

Multiscale Hierarchical Modeling of Fiber Reinforced Composites by Asymptotic Homogenization Method

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Abstract

The paper is devoted to development of the asymptotic averaging method for periodic structures being multilevel hierarchical ones. A determination method for effective elastic characteristics of composites with a multiscale hierarchical structure is suggested. Recurrent sequences of local and averaged problems of the elasticity theory are formulated over periodicity cells of different structural levels. The example of finite-element solving the number of local problems is given for the two-level structure model of a textile composite.

Keywords: multiscale homogenization method, finite element analysis, effective elastic properties of composite materials

Introduction

Advanced composite materials, as a rule, have a multiscale hierarchical structure (MHS), where each the next structural level contains the preceding level beginning with the least one. This structure is especially clear in composites with reinforcing fibers of different weaving: textile, winding, spatial, etc., where the fibers are bundles of a great number of monofibers. Traditional computation methods [6] for effective elastic characteristics of multiscale composites are based on approximate-analytical approaches, when we assume that there appears a uniform stress-strain state in their matrix and fibers. The

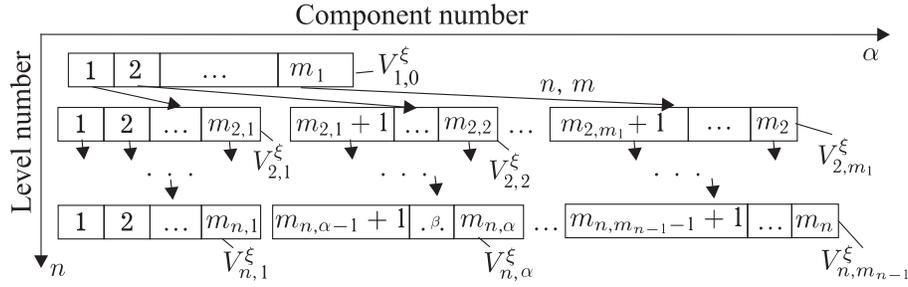


Fig. 1. The general scheme of a MHS

methods, as a rule, lead to appropriate results only in computations of effective characteristics in the reinforcing direction, but show a considerable error for characteristics in transverse directions and in shear. Moreover, calculations of microstresses (stresses in components of a composite) by the methods are rather approximate because an actual geometrical shape of components is not taken into account. The considerable success [1, 7–9] in calculations of composite characteristics was achieved after creation of the homogenization (asymptotic averaging) method and development of effective computational algorithms of solving the local problems appearing in the method [4, 5]. However, the method was created only for a classical one-level structure: matrix+fiber is a composite. Our purpose is to generalize the method for composites with an arbitrary number of structural levels and to develop the finite-element method of solving the recurrent sequence of local problems over periodicity cells (PCs).

Geometry of a MHS.

To describe a geometric composite structure with an arbitrary number of levels n , where each the lower n th level is included in the higher $(n - 1)$ th level, we apply the method suggested in [5] (Fig.1).

Let L be a characteristic scale of the whole domain V occupied by a composite, and l_n be characteristic scales of PCs $V_{n\alpha}^\xi$ of the n th level. We assume that in the frames of a single level all the cells have the same characteristic scale. Due to the assumption that the MHS and each the component $V_{n-2,\alpha}$ consists of a great number N_{0n} of PCs $V_{n\alpha}^\xi$, the following scale relations holds: $l_N \ll l_{N-1} \ll \dots \ll l_n \ll \dots \ll l_1 \ll L = l_0$, where $L = N_{01}l_1$, and, in general, $l_{n-1} = N_{0n}l_n$. Then we can introduce N small parameters \varkappa_n : $\varkappa_n = l_n/l_{n-1} = 1/N_{0n} \ll 1$ ($n = 1, \dots, N$). Let $\mathbf{x} = x^i \mathbf{e}_i$ be the radius-vector and x^i be Cartesian coordinates of a point in domain V . Then we can introduce dimensionless coordinates $\bar{x}^i = x^i/L$ called global. Local coordinates $\boldsymbol{\xi}_n$ of the n th level are introduced as follows: $\boldsymbol{\xi}_n = \boldsymbol{\xi}_{n-1}/\varkappa_n$ ($n = 1, \dots, N$; $\boldsymbol{\xi}_0 \equiv \mathbf{x}$), where $\boldsymbol{\xi}_n = \xi_n^i \mathbf{e}_i$ and ξ_n^i take values from $[0, 1]$, while x^i vary in cells $V_{n\alpha}^\xi$.

