) do not change sign, \( e(\xi, \eta) \equiv 0 \). Then the difference of two solutions of the diffraction problem has zero traces on the whole plane \( x, y \), and this difference is identically zero in the half-space.

A similar statement takes place in the case of a boundary value problem for Maxwell equations in the lower half-space.

**Corollary 4.2** Diffraction problem (1)-(3) can have only one solution.

**Corollary 4.3** If \( e(x, y), h(x, y) \) are periodic functions, the mixed type boundary value problem (1), (13) can have only periodic solution by variables \( x, y \) with the same periods.
Proof. Let \( \mathbf{E}(x, y, z), \mathbf{H}(x, y, z) \) be a solution of the boundary value problem (1), (13). Let us consider vector-functions \( \tilde{\mathbf{E}}(x, y, z) = \mathbf{E}(x + mp, y + nq, z) \), \( \tilde{\mathbf{H}}(x, y, z) = \mathbf{H}(x + mp, y + nq, z) \), where \( m \) and \( n \) are integer numbers (it is sufficient to explore only two cases: \( m = 1, n = 0 \) and \( m = 0, n = 1 \)). It is easy to see that \( \tilde{\mathbf{E}}, \tilde{\mathbf{H}} \) is a solution of set of equations (1) that satisfies the boundary conditions (13). By uniqueness theorem \( \tilde{\mathbf{E}} = \mathbf{E}, \tilde{\mathbf{H}} = \mathbf{H} \). 

Let us prove now more general statement.

**Theorem 4.4** If \( e(x, y), h(x, y) \) are quasi-periodic functions, the oriented solution of the mixed type boundary value problem (1), (13) can be only quasi-periodic by variables \( x, y \) with the same parameters.

Proof. Let \( \mathbf{E}(x, y, z), \mathbf{H}(x, y, z) \) be a solution of the set of equations (1) satisfying the boundary conditions (13). Let \( e(x, y) = e^{i\alpha x + i\beta y}e_0(x, y), h(x, y) = e^{i\alpha x + i\beta y}h_0(x, y) \), where \( e_0(x, y), h_0(x, y) \) are periodic vector-functions. Then vector-functions

\[
\tilde{\mathbf{E}}(x, y, z) = e^{-i\alpha x - i\beta y}\mathbf{E}(x, y, z) \quad \tilde{\mathbf{H}}(x, y, z) = e^{-i\alpha x - i\beta y}\mathbf{H}(x, y, z)
\]

satisfy the boundary conditions of the form (13), but with periodic right-hand sides.

By this \( \tilde{\mathbf{E}} \) and \( \tilde{\mathbf{H}} \) are not the solution of the set of Maxwell equations (1). A new system of differential equations is easy to get. But you can reason as follows. If, for example, the Fourier pre-image is multiplied by \( e^{i\alpha x} \), then in the expression of Fourier transform the values of the variable \( \xi \) are shifted: instead of \( \xi \) you should substitute \( \xi + \alpha \). It follows that the solvability condition in the class of positively oriented solutions of the over-determined problem for a new set of equations will have the form

\[
\omega e_0 \gamma(\xi + \alpha, \eta + \beta) e^+\gamma(\xi, \eta) = h^+\gamma(\xi, \eta) R(\xi + \alpha, \eta + \beta).
\]

As real and imaginary parts of the function \( \gamma(\xi + \alpha, \eta + \beta) \) do not change sign, the mixed type boundary value problem for a new set of differential equations can have only one solution in this case. By this you can easily get also that if the given functions in boundary conditions are periodic, then the solution of the boundary value problem \( \tilde{\mathbf{E}}, \tilde{\mathbf{H}} \) is periodic also. 

5 Quasi-periodic solutions of Maxwell equations

We will look for solution of the diffraction problem (1)-(3) for \( z > 0 \) and for \( z < 0 \) in the form of oriented Floquet waves. Let each component of electric
and of magnetic vectors have the form

\[ A(x, y, z) = \sum_m \sum_n A_{m,n}(z) e^{ipmx+iqny}, \quad p_m = \frac{2\pi}{p} m + \alpha, \quad q_n = \frac{2\pi}{q} n + \beta. \]

If we pass to Fourier transforms, we will have decompositions of the desired distributions in the series for \( \delta \)-functions. But it is uncomfortable to use directly the tangential components of vectors \( \mathbf{E} \) and \( \mathbf{H} \) obtained above. The fact is that, for example, in the case of positively oriented solutions two types of waves for \( \mathbf{a}(\xi, \eta) = (a_x(\xi, \eta), 0) \) and \( \mathbf{a}(\xi, \eta) = (a_x(\xi, \eta), 0) \) are independent, but are not orthogonal. Then there is some complexity for calculating the energy flows through rectangle of periods: the energy flow of the sum of harmonics is not the sum of the individual energy flows. So it is reasonable to choose components \( E_y \) and \( H_y \) as basic components rather than \( E_x \) and \( E_y \).

