A Solution of the Two-dimensional Boltzmann Transport Equation

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Abstract

The present paper deal with solving the two-dimensional Boltzmann equation using a coordinate transformation. We find the analytical expression for the temporal evolution of the electron distribution function in the first Brilloune zone using the tight binding approximation. Also, we analyze several limits where the statistical equilibrium distribution is the limit behaviour.

Keywords: 2d Boltzmann-Equation, coordinate transformation

1 Introduction

The semiclassical transport is given by the Boltzmann transport equation (BTE):

\[
\frac{\partial f}{\partial t} + (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{k}} + \vec{v}(\vec{k}) \cdot \frac{\partial f}{\partial \vec{r}} = \sum_{\vec{k}'} [S(\vec{k}', \vec{k}) f(\vec{r}, \vec{k}', t)] \times [1 - f(\vec{r}, \vec{k}, t)] - [S(\vec{k}, \vec{k}') f(\vec{r}, \vec{k}, t)] [1 - f(\vec{r}, \vec{k}', t)]
\]

(1)

Basically, this equation accounts for the spatiotemporal evolution of a distribution function. The left hand side, is the temporal evolution in phase space
of an electron. The position is given by $\vec{r}$, momentum by $\vec{k}$, $\vec{v}$ is the velocity group, $t$ is time and $f(\vec{r}, \vec{k}, t)$ is the Fermi-Dirac distribution function. $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. The right-hand side is the collision term, and the summation accounts for the scattering process. The term $S(\vec{k}, \vec{k}')$ is the transition probability between the momentum states $\vec{k}$ and $\vec{k}'$, and $[1 - f(\vec{r}, \vec{k}', t)]$ is the probability of non-occupation for a momentum state $\vec{k}'$.

In general, the question of a general solution for the full Boltzmann transport equation is of great importance in Physics and Mathematics. Up to now, there have been important studies that address the issue; e.g. some authors study the existence and uniqueness of global solutions of the Boltzmann equation close to a Maxwellian statistical dynamics [6]. Also, there is a collection of particular solutions which correspond to specific conditions of the physical systems. For instance, an exact solution for the relativistic BTE with some relativistic invariant properties has been applied to a Gubser flow. This innovative analytical solution could be applied to fine tuning approximative models, where there are no analytical treatments, e.g., heavy-ion relativistic collisions. Additionally, exact solutions using the Bobylev approach of a two-particle Boltzmann equation are given in reference [1]. Furthermore, over the last two decades, we have witnessed the emergence of a plethora of analytical methods applied to nonlinear partial differential equations, that have appeared from fields such as the physics of complex phenomena, mechanics, chemistry, engineering, and biology. Most of those solutions can be grouped into what are called solitary wave solutions, e.g. the F-expansion method, Exp-function method, Painlevé method, tanh method, Riccati equation mapping method, tanh-coth method, sine-cosine method, G'/G-expansion method. In general, in all those methods, one of the main ideas is the need for a coordinate transformation that reduces the partial differential equation to an ordinary differential one. This concept is the main idea that underlies this manuscript.

On the other hand, because of the complexity of the BTE the numerical approach are of greatest importance in physics. Methods like Monte-Carlo [8], lattice Boltzmann, which is numerically stable and straightforward to parallel implementations [4] and [11], finite difference which a simpler approach [2], are very popular techniques in order to solve BTE.

This article presents a particular solution for the time-dependent transport Boltzmann equation in two-dimensional momentum space, using a coordinate transformation method. This paper is organized as follows. Section 2 presents the dimensionless time-dependent BTE in a two-dimensional momentum space, in the tight binding approximation. In section 3, the BTE is solved using the coordinate transformation, and an explicit solutions for the time dependent electron distribution function in momentum space is presented. For section 4, we analyze some limits.
2 Two-dimensional Boltzmann transport equation

In Cartesian coordinates, the governing equations for incompressible conservative three-dimensional flows are [7]:

\[
\frac{\partial f}{\partial t} + \frac{e}{\hbar} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{k}} + \vec{v}(\vec{k}) \cdot \frac{\partial f}{\partial \vec{r}} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \tag{2}
\]

Using the B.G.K. treatment of the collision term [10], we have:

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = f_0 - f \tag{3}
\]

Here \( f_0 \) is the equilibrium electron distribution function and \( \tau \) is the characteristic relaxation time for reaching the statistical equilibrium. Also, considering a spatial homogeneous distribution function, \( \frac{\partial f}{\partial \vec{r}} \approx 0 \), and using the dispersion relation in the tight binding approximation for the transport of an electron in a miniband, we have:

\[
\epsilon = \frac{\hbar^2 k_x^2}{2m} - \frac{\Delta_1}{2} \cos (k_x d) \tag{4}
\]

Here, \( \hbar \) is the Planck reduced constant, \( m \) is the effective electron mass, \( \Delta_1 \) is the width of the miniband. The electron velocity is:

\[
\vec{v}_k = \frac{1}{\hbar} \frac{\partial \epsilon}{\partial \vec{k}} \tag{5}
\]

Also, choosing the velocity components as:

\[
\vec{v} = (v_x, -v_y, 0) \tag{6}
\]

and the magnetic field

\[
\vec{B} = (0, 0, B) \tag{7}
\]