Differentiation and integration of periodic functions in MHS.

Different physical properties and processes occurring in a MHS are described with the help of certain functions $g = g(\xi_1, \dots, \xi_N)$ of N local variables satisfying the periodicity conditions in each the argument:

$$g(\xi_1, \dots, \xi_n + a_i e_i, \dots, \xi_N) = g(\xi_1, \dots, \xi_n, \dots, \xi_N) \quad \forall \xi_n \in V_n^\xi, \quad (1)$$

where a_i are arbitrary integers. Functions satisfying (1) are called multilevel periodic. If function $g = g(x, \xi_1, \dots, \xi_N)$ is multilevel periodic and depends on global coordinates, it is called multilevel quasiperiodic (MQ). The gradient of a MQ-function is determined by the rule of differentiation of a composite function: $\nabla g \rightarrow \nabla_x g + (1/\varkappa_1)\nabla_1 g$, $\nabla_1 g \rightarrow \nabla_1 g + (1/\varkappa_2)\nabla_2 g$, $\nabla g_{n-1} \rightarrow \nabla_{n-1} g + (1/\varkappa_n)\nabla_n g$. Here $n = 1, \dots, N$; $\nabla_n = \mathbf{e}_i(\partial/\partial \xi_n^i)$ and $\nabla_x = \mathbf{e}_i(\partial/\partial x^i)$ are nabla-operators of differentiation with respect to local coordinates of the n th level and global coordinates, respectively, and $\nabla g = \mathbf{e}_i(\partial/\partial x^i)$. The integral of a MQ-function is determined by the recurrent way:

$$\langle g \rangle_1 = \sum_{1 \leq \alpha \leq m_1} \int_{V_{1\alpha}} \langle g \rangle_{2\alpha} dV_{\xi_1}, \quad \langle g \rangle_{N\beta} = \sum_{m_{N,\beta-1} \leq \gamma \leq m_{N\beta}} \int_{V_{N\gamma}} g dV_{\xi_N}. \quad (2)$$

Statement of the elasticity problem for MHS.

Consider the linear elasticity problem statement for MHS described above and occupied the domain V :

$$\nabla \cdot \boldsymbol{\sigma} \equiv \mathbf{f} = 0, \quad 2\boldsymbol{\varepsilon} = \nabla \otimes \mathbf{u} + \nabla \otimes \mathbf{u}^T, \quad \mathbf{x} \in V; \quad \boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon}, \quad \mathbf{x} \in V \cup \Sigma; \quad (3)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{S}_e, \quad \mathbf{x} \in \Sigma_\sigma; \quad \mathbf{u} = \mathbf{u}_e, \quad \mathbf{x} \in \Sigma_u.$$

Here $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ is the small strain tensor, \mathbf{u} is the displacement vector, \mathbf{n} is the external normal vector to the surface Σ of domain V , \mathbf{S}_e is the force vector given at the surface part Σ_σ , \mathbf{u}_e is the displacement vector given at the surface part Σ_u . At all interfaces $\Sigma_{n\alpha\alpha'}$ of components $V_{n\alpha}$ and $V_{n\alpha'}$ in the MHS the ideal contact conditions are assumed to be satisfied:

$$[\boldsymbol{\sigma}] \cdot \mathbf{n} = 0, \quad [\mathbf{u}] = 0, \quad \mathbf{x} \in \Sigma_{n\alpha\alpha'}, \quad (4)$$

where $[\mathbf{u}]$ is the jump of functions. Due to the structure periodicity, tensor \mathbf{C} may be considered as a multilevel periodic function (1): $\mathbf{C} = \mathbf{C}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N)$.

