The $g$-Extra Conditional Diagnosability of Folded Hypercubes

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Abstract

Diagnosability plays a crucial role in measuring the reliability and fault tolerance of interconnection networks. The $g$-extra conditional diagnosability of multiprocessor systems is a new diagnosability, which is more accurate than the classical diagnosability. In this paper, we show that the $g$-extra conditional diagnosability $\tilde{t}_g(FQ_n)$ of the folded hypercube $FQ_n$ is $(g + 1)n - \left(\frac{g}{2}\right) + 1$ under the PMC model and the MM* model in some cases, which is several times larger than the classical diagnosability of folded hypercubes.

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1 Introduction

As an underlying topology of a multiprocessor system, an interconnection network is usually modeled by a connected graph $G$, whose vertices represent processors and edges represent communication links. The process of identifying faulty processors is called the diagnosis of the system. Diagnosability is
defined as the maximum number of faulty processors which the system can guarantee to identify, which is an important parameter to measure the reliability and fault-tolerance of a multiprocessor system. The hypercube structure [8] is a well-known interconnection network model for multiprocessor systems. As a variant of the hypercube, the folded hypercube [3] can be constructed from an \(n\)-dimensional hypercube by adding \(2^n-1\) edges, called complementary edges. The \(n\)-dimensional folded hypercube \(FQ_n\) has recently received considerable attention [3], [4], [10], [12], [13]. The PMC model [7] and the MM model [6] are two widely adopted as the fault diagnosis model. In 2015, Zhang et al. [11] defined \(g\)-extra conditional diagnosability under assumption that every component of \(G - F\) has at least \(g + 1\) vertices, where \(F\) is a faulty subset of \(G\). Moreover, they also obtained \(g\)-extra conditional diagnosability results for hypercubes under the PMC model and the MM* model. In this paper, we show that the \(g\)-extra conditional diagnosability \(\tilde{t}_g(FQ_n)\) of the folded hypercube \(FQ_n\) is \((g + 1)n - \left(\binom{g}{2}\right) + 1\) under the PMC model and the MM* model in some cases, which is several times larger than the classical diagnosability of folded hypercubes. For graph-theoretical terminology and notation not defined here we follow [1].

**Definition 1.1 ([11])** A system \(G\) is \(g\)-extra conditionally \(t\)-diagnosable if and only if for each pair of distinct faulty \(g\)-extra vertex subsets \(F_1, F_2 \subseteq V(G)\) such that \(|F_1| \leq t, |F_2| \leq t, F_1\) and \(F_2\) are distinguishable. The \(g\)-extra conditional diagnosability of \(G\), denoted as \(\tilde{t}_g(G)\), is the maximum value of \(t\) such that \(G\) is \(g\)-extra conditionally \(t\)-diagnosable.

## 2 Results

**Lemma 2.1 ([10])** Let \(A\) be a subgraph of \(FQ_n\) with \(|V(A)| = g + 1 \leq n-1\) for \(n \geq 4\) and \(0 \leq g \leq n-4\). Then \(|N_{FQ_n}(V(A))| \geq (g+1)(n+1) - 2g - \left(\binom{g}{2}\right)\).

**Lemma 2.2 ([10])** Let \(FQ_n = D_0 \bigotimes D_1\), where \(n \geq 8\) and \(0 \leq g \leq n-4\), and let \(A \subseteq D_1\) and \(A \cong K_1.g\). Then the following two statements hold: (i) \(|N_{FQ_n}(V(A))| = (g+1)(n+1) - 2g - \left(\binom{g}{2}\right)\), (ii) \(FQ_n - C_{FQ_n}(V(A))\) is a connected subgraph of \(FQ_n\) with at least \(g + 1\) vertices.

**Lemma 2.3 ([10])** \(\kappa_g(FQ_n) = (g + 1)(n + 1) - 2g - \left(\binom{g}{2}\right),\) for \(n \geq 8\) and \(0 \leq g \leq n-4\).