Defining the next set of variables in order to make dimensionless eq. (1) [5], we have:
\[ l_x = k_x d \quad l_y = k_y d / \sqrt{\alpha} \]  
\[ E / E_s \rightarrow E \quad E_s = h / e d \tau \]  
\[ eB / \tau / \sqrt{m m_x} = B \quad t \tau \rightarrow t \]  
\[ \alpha = m / m_x \quad m_* = \hbar^2 / \Delta_1 d^2 \]

where \( d \) is the period of superlattice semiconductor. Under above conditions, we obtain:

\[ \frac{\partial f}{\partial t} + (E - B l_y) \frac{\partial f}{\partial l_x} - B \sin (l_x) \frac{\partial f}{\partial l_y} = f_0 - f \]  

\[ \text{(9)} \]

3 Solution

We use the next transformation

\[ \xi = l_x + l_y + \xi_0 \]  

\[ \text{(10)} \]

So the momentum and temporal coordinates dependence on the distribution function is:

\[ f(l_x, l_y, t) = f(\xi, t) \]  

\[ \text{(11)} \]

using the approximation

\[ \sin (l_x) \approx l_x \]  

\[ \text{(12)} \]

The momentum coordinates derivatives change as:

\[ \frac{\partial}{\partial l_x} = \frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial l_y} = \frac{\partial}{\partial \xi} \]  

\[ \text{(13)} \]

Therefore, the two-dimensional Boltzmann transport equation eq. (9), using eqs. (10-13), is:

\[ \frac{\partial f}{\partial t} + (E - B(\xi)) \frac{df}{d\xi} = f_0 - f \]  

\[ \text{(14)} \]

Redefining \( f \)
Figure 1: Distribution function corresponding to eq. (24). The system momentum size is $l_x \times l_y = 100 \times 100$, $k = 0.25$, $t = 3.0$ and $E = 1.0$ and $B = 0.018$.

\[ u = f - f_0 \]  

(15)

and applying variables separation method, we have

\[ u(\xi, t) = u(\xi) u(t) \]  

(16)

Then, eq. (14) is:

\[ \frac{\partial u(\xi) u(t)}{\partial t} + (E - B\xi) \frac{du(\xi) u(t)}{d\xi} = -u \]  

(17)

and the equation for $u(t)$ is:

\[ \frac{1}{u(t)} \frac{\partial u(t)}{\partial t} = -k \]  

(18)
We choose a negative sign for $k$, in order to converge the electron Boltzmann distribution function, at time infinity, to a finite value, $f_0$. Its solution is

$$f(t) = f_0 - f_0 \exp(-k(t)) \rightarrow f(t) = f_0(1 - \exp(-k(t)))$$ \hspace{1cm} (19)

as $t \rightarrow \infty$ we have $f(t) \rightarrow f_0$. The equation for $u(\xi)$ in eq. (17) is:

$$-(E - B\xi) \frac{du(\xi)}{d\xi} + u(\xi) (k - 1) = 0$$ \hspace{1cm} (20)

and

$$u(\xi) = u(0) \left( \frac{E - B\xi}{E} \right)^{-(k-1)B}$$ \hspace{1cm} (21)

Using eq. (16), $u_0$ is:

$$u = u(\xi) u(t) \rightarrow u_0 = u(0)_{\xi} u(0)_{t} \rightarrow u_0 = -f_0$$ \hspace{1cm} (22)

Therefore, the total solution is:

$$f(\xi, t) = f_0 \left( 1 - \left( \frac{E - B\xi}{E} \right)^{-(k-1)B} e^{-k(t)} \right)$$ \hspace{1cm} (23)

or in coordinates $(l_x, l_y, t)$

$$f(l_x, l_y, t) = f_0 \left( 1 - \left( \frac{E - B(l_x + l_y)}{E} \right)^{-(k-1)B} e^{-k(t)} \right)$$ \hspace{1cm} (24)

4 Limits

It is convenient to analyze some limit behaviours of the solution (24). So, at the limit where $t \rightarrow \infty$

$$f(l_x, l_y, \infty) \approx f_0$$ \hspace{1cm} (25)

For the characteristic time $t = k^{-1}$, we have:
Figure 2: Distribution function corresponding to eq. (24). We have \( t = l_y = 1 \times 10, \, k = 0.5, \, l_x = 3.1 \) and \( E = 1.1 \) and \( B = 0.14 \).

\[
f(\xi, k) = f_0 \left( 1 - \left( \frac{E - B\xi}{E} \right)^{-(k-1)B} \right)^{-1} \tag{26}
\]

If \( l_x = -l_y \rightarrow \xi = 0 \)

\[
f(0, t) = f_0 \left( 1 - e^{-(k(t))} \right) \tag{27}
\]

In the case where \( l_x = -l_y \rightarrow \xi = 0 \) and \( t = k^{-1} \), we have:

\[
f(0, k) = f_0 \left( 1 - e^{-1} \right) \tag{28}
\]

So \( f(0, k) \) is reduced to 63\% of \( f_0 \). Also, if \( l_x = -l_y \rightarrow \xi = 0 \), and \( t = 0 \), we obtain:

\[
f(0, 0) = f_0 \tag{29}
\]
5 Conclusions

This work presents a solution for the Boltzmann transport equation using a co-ordinates transformation method. We find an explicit solution for the temporal electron Boltzmann distribution in the two-dimensional momentum space, using tight binding approximation and assuming small values for \( l_x \). Also, we determine several limit behaviours, especially when \( t \to \infty \) we recover the equilibrium distribution function. Likewise, if \( l_x = -l_y \to \xi = 0 \) we have the classical exponential behaviour, where \( f_0 \) is reduced to 63% for a characteristic time of \( t = k^{-1} \). As far as we know, this solution has not been published in the current literature.

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References


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