Asymptotic solution of the elasticity problem for a MHS.

A solution of the problem (3), (4) with respect to the displacement vector is sought in the form of asymptotic expansions in terms of small parameters \varkappa_n . The asymptotic solution of problem (3), (4) taking into account the structure nonuniformity accurate only up to the first level is sought in the form

$$\mathbf{u} = u^{(0)}(\mathbf{x}) + \varkappa_1 u_{1\alpha}^{(1)}(\mathbf{x}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N) + \varkappa_1^2 u_{1\alpha}^{(2)}(\mathbf{x}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N) + \dots; \quad \boldsymbol{\xi}_1 \in V_{1\alpha}; \quad (5)$$

where $\alpha = 1, \dots, m_1$. Gradient of functions (5) is determined has the form

$$\nabla \otimes \mathbf{u} = (\nabla_x \otimes \mathbf{u}^{(0)} + \nabla_1 \otimes \mathbf{u}_{1\alpha}^{(1)}) + \varkappa_1 (\nabla_x \otimes \mathbf{u}_{1\alpha}^{(0)} + \nabla_1 \otimes \mathbf{u}_{1\alpha}^{(2)}) + \varkappa^2 \dots \quad (6)$$

Substituting (6) into system (3), we obtain the asymptotic expansion for all the remaining functions of the initial problem:

$$\Omega = (1/\varkappa_1)\Omega_{1\alpha}^{(-1)} + \Omega_{1\alpha}^{(0)} + \varkappa_1\Omega_{1\alpha}^{(1)} + \varkappa_1^2\Omega_{1\alpha}^{(2)} + \dots, \quad \boldsymbol{\xi}_1 \in V_{1\alpha}, \quad \Omega = \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{f}\}, \quad (7)$$

where $\boldsymbol{\varepsilon}_{1\alpha}^{(-1)} = 0$, $\boldsymbol{\sigma}_{1\alpha}^{(-1)} = 0$, $\mathbf{f}_{1\alpha}^{(-1)} = \nabla_1 \cdot \boldsymbol{\sigma}_{1\alpha}^{(0)}$.

Local and averaged problems of the first level.

Substituting (7) for \mathbf{f} into equilibrium equation (1), we find that all the terms $f_{1\alpha}^{(k)}$ ($k = -1, 0, 1, \dots$) are zero. The equation $f_{1\alpha}^{(-1)} = 0$ complemented by (7) for $\boldsymbol{\sigma}_{1\alpha}^{(0)}$, $\boldsymbol{\varepsilon}_{1\alpha}^{(0)}$ leads to the local problem of the class (-1) over PC V_1^ξ for functions $\mathbf{u}_\alpha^{(1)}$:

$$\begin{cases} \nabla_1 \cdot \boldsymbol{\sigma}_{1\alpha}^{(0)} \equiv f_{1\alpha}^{(-1)} = 0, & \boldsymbol{\xi}_1 \in V_{1\alpha}, \alpha = 1, \dots, m_1; \\ \boldsymbol{\sigma}_{1\alpha}^{(0)} = \mathbf{C} \cdot \boldsymbol{\varepsilon}_{1\alpha}^{(0)}, & 2\boldsymbol{\varepsilon}_{1\alpha}^{(0)} = 2\bar{\boldsymbol{\varepsilon}}_1 + \nabla_1 \otimes \mathbf{u}_{1\alpha}^{(1)} + \nabla_1 \otimes \mathbf{u}_{1\alpha}^{(1)\text{T}}, \\ [\boldsymbol{\sigma}_{1\alpha}^{(0)}] \cdot \mathbf{n} = 0, & [\mathbf{u}_\alpha^{(1)}] = 0, \quad \boldsymbol{\xi}_1 \in \Sigma_{1\alpha\beta}; \quad \langle \mathbf{u}_{1\alpha}^{(1)} \rangle_1 = 0. \end{cases} \quad (8)$$

The condition $\langle \mathbf{u}_{1\alpha}^{(1)} \rangle_1 = 0$ is added to ensure the uniqueness of the solution.