**Lemma 2.4 ([12])** Any two vertices in \(V(FQ_n)\) exactly have two common neighbors for \(n \geq 4\) if they have.

**Lemma 2.5 ([2])** For a system \(G\) and any two distinct subsets \(F_1\) and \(F_2\) of \(V, F_1\) and \(F_2\) are distinguishable under the PMC model if and only if there exist two vertices \(u \in V\setminus(F_1 \cup F_2)\) and \(v \in F_1 \Delta F_2\) such that \((u,v) \in E(G)\).
Lemma 2.6 ([9]) For a system $G$ and any two distinct subsets $F_1$ and $F_2$ of $V$, $F_1$ and $F_2$ are distinguishable under the MM* model if and only if at least one of the following conditions is satisfied:

(i). There are three vertices $u, v, w \in V(G) \setminus (F_1 \cup F_2)$ and $v \in F_1 \setminus F_2$ such that $uw \in E(G)$, $vw \in E(G)$.

(ii). There are three vertices $u, v \in F_1 \setminus F_2$ and $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw \in E(G)$, $vw \in E(G)$.

(iii). There are three vertices $u, v \in F_2 \setminus F_1$ and $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw \in E(G)$, $vw \in E(G)$.

Lemma 2.7 Assume that $n \geq 8$ and $0 \leq g \leq n - 4$. Then $\tilde{\Phi}_g(FQ_n) \leq (g + 1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$ under the PMC model and the MM* model.

Proof. Suppose that $A$ is a connected subgraph of $FQ_n$, $A \subseteq D_1$ and $A \cong K_{1,g}$. Let $F_1 = N_{FQ_n}(V(A))$ and $F_2 = C_{FQ_n}(V(A))$. Then, by Lemma 2.2, $|F_1| = (g+1)(n+1) - 2g - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right)$, $|F_2| = (g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 2$, and $FQ_n - C_{FQ_n}(V(A))$ is a connected subgraph of $FQ_n$ with at least $g + 1$ vertices. Thus, $|F_1| \leq (g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 2$, $|F_2| \leq (g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 2$, and both $F_1$ and $F_2$ are $g$-extra vertex subsets. Since there is no edge between $F_1 \triangle F_2$ and $V(FQ_n) \setminus (F_1 \cup F_2)$, by Lemmas 2.5 and 2.6, $F_1$ and $F_2$ are indistinguishable under the PMC model and the MM* model. By Definition 1.1, $\tilde{\Phi}_g(FQ_n) \leq (g + 1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$ under the PMC model and the MM* model. This completes the proof.

Lemma 2.8 Assume that $n \geq 8$ and $0 \leq g \leq n - 4$. Then $\tilde{\Phi}_g(FQ_n) \geq (g + 1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$ under the PMC model.

Proof. By contradiction. Suppose that $\tilde{\Phi}_g(FQ_n) \leq (g + 1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$. Let $F_1$ and $F_2$ be two distinct $g$-extra faulty vertex subsets of $FQ_n$, such that $|F_1| \leq (g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$, $|F_2| \leq (g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$, and $F_1$ and $F_2$ are indistinguishable under the PMC model. Let $\Phi_n(g) = (g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1$, where $n \geq 8$ and $0 \leq g \leq n - 4$. Then $\Phi_n(g)$ is strictly monotonic increasing for $0 \leq g \leq n - 4$. Therefore, $|V(FQ_n)| - |F_1 \cup F_2| \geq |V(FQ_n)| - |F_1| - |F_2| + |F_1 \cap F_2| \geq 2^n - 2[(g+1)n - \left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right) + 1] \geq 2^n - 2[(n-4+1)n - \left(\begin{smallmatrix} n-4 \\ 2 \end{smallmatrix}\right) + 1] = 2^n - (n^2 + 3n - 18)$. Whereby $n \geq 8$, then $|V(FQ_n)| - |F_1 \cup F_2| > 0$. Therefore, $V(FQ_n) \setminus (F_1 \cup F_2) \neq \emptyset$. Since $F_1$ and $F_2$ are indistinguishable under the PMC model, and by Lemma 2.5, there is no edge between $F_1 \triangle F_2$ and $V(FQ_n) \setminus (F_1 \cup F_2)$. So all neighbors of vertices in $F_1 \triangle F_2$ and $V(FQ_n) \setminus (F_1 \cup F_2)$ are located in $F_1 \cap F_2$. Thus, the neighbors of all these vertices in the components of $FQ_n[F_1 \triangle F_2]$ and $FQ_n[V(FQ_n) \setminus (F_1 \cup F_2)]$ are located in $F_1 \cap F_2$. By the definitions of $F_1$ and $F_2$, we conclude that every component of $FQ_n - F_1$ and $FQ_n - F_2$ has at least $g + 1$ vertices. If $F_1 \cap F_2$ is deleted, then $FQ_n - F_1 \cap F_2$ will be disconnected and all the components of $FQ_n - F_1 \cap F_2$ have at least $g + 1$ vertices. Thus, $F_1 \cap F_2$ is a $g$-extra vertex cut of $FQ_n$. By Lemma 2.3, we conclude that
$|F_1 \cap F_2| \geq (g + 1)(n - 1) - 2g - \binom{n}{2}$. Noting that $F_1 \neq F_2$, without loss of generality, we assume that $F_2 \setminus F_1 \neq \emptyset$. Observing that $F_1$ is a $g$-extra vertex cut, thus every component of $FQ_n - F_1$ has at least $g + 1$ vertices. Since $F_2 \setminus F_1$ is not connected with $V(FQ_n) \setminus (F_1 \cup F_2)$, every component of $FQ_n[F_2 \setminus F_1]$ has at least $g + 1$ vertices, i.e., $|F_2 \setminus F_1| \geq g + 1$. Hence $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq g + 1 + (g + 1)(n - 1) - 2g - \binom{n}{2} = (g + 1)n - \binom{n}{2} + 2$, which contradicts to $|F_2| \leq (g + 1)n - \binom{n}{2} + 1$. This completes the proof. 