Let

\[ E_y = \sum_m \sum_n (k^2 - q_n^2) [a_{m,n}^+ e^{i\gamma_{m,n}z} + a_{m,n}^- e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}, \]

\[ H_y = \sum_m \sum_n (k^2 - q_n^2) [b_{m,n}^+ e^{i\gamma_{m,n}z} + b_{m,n}^- e^{-i\gamma_{m,n}z}] e^{ipmx+iqny} \]

(multiplier \( k^2 - q_n^2 \) is used to simplify further formulas). Then

\[ E_x = \sum_m \sum_n ((-p_m q_n a_{m,n}^+ + \omega \mu_0 \mu_0 \gamma_{m,n} b_{m,n}^+) e^{i\gamma_{m,n}z} \]

\[ - (p_m q_n a_{m,n}^- + \omega \mu_0 \mu_0 \gamma_{m,n} b_{m,n}^-) e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}, \]

\[ H_x = \sum_m \sum_n ((-\omega \xi e^{i\gamma_{m,n}z} a_{m,n}^+ m, n - p_m q_n b_{m,n}^+) e^{i\gamma_{m,n}z} \]

\[ + (\omega \xi e^{i\gamma_{m,n}z} a_{m,n}^- m, n - p_m q_n b_{m,n}^-) e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}, \]

\[ E_z = \sum_m \sum_n ((-q_n \gamma_{m,n} a_{m,n}^+ - \omega \mu_0 \mu_0 \mu_0 b_{m,n}^+) e^{i\gamma_{m,n}z} \]

\[ + (q_n \gamma_{m,n} a_{m,n}^- - \omega \mu_0 \mu_0 \mu_0 b_{m,n}^-) e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}, \]

\[ H_z = \sum_m \sum_n ((\omega \xi e^{i\gamma_{m,n}z} p_m a_{m,n}^+ m, n - q_n \gamma_{m,n} b_{m,n}^+) e^{i\gamma_{m,n}z} \]

\[ + (\omega \xi e^{i\gamma_{m,n}z} p_m a_{m,n}^- m, n + q_n \gamma_{m,n} b_{m,n}^-) e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}. \]

In vector-matrix form we have

\[ \mathbf{E}(x, y, z) = \sum_m \sum_n [c_{m,n}^+ A_{m,n}^+ e^{i\gamma_{m,n}z} + c_{m,n}^- A_{m,n}^- e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}, \]

\[ \mathbf{H}(x, y, z) = \sum_m \sum_n [c_{m,n}^+ B_{m,n}^+ e^{i\gamma_{m,n}z} + c_{m,n}^- B_{m,n}^- e^{-i\gamma_{m,n}z}] e^{ipmx+iqny}, \quad (14) \]
where vectors-strings \( c_{m,n}^\pm = (a_{m,n}^\pm, b_{m,n}^\pm) \) and matrices

\[
A_{m,n}^\pm = \begin{pmatrix}
-p_m q_n & k^2 - q_n^2 & \mp q_n \gamma_{m,n} \\
\pm \omega \mu_0 \gamma_{m,n} & 0 & -\omega \mu_0 \mu p_m
\end{pmatrix},
\]

\[
B_{m,n}^\pm = \begin{pmatrix}
\mp \omega \varepsilon_0 \varepsilon \gamma_{m,n} & 0 & \omega \varepsilon_0 \varepsilon p_m \\
-p_m q_n & k^2 - q_n^2 & \mp q_n \gamma_{m,n}
\end{pmatrix}.
\]

In formulas (14) addends with multipliers \( c_{m,n}^+ \) define positively oriented waves and addends with multipliers \( c_{m,n}^- \) define negatively oriented waves.

In tangential components of vectors \( \mathbf{E} \) and \( \mathbf{H} \) you should leave in the matrices \( A_{m,n}^\pm \) and \( B_{m,n}^\pm \) only the first two columns, such truncated matrices we denote by \( (A_{m,n}^\pm)_{\tau} \) \( (B_{m,n}^\pm)_{\tau} \).

A particular case of Floquet wave is a plane wave. To such a wave corresponds one addend in the sums (14) with real number \( \gamma_{m,n} \), for example, if \( m = 0, n = 0 \).

### 6 Infinite set of linear algebraic equations

Let the complex amplitudes of vectors \( \mathbf{E} \) and \( \mathbf{H} \) waves from an external source be

\[
\mathbf{E}^0 = c^0 A^0 e(x,y) e^{-i\gamma z}, \quad \mathbf{H}^0 = c^0 B^0 e(x,y) e^{-i\gamma z},
\]

where

\[
A^0 = A_{0,0}^-, \quad B^0 = B_{0,0}^-, \quad e(x,y) = e^{i\alpha x + i\beta y} = e_{0,0}(x,y),
\]

\[
\gamma = \sqrt{k^2 - \alpha^2 - \beta^2} = \gamma_{0,0}.
\]