In the same way, we can formulate problems of a higher class when considering the equation $f_{1\alpha}^{(k)} = 0$ at $k = 0, 1, 2, \dots$. However, with writing explicitly only terms up to the first-order smallness, it is sufficient to consider only the equation $f_{1\alpha}^{(0)} = \nabla_x \cdot \boldsymbol{\sigma}_{1\alpha}^{(0)} + \nabla_1 \cdot \boldsymbol{\sigma}_{1\alpha}^{(1)} = 0$. There exists a periodic in $\boldsymbol{\xi}_1$ solution of the equation if the averaged value is zero: $\langle f_{1\alpha}^{(0)} \rangle_1 = 0$, or $\nabla_x \cdot \langle \boldsymbol{\sigma}_{1\alpha}^{(0)} \rangle_1 = 0$ ($\mathbf{x} \in V$), because $\langle \nabla_1 \cdot \boldsymbol{\sigma}_{1\alpha}^{(1)} \rangle_1 = 0$. Complementing this equation by the averaged constitutive relation from (8): $\langle \boldsymbol{\sigma}_{1\alpha}^{(0)} \rangle_1 = \langle \mathbf{C} \cdot \boldsymbol{\varepsilon}_{1\alpha}^{(0)} \rangle_1 = \bar{\mathbf{C}} \cdot \bar{\boldsymbol{\varepsilon}}_0$ together with the boundary conditions for $\mathbf{x} \in \Sigma$: $\langle \boldsymbol{\sigma} \rangle_1 \cdot \mathbf{n} = \langle \mathbf{S}_e \rangle$, $\langle \mathbf{u}^{(0)} \rangle = \langle u_e \rangle$, we obtain the averaged problem for $\mathbf{u}^{(0)}(\mathbf{x})$, which is called the problem A_0 .

Local and averaged problems of the n th level.

Conducting n times the similar procedure, we get local and averaged problems of an arbitrary n th level ($n = 1, \dots, N$). Asymptotic expansion is sought in the form

$$u_{n-1,\alpha}^{(1)} = u_{n\alpha}^{(0)}(\mathbf{x}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}) + \varkappa_n u_{n\beta}^{(1)}(\mathbf{x}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N) + \varkappa_n^2 u_{n\beta}^{(2)} \dots, \quad (9)$$

where $\boldsymbol{\xi}_{n-1} \in V_{n-1,\alpha}$; $\boldsymbol{\xi}_n \in V_{n\beta}$; $m_{n,\alpha-1} \leq \beta \leq m_{n,\alpha}$. To find functions $u_{n\beta}^{(1)}$ we have local problems over PC $V_{n\beta}^\xi$ called the problems $L_{n\beta}$:

$$\begin{cases} \nabla_n \cdot \boldsymbol{\sigma}_{n\beta}^{(0)} = 0, & \boldsymbol{\xi}_n \in V_{n\beta}; \quad \boldsymbol{\sigma}_{n\beta}^{(0)} = \mathbf{C} \cdot \boldsymbol{\varepsilon}_{n\beta}^{(0)} \quad 2\boldsymbol{\varepsilon}_{n\beta}^{(0)} = 2\bar{\boldsymbol{\varepsilon}}_{n-1,\alpha} + \nabla_n \otimes \mathbf{u}_{n\beta}^{(1)} + \nabla_n \otimes \mathbf{u}_{n\beta}^{(1)\text{T}}, \\ [\boldsymbol{\sigma}_{n\beta}^{(0)}] \cdot \mathbf{n} = 0, & [\mathbf{u}_{n\beta}^{(1)}] = 0, \quad \boldsymbol{\xi}_n \in \Sigma_{n\beta\beta'}, \\ \langle \mathbf{u}_{n\beta}^{(1)} \rangle_{n\alpha} = 0, & m_{n,\alpha-1} \leq \beta, \beta' \leq m_{n,\alpha}, \quad n = N, \dots, 1. \end{cases} \quad (10)$$