Lemma 2.9 Let $n \geq 32$, $3 \leq g \leq \frac{n}{4}$ and $F$ be a $g$-extra faulty vertex subset of $FQ_n$. If $FQ_n - F$ has at most $\binom{n}{2}$ vertices of degree 1, then $\tilde{t}_g(FQ_n) \geq (g + 1)n - \binom{n}{2} + 1$ under the MM* model.

Proof. By contradiction. Suppose that the $g$-extra conditional diagnosability of $FQ_n$, $\tilde{t}_g(FQ_n) \leq (g + 1)n - \binom{n}{2}$ under the MM* model. Assume that there exist two distinct $g$-extra faulty vertex subsets $F_1$ and $F_2$, such that $|F_1| \leq (g + 1)n - \binom{n}{2} + 1$, $|F_2| \leq (g + 1)n - \binom{n}{2} + 1$, every component of $FQ_n - F_1$ and $FQ_n - F_2$ has at most $\frac{n}{2}$ vertices of degree 1, and $F_1$ and $F_2$ are indistinguishable under the MM* model. Whereby $n \geq 32$, then $g \leq \frac{n}{4} < n - 4$. Since $\Phi_n(g) = (g + 1)n - \binom{n}{2} + 1$ is strictly monotonic increasing for $0 \leq g \leq n - 4$, $|V(FQ_n)| - |F_1 \cup F_2| = |V(FQ_n)| - |F_1| - |F_2| + |F_1 \cap F_2| \geq 2^n - 2[(g + 1)n - \binom{n}{2} + 1] > 2^n - 2[(n - 4 + 1)n - \binom{n - 4}{2} + 1] > 0$. So $V(FQ_n)(F_1 \cup F_2) \neq \emptyset$.