We will seek Floquet waves when \( z > 0 \) and when \( z < 0 \) as

\[
\mathbf{E}^+ = \sum_m \sum_n c_{m,n}^+ A_{m,n}^+ e_{m,n}(x,y) e^{i\gamma_{m,n} z},
\]

\[
\mathbf{H}^+ = \sum_m \sum n c_{m,n}^+ B_{m,n}^+ e_{m,n}(x,y) e^{i\gamma_{m,n} z},
\]

\[
\mathbf{E}^- = \sum_m \sum n c_{m,n}^- A_{m,n}^- e_{m,n}(x,y) e^{-i\gamma_{m,n} z},
\]

\[
\mathbf{H}^- = \sum_m \sum n c_{m,n}^- B_{m,n}^- e_{m,n}(x,y) e^{-i\gamma_{m,n} z}.
\]

As it follows from Theorem 2, to determine the coefficients \( c_{m,n}^\pm \) we have a dual summatorial functional equation (11), i.e.

\[
\sum_m \sum n c_{m,n}^+ (A_{m,n}^+)_{\tau} e_{m,n}(x,y) = -c^0 A^0 e(x,y) \quad \text{on} \quad \mathcal{M}, \quad (15)
\]
\[ \sum_m \sum_n c_{m,n}^+ (B_{m,n}^+)_{\tau} e_{m,n}(x, y) = 0 \quad \text{on} \quad \mathcal{N}. \]  

(16)

It is easy to project this dual equation on functions \( e_{j,k}(x, y) \) and to get an infinite set of linear algebraic equations for the desired coefficients \( c_{m,n}^+ \). But truncation method can’t be used to get its approximate solution as the sequence of solutions of truncated equations does not tend to any limit when we increase the number of dimensions. We should use one of the methods of regularization.

We use the method of integral-summatorial identities. It is easy to see that there is equality

\[
\sum_m \sum_n c_{m,n}^+ (A_{m,n}^+)_{\tau} e_{m,n}(x, y) = \int_0^p \int_0^q \left( \sum_m \sum_n c_{m,n}^+ (B_{m,n}^+)_{\tau} e_{m,n}(x', y') \right) \times \left( \frac{1}{pq} \sum_{m'} \sum_{n'} (B_{m',n'}^+)_{\tau}^{-1} (A_{m',n'}^+)_{\tau} e_{m',n'}(x - x', y - y') \right) dx' dy'.
\]

(17)

Note that equality is a necessary and sufficient condition for the solvability of over-determined boundary value problem for Maxwell equations in the upper half-space in the class of quasi-periodic functions.

Let us denote by

\[
I_{m,n} = \int_M e_{m,n}(x', y') \, dx' \, dy', \quad J_{m,n} = \int_{\mathcal{N}} e_{m,n}(x', y') \, dx' \, dy'.
\]

Integration in (17) is held only by \( M \) by virtue of equality (16). Then from the identity (17) it follows that on \( \mathcal{N} \)

\[
\sum_m \sum_n c_{m,n}^+ (A_{m,n}^+)_{\tau} e_{m,n}(x, y) = \frac{1}{pq} \sum_m \sum_n c_{m,n}^+ (B_{m,n}^+)_{\tau} \times \sum_{m'} \sum_{n'} (B_{m',n'}^+)_{\tau}^{-1} (A_{m',n'}^+)_{\tau} e_{m',n'}(x, y) I_{m-m',n-n'}. \]

(18)

We project new dual equation (15), (18) on functions \( e_{j,k}(x, y) \) and get an infinite set of linear algebraic equations

\[
pq c_{j,k}^+ (A_{j,k}^+)_{\tau} = -c_0^0 (A_0^0)_{\tau} I_{-j,-k}
\]

\[
+ \frac{1}{pq} \sum_m \sum_n c_{m,n}^+ (B_{m,n}^+)_{\tau} \sum_{m'} \sum_{n'} (B_{m',n'}^+)_{\tau}^{-1} (A_{m',n'}^+)_{\tau} I_{m-m',n-n'} J_{m'-j,n'-k},
\]

\[ j, k = 0, \pm 1, \ldots \]

(19)

As a result, we have the following statement.
Theorem 6.1 The electromagnetic wave diffraction problem by the bi-periodical grating is equivalent to an infinite set of linear algebraic equations (19), approximate solution of which can be found by the reduction method.

Note that integrals $J_{m,n}$ can be easily expressed through integrals $I_{m,n}$. If, for example, domain $\mathcal{M}$ is a rectangle or consists of multiple rectangles, the values of these integrals can be calculated explicitly.

7 Conclusion

Thus, by the method of the over-determined boundary value problem it is proved that the plane electromagnetic wave diffraction problem by bi-periodical plane grating of thin conductive strips can have only one solution in the form of Floquet wave. By integral-summatorial identity the initial diffraction problem is reduced to an infinite set of linear algebraic equations. In these equations the unknowns are vectors and coefficients in the unknowns are square matrices. The numerical algorithm of approximate solution allows the effective parallelization on the computation phase of calculating coefficients in linear equations.

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References


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