To determine functions $u_{n-1,\beta}^{(0)}$ we obtain the averaged problems:

$$\begin{cases} \nabla_{n-1} \cdot \langle \boldsymbol{\sigma}_{n\gamma}^{(0)} \rangle_{n\beta} = 0, & \boldsymbol{\xi}_{n-1} \in V_{n-1,\beta}; & \langle \boldsymbol{\sigma}_{n\gamma}^{(0)} \rangle_{n\beta} = \langle \mathbf{C} \cdot \boldsymbol{\varepsilon}_{n\gamma}^{(0)} \rangle_{n\beta} = \bar{\mathbf{C}}_{n\beta} \cdot \bar{\boldsymbol{\varepsilon}}_{n-1,\beta}, \\ 2\bar{\boldsymbol{\varepsilon}}_{n-1,\beta} = 2\bar{\boldsymbol{\varepsilon}}_{n-2,\alpha} + \nabla_{n-1} \otimes \mathbf{u}_{n-1,\beta}^{(0)} + \nabla_{n-1} \otimes \mathbf{u}_{n-1,\beta}^{(0)\text{T}}, \\ [\langle \boldsymbol{\sigma}_{n\gamma}^{(0)} \rangle_{n\beta}] = 0, & [\mathbf{u}_{n-1,\beta}^{(0)}] = 0, & \boldsymbol{\xi}_{n-1} \in \Sigma_{n-1,\beta\beta'}, \\ \langle \mathbf{u}_{n-1,\beta}^{(0)} \rangle_{n-1,\alpha} = 0; & m_{n-1,\alpha-1} \leq \beta, \beta' \leq m_{n-1,\alpha}; & m_{n,\beta-1} \leq \gamma \leq m_{n,\beta}; & n=2, \dots, N, \end{cases} \quad (11)$$

which are called the problems $A_{n-1,\alpha}$.

The recurrent sequence of problems.

As a result of the procedure constructed above, we have the following recurrent sequence of problems:

$$L_{N\beta} \rightarrow A_{N-1,\alpha_{N-1}} \rightarrow A_{N-2,\alpha_{N-2}} \rightarrow \dots \rightarrow A_{n,\alpha_n} \rightarrow \dots \rightarrow A_{1,\alpha_1} \rightarrow A_0. \quad (12)$$

At first, there is a need to solve local problems $L_{N\beta}$ for the lowest N th level of the MHS, for all the PCs $V_{n\beta}^\xi$ with $\beta = 1, \dots, m_N$, thus, we determine functions $\mathbf{u}_{N\beta}^{(1)}(\mathbf{x}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N)$, depending parametrically on initial data of the problem being functions $\bar{\boldsymbol{\varepsilon}}_{N-1,\alpha}$. We then solve the averaged problems $A_{N-1,\alpha_{N-1}}$ for all PCs $V_{N-1,\alpha_{N-1}}^\xi$ ($\alpha_{N-1} = 1, \dots, m_{N-1}$), where unknowns are the functions $\mathbf{u}_{N-1,\alpha}^{(0)}(\mathbf{x}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{N-1})$, and initial data — functions $\bar{\boldsymbol{\varepsilon}}_{N-2,\alpha}$. In the similar way, we solve all the problems A_{n,α_n} up to $n = 1$. The last solved problem is the problem A_0 for functions $\mathbf{u}_{0,\alpha}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x})$, which depends only on global coordinates.