Claim 1. There is no isolated vertex in $FQ_n - (F_1 \cup F_2)$. Assume that $S$ is a set of all the isolated vertices in $FQ_n - (F_1 \cup F_2)$. For any vertex $w \in S$, $N_{FQ_n}(w) \subseteq F_1 \cup F_2$. We prove that $F_1 \setminus F_2 \neq \emptyset$ and $F_2 \setminus F_1 \neq \emptyset$. If $F_2 \setminus F_1 = \emptyset$, then $N_{FQ_n}(w) \subseteq F_1$. Then $w$ is an isolated vertex in $FQ_n - F_1$ and so $g = 0$, a contradiction. Similarly, $F_1 \setminus F_2 \neq \emptyset$. Assume that there exist two distinct vertices $u_1, v_1 \in F_2 \setminus F_1$ such that $u_1w, v_1w \in E(FQ_n)$, by Lemma 2.6 (iii), $F_1$ and $F_2$ are distinguishable, a contradiction. If there exists no vertex $u_1 \in F_2 \setminus F_1$ such that $u_1w \in E(FQ_n)$, then $N_{FQ_n}(w) \subseteq F_1$ and so $w$ is an isolated vertex of $FQ_n - F_1$. Thus $g = 0$, a contradiction. Therefore, $|N_{FQ_n}(w) \cap (F_2 \setminus F_1)| = 1$. Similarly, $|N_{FQ_n}(w) \cap (F_1 \setminus F_2)| = 1$. Since $FQ_n$ is $(n + 1)$ regular, $|N_{FQ_n}(w) \cap (F_1 \cap F_2)| = |N_{FQ_n}(w)| - |N_{FQ_n}(w) \cap (F_1 \cap F_2)| - |N_{FQ_n}(w) \cap (F_1 \setminus F_2)| = n + 1 - 1 - 1 = n - 1$. Thus $F_1 \cap F_2 \neq \emptyset$. If $FQ_n - F_1$ has at most $\frac{n}{2}$ vertices of degree 1, then $|S| \leq \frac{n}{2}$. Let $|F_1 \setminus F_2| = p$, $|F_2 \setminus F_1| = q$ and $V' = (F_1 \Delta F_2) \subseteq S$. Then $|V'| = p + q + |S|$ and $p, q \geq 1$. We prove that $N_{FQ_n}(V') \subseteq F_1 \cap F_2$. If $F_1 \cup F_2 \cup S = V(FQ_n)$, then $N_{FQ_n}(V') \subseteq F_1 \cap F_2$. If $F_1 \cup F_2 \cup S \subset V(FQ_n)$, then let $R = V(FQ_n) \setminus (F_1 \cup F_2 \cup S)$, then $R \neq \emptyset$. Observing that all the isolated vertices of $FQ_n - F_1 \cup F_2$ are included in $S$, so there is no isolated vertex in $FQ_n[R]$. We claim that there is no edge between $F_1 \Delta F_2$ and $R$. Otherwise, noting that $FQ_n[R]$ has no isolated vertex, thus there exists an edge $uv$ in $FQ_n[R]$ such that $u(v)$ is adjacent to some vertex in $F_1 \Delta F_2$. By Lemma 2.6 (i), $F_1$ and $F_2$ are distinguishable, a contradiction. Since $S$ is a set of all the isolated vertices in $FQ_n - F_1 \cup F_2$, we conclude that
there is no edge between $S$ and $R$. Furthermore, there is no edge between $V'$ and $R$. Therefore, $N_{FQ_n}(V') \subseteq F_1 \cap F_2$

Case 1. $|S| > 2g$.

Subcase 1.1. Either $p \geq g + 1$ or $q \geq g + 1$.

Without loss of generality, suppose $p \geq g + 1$. For an arbitrary vertex $w \in S$, $|N_{FQ_n}(w) \cap (F_1 \cap F_2)| = |N_{FQ_n}(w)\setminus |N_{FQ_n}(w) \cap (F_1 \setminus F_2)| = (n+1)-1 = n-1$. By Lemma 2.4, $|F_1 \cap F_2| \geq (n-1)|S| - 2|\{S\}|^{2/3}$

Let $|S| = t$, then $h_n(|S|) = h_n(t) = (n-1)t - 2(t^3) = nt$ and $|S| = 2n-1$ is strictly monotonic increasing for $2g+1 < |S| < \frac{n}{2}$. Thus, $|F_1 \cap F_2| \geq h_n(2g+1)$. Since $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| = q + |F_1 \cap F_2|$ and $|F_2| \leq (g+1)n - \left(\frac{g}{2}\right) + 1 - q \leq (g+1)n - \left(\frac{g}{2}\right) + 1 - (g+1) = (g+1)n - \left(\frac{g}{2}\right) - g$. Therefore, $h_n(2g+1) \leq |F_1 \cap F_2| \leq (g+1)n - \left(\frac{g}{2}\right) - g$. If $n \geq 32$ and $3 \leq g \leq \frac{n}{4}$, then $\frac{2n-8}{2} \geq \frac{n}{4} \geq g$. Thus, $h_n(2g+1) - [(g+1)n - \left(\frac{g}{2}\right) - g] = (n-1)(2g+1) - 2\left(\frac{2g+1}{2}\right) - [(g+1)n - \left(\frac{g}{2}\right) - g] = g\left(\frac{2n-7-7g}{2}\right) - 1 \geq 3 \times \frac{1}{2} - 1 > 0$, a contradiction.

Subcase 1.2. $p \leq g$ and $q \leq g$.

Since $|V'| = |(F_1 \Delta F_2) \cup S| = |F_1 \setminus F_2| + |F_2 \setminus F_1| + |S| = p + q + |S|$ and $|S| \leq \frac{n}{2}$, $|V'| - 1 = p + q + |S| - 1 \leq g + g + \frac{n}{2} - 1 = 2g + \frac{(n-2)}{2}$. Whereby $3 \leq g \leq \frac{n}{4}$, i.e., $4g - n \leq 0$, then $(2g + \frac{(n-2)}{2}) - (n-1) = \frac{1}{2}(4g - n) \leq 0$. So $|V'| - 1 \leq 2g + \frac{(n-2)}{2} \leq n - 1$. Since $p \geq 1$, $q \geq 1$ and $|S| > 2g$, $|V'| - 1 = p + q + |S| - 1 > p + q + 2g-1 \geq 2g+1 > 2g$. Then $2g < |V'| - 1 \leq n - 1$. Let $f_n(g) = (g+1)(n+1) - 2g - \left(\frac{g}{2}\right)$, then $f_n(g)$ is strictly monotonic increasing when $0 \leq g \leq n-1$. Since $N_{FQ_n}(V') \subseteq F_1 \cap F_2$, by Lemma 2.1, $|F_1 \cap F_2| \geq |N_{FQ_n}(V')| \geq (|V'| - 1 + 1)(n+1) - 2(|V'| - 1) \left(\frac{1}{2}\right) > (2g+1)(n+1) - 4g - \left(\frac{2g}{2}\right)$. Since $n \geq 32$ and $3 \leq g \leq \frac{n}{4} < \left(\frac{2n-3}{3}\right)$, $|F_1 \cap F_2| - [(g+1)n - \left(\frac{g}{2}\right) + 1] > [(2g+1)(n+1) - 4g - \left(\frac{2g}{2}\right)] - [(g+1)n - \left(\frac{g}{2}\right) + 1] = g\left(\frac{2n-7-7g}{2}\right) > 0$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| = q + |F_1 \cap F_2| > 1 + (g+1)n - \left(\frac{g}{2}\right) + 1 > (g+1)n - \left(\frac{g}{2}\right) + 1$, a contradiction.

Case 2. $|S| \leq 2g$.

If $n \geq 32$, then $3 \leq g \leq \frac{n}{4} < n - 4$. Let $I_n(g) = (g+1)n + g - \left(\frac{g}{2}\right) + 1$, then $I_n(g)$ is strictly monotonic increasing for $3 \leq g < n - 4$. Therefore, $|V(FQ_n)| - |F_1 \cup F_2| - |S| \geq 2^n - 2[(g+1)n - \left(\frac{g}{2}\right) + 1] - 2g > 2^n - 2(n-4+1)n + n-4 - \left(\frac{n-4}{2}\right) + 1 = 2^n - (n^2 + 5n - 26) > 0$. So $R = V(FQ_n) \setminus (F_1 \cup F_2 \cup S) \neq \emptyset$.