Since problems $L_{N\beta}$ are linear, we can write formally their solution as a linear function of $\bar{\boldsymbol{\varepsilon}}_{N-1,\alpha}$ and separate the summand depending linearly on $\boldsymbol{\xi}_N$:

$$\mathbf{u}_{N\beta}^{(1)} = -\bar{\boldsymbol{\varepsilon}}_{N-1,\alpha} \cdot \boldsymbol{\xi}_N + \mathbf{N}_{N\beta}^{(1)}(\boldsymbol{\xi}_N) \cdot \bar{\boldsymbol{\varepsilon}}_{N-1,\alpha}; \quad m_{N,\alpha-1} + 1 \leq \beta \leq m_{N,\alpha}, \quad (13)$$

where $\mathbf{N}_{N\beta}^{(1)}(\boldsymbol{\xi}_N)$ are third-order tensors being new unknown functions and called pseudodisplacements, then

$$\boldsymbol{\varepsilon}_{N\beta}^{(0)} = \boldsymbol{\Delta}_{N\beta} \cdot \bar{\boldsymbol{\varepsilon}}_{N-1,\alpha}, \quad \boldsymbol{\Delta}_{N\beta} = (1/2)(\nabla_N \otimes \mathbf{N}_{N\beta}^{(1)} + \nabla_N \otimes \mathbf{N}_{N\beta}^{(1)\text{T}}), \quad (14)$$

Substituting (13) and (14) into system (10) at $n = N$, we get the local problem for components $N_{N\beta jpq}^{(1)}$ of tensors $\mathbf{N}_{N\beta}^{(1)}$:

$$\begin{cases} \nabla_N \cdot \mathbf{T}_{N\beta}^{(0)} = 0, & \boldsymbol{\xi}_N \in \tilde{V}_{N\beta}, & m_{N,\alpha-1} \leq \beta, \beta' \leq m_{N,\alpha}, \\ \mathbf{T}_{N\beta}^{(0)} = \mathbf{C} \cdot \mathbf{E}_{N\beta}^{(0)}, & 2\mathbf{E}_{N\beta}^{(0)} = \nabla_N \otimes \mathbf{N}_{N\beta}^{(1)} + (\nabla_N \otimes \mathbf{N}_{N\beta}^{(1)})\mathbf{T}, \\ [\mathbf{T}_{N\beta}^{(0)}] \cdot \mathbf{n} = 0, & [\mathbf{N}_{N\beta}^{(1)}] = 0, & \boldsymbol{\xi}_N \in \Sigma_{N\beta\beta'}, \\ \mathbf{n} \cdot \mathbf{N}_{N\beta}^{(1)} = \mathbf{n} \cdot \boldsymbol{\Delta} \cdot \boldsymbol{\xi}_N, & \mathbf{e}_j \cdot \mathbf{T}_{N\beta}^{(0)} = 0, & \boldsymbol{\xi}_N \in \Sigma'_{iN\beta}, \quad j \neq i, \\ \mathbf{n} \cdot \mathbf{N}_{N\beta}^{(1)} = 0, & \mathbf{e}_j \cdot \mathbf{T}_{N\beta}^{(0)} = 0, & \boldsymbol{\xi}_N \in \Sigma_{iN\beta}. \end{cases} \quad (15)$$