Since all the isolated vertices are in $S$, there is no isolated vertex in $FQ_n[R]$. Then there exists one edge $vw \in E(FQ_n[R])$. If $u(v)$ is adjacent to some vertex in $F_1 \Delta F_2$, then by Lemma 2.6 (i), $F_1$ and $F_2$ are distinguishable, a contradiction. If there is a vertex $u(v) \in R$ which is adjacent to some vertex $w$ in $S$, and let $u_0 \in N_{F_1 \cup F_2}(w)$. By Lemma 2.6 (i), $F_1$ and $F_2$ are distinguishable, a contradiction. Thus there exists no edge between $V'$ and $R$. Moreover, there exists no edge between $(F_1 \setminus F_2) \cup S$ and $R$, and there exists no edge between $(F_2 \setminus F_1) \cup S$ and $R$. Therefore, $N_{FQ_n}(R) \subseteq F_1 \cap F_2$. Since every component
of $FQ_n - F_1$ and $FQ_n - F_2$ has at least $g + 1$ vertices, we conclude that every component of $FQ_n[R]$, $FQ_n[(F_1 \setminus F_2) \cup S]$ and $FQ_n[(F_2 \setminus F_1) \cup S]$ has at least $g + 1$ vertices. We claim that every component of $FQ_n[V']$ has at least $g + 1$ vertices. If every component of $FQ_n[F_2 \setminus F_1]$ has at least $g + 1$ vertices, then the claim holds because every component of $FQ_n[(F_1 \setminus F_2) \cup S]$ has at least $g + 1$ vertices. Suppose that there is a component $C_0$ of $FQ_n[F_2 \setminus F_1]$ with less than $g + 1$ vertices. If there exists no edge between $V(C_0)$ and $(F_1 \setminus F_2) \cup S$, then $C_0$ is also a component of $FQ_n[(F_2 \setminus F_1) \cup S]$, a contradiction. Thus there exists at least one edge between $V(C_0)$ and $(F_1 \setminus F_2) \cup S$. Without loss of generality, we assume that $uv$ is an edge between $V(C_0)$ and $(F_1 \setminus F_2) \cup S$, such that $u \in V(C_0)$ and $v \in V(C_1)$, where $C_1$ is a component of $FQ_n[(F_1 \setminus F_2) \cup S]$. Since every component of $FQ_n[(F_1 \setminus F_2) \cup S]$ has at least $g + 1$ vertices, there are at least $g + 1$ vertices in $C_1$. Assume that $C_2$ is an arbitrary component of $FQ_n[V']$, such that $FQ_n[V(C_0) \cup V(C_1)]$ is a connected subgraph of $C_2$, then there are at least $g + 1$ vertices in $C_2$. Therefore, every component of $FQ_n[V']$ has at least $g + 1$ vertices.

Observing that every component of $FQ_n[V']$ and $FQ_n[R]$ has at least $g + 1$ vertices, and there is no edge between $V'$ and $R$, thus $F_1 \cap F_2$ is a $g$-extra vertex cut of $FQ_n$. By Lemma 2.3, we have $|F_1 \cap F_2| \geq (g + 1)(n + 1) - 2g - \binom{g}{2}$. Assume $q = |F_2 \setminus F_1| \geq g + 1$, then $|F_2 = |F_2 \setminus F_1| + |F_1 \cap F_2| = q + |F_1 \cap F_2| \geq g + 1 + (g + 1)(n + 1) - 2g - \binom{g}{2} = (g + 1)n - \binom{g}{2} + 2 > (g + 1)n - \binom{g}{2} + 1$, a contradiction. So $q \leq g$. Similarly, $p \leq g$. Since there exists no edge between $(F_2 \setminus F_1) \cup S$ and $R$, and every component of $FQ_n - F_1$ has at least $g + 1$ vertices, every component of $FQ_n[(F_2 \setminus F_1) \cup S]$ has at least $g + 1$ vertices. Thus, $g + 1 \leq |V(F_2 \setminus F_1) \cup S| = |F_2 \setminus F_1| + |S| = q + |S|$. Similarly, $g + 1 \leq p + |S|$. Let $r = \min\{p, q\}$, then $g + 1 \leq \min\{p + |S|, q + |S|\}$, which implies that $r + |S| \leq g$. Since $F_2 \cap F_2 \geq N_{FQ_2}(V') \subseteq F_1 \cap F_2$ and $|V'| = p + q + |S|$, by Lemma 2.1, $|F_1 \cap F_2| \geq |N_{FQ_2}(V')| \geq (p + q + |S| - 1 + 1)(n + 1) - 2(p + q + |S| - 1) - (p + q + |S| - 1) - (q + |S| - 1)$.

Noting that $g \leq \frac{n}{2}$ and $|S| \leq 2g$, thus $g + r = r + r + (g - r) \leq p + q + g - r \leq p + q + |S| - 1 \leq 2g + |S| - 1 \leq 4g - 1 \leq n - 1$. Since $f_n(g) = (g + 1)(n + 1) - 2g - \binom{g}{2}$ is strictly monotonic increasing for $0 \leq g \leq n - 1$, $|F_1 \cap F_2| \geq (p + q + |S| - 1)(n + 1) - 2(p + q + |S| - 1) - (p + q + |S| - 1)(n + 1) - 2g + r$, it follows that $r > 2g$. Noting that $p \leq g$, $q \leq g$ and $r = \min\{p, q\}$, so $r \leq g$. Since $gr + 2g \geq r^2 + 2g \geq 6gr + r$, $2g \geq 5gr + r = (5g + 1)r$, $\frac{2}{g} \geq \frac{5gr + r}{g} = 5 + \frac{1}{g}$. Whereby $r \geq 1$, then $\frac{2}{g} \leq 2 < 5 + \frac{1}{g}$, a contradiction. Therefore $S = \emptyset$. This completes the proof of Claim 1.

By Claim 1, there is no isolated vertex in $FQ_n[V(FQ_n) \setminus (F_1 \cup F_2)]$. That is to say, for any vertex $u \in V(FQ_n)(F_1 \cup F_2)$, there exists one vertex
$v_1 \in V(FQ_n) \setminus (F_1 \cup F_2)$ which is adjacent to $u_1$. Assume that there exists one vertex $w \in F_1 \Delta F_2$ that is adjacent to $u_1$. By Lemma 2.6 (i), $F_1$ and $F_2$ are distinguishable, a contradiction. So there exists no edge between $F_1 \Delta F_2$ and $V(FQ_n) \setminus (F_1 \cup F_2)$. Since both $F_1$ and $F_2$ are $g$-extra vertex subsets, every component of $FQ_n[F_1 \setminus F_2]$, $FQ_n[F_2 \setminus F_1]$ and $FQ_n[V(FQ_n) \setminus (F_1 \cup F_2)]$ has at least $g + 1$ vertices. Hence, all the neighbors of the vertices in $F_1 \Delta F_2$ and $V(FQ_n) \setminus (F_1 \cup F_2)$ are located in $F_1 \cap F_2$. If we delete $F_1 \cap F_2$, then $FQ_n - F_1 \cap F_2$ will be disconnected and all components of $FQ_n - F_1 \cap F_2$ have at least $g + 1$ vertices. Therefore, $F_1 \cap F_2$ is a $g$-extra vertex cut of $FQ_n$. By Lemma 2.3, $|F_1 \cap F_2| \geq (g + 1)(n + 1) - 2g - \binom{g}{2}$. Since every component of $FQ_n[F_2 \setminus F_1]$ has at least $g + 1$ vertices, $|F_2 \setminus F_1| \geq g + 1$. Hence, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq g + 1 + (g + 1)(n + 1) - 2g - \binom{g}{2} = (g + 1)n - \binom{g}{2} + 2$, a contradiction. The proof is completed.

By Lemmas 2.7, 2.8, 2.9, we have the $g$-extra conditional diagnosability of $FQ_n$ as follows:

**Theorem 2.10** Assume that $n \geq 8$ and $0 \leq g \leq n - 4$. Then $\tilde{t}_g(FQ_n) = (g + 1)n - \binom{g}{2} + 1$ under the PMC model.

**Theorem 2.11** Assume that $n \geq 32$ and $3 \leq g \leq \frac{n}{4}$. If $FQ_n - F$ has at most $\frac{g}{2}$ vertices of degree 1, where $F$ is a $g$-extra vertex subset of $FQ_n$, then $\tilde{t}_g(FQ_n) = (g + 1)n - \binom{g}{2} + 1$ under the MM$^*$ model.

### 3 Conclusion

In this study, we show that the $g$-extra conditional diagnosability $\tilde{t}_g(FQ_n)$ of the folded hypercube $FQ_n$ is $(g + 1)n - \binom{g}{2} + 1$ under the PMC model and the MM$^*$ model in some cases. $(g + 1)n - \binom{g}{2} + 1$ is several times larger than the conditional diagnosability of the folded hypercube given by Zhu et al. [13] under the PMC model and the conditional diagnosability of the folded hypercube given by Hsieh et al. [4] under the MM$^*$ model.

### References


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