The problem (14) is a collection of six independent local problems denoted by $L_{N\beta}^{pq}$. This problem is stated for domain $\tilde{V}_{N\beta}$ being the 1/8 part of PC $V_{N\beta}^\xi$ bounded by coordinate surfaces $\Sigma_{iN\beta} = \{\xi_N^i = 0\}$ and external boundaries of the PC $\Sigma'_{iN\beta} = \{\xi_N^i = 1/2\}$, where we introduced the fourth-order tensor $\mathbf{T}_{N\beta}^{(0)}$ called the pseudostress tensor, which is connected to $\boldsymbol{\sigma}_{N\beta}^{(0)}$ by the relation $\boldsymbol{\sigma}_{N\beta}^{(0)} = \mathbf{T}_{N\beta}^{(0)} \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{N-1,\alpha}$; $\mathbf{E}_{N\beta}^{(0)}$ is the fourth-order tensor called the pseudostrain tensor; $(\nabla_N \otimes \mathbf{N}_{N\beta}^{(1)})^T$ is transposition of the fourth-order tensor relative to the first two indices; $\boldsymbol{\Delta}$ is the unit fourth-order tensor [2, 3]. The solution $\mathbf{N}_{N\beta}^{(1)}(\boldsymbol{\xi}_N)$ is continued into the remaining part of PC $\tilde{V}_{N\beta}$ in a symmetric or antisymmetric way (for different components of tensor $\mathbf{N}_{N\beta}^{(1)}$ we use either one or another way) described in [5]. The last two conditions in (14) are the consequences of continuation and periodicity conditions at the PC boundary. Averaging stresses $\boldsymbol{\sigma}_{N\beta}^{(0)}$ over PC $\tilde{V}_{N\beta}$, we obtain constitutive relations of elasticity for the next $(N-1)$ th structural level: $\langle \boldsymbol{\sigma}_{N\beta}^{(0)} \rangle_{N\beta} = \bar{\mathbf{C}}_{N\beta} \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{N-1,\alpha}$; $\bar{\mathbf{C}}_{N\beta} = \langle \mathbf{T}_{N\beta}^{(0)} \rangle_{N\beta}$.

Averaged problems $\hat{A}_{n,\alpha}$.

Local $L_{n\beta}$ (10) and averaged $A_{n-1,\alpha}$ (11) problems have the same type: these are contact problems of the nonuniform elasticity theory over PCs with periodicity and norming conditions. Thus, solving the problems $A_{n-1,\alpha}$ can be conducted by the method given above, their solutions are sought in the form of sums (for $n = N, \dots, 2$):

$$\begin{aligned} \mathbf{u}_{n-1,\beta}^{(0)} &= -\bar{\boldsymbol{\varepsilon}}_{n-2,\alpha} \cdot \boldsymbol{\xi}_{n-1} + \mathbf{N}_{n-1,\beta}^{(0)}(\boldsymbol{\xi}_{n-1}) \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{n-2,\alpha}; \quad m_{n,\alpha-1} \leq \beta \leq m_{n,\alpha}, \\ \bar{\boldsymbol{\varepsilon}}_{n-1,\beta}^{(0)} &= \boldsymbol{\Delta}_{n-1,\beta} \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{n-2,\alpha}, \quad \boldsymbol{\Delta}_{n-1,\beta} = (1/2)(\nabla_{n-1} \otimes \mathbf{N}_{n-1,\beta}^{(0)} + \nabla_{n-1} \otimes \mathbf{N}_{n-1,\beta}^{(0)T}). \end{aligned}$$

Substitution of the equation into (11) gives the problem for $\mathbf{N}_{n-1,\beta}^{(0)}(\boldsymbol{\xi}_{n-1})$:

$$\begin{cases} \nabla_{n-1} \cdot \mathbf{T}_{n-1,\beta}^{(0)} = 0, & \boldsymbol{\xi}_{n-1} \in \tilde{V}_{n-1,\beta}, \\ \mathbf{T}_{n-1,\beta}^{(0)} = \bar{\mathbf{C}}_{n\beta} \cdot \cdot \mathbf{E}_{n-1,\beta}^{(0)}, & 2\mathbf{E}_{n-1,\beta}^{(0)} = \nabla_{n-1} \otimes \mathbf{N}_{n-1,\beta}^{(1)} + (\nabla_{n-1} \otimes \mathbf{N}_{n-1,\beta}^{(0)})^T, \\ [\mathbf{T}_{n-1,\beta}^{(0)}] \cdot \mathbf{n} = 0, & [\mathbf{N}_{n-1,\beta}^{(1)}] = 0, \quad \boldsymbol{\xi}_{n-1} \in \Sigma_{n-1,\beta\beta'}, \\ \mathbf{n} \cdot \mathbf{N}_{n-1,\beta}^{(0)} = \mathbf{n} \cdot \boldsymbol{\Delta} \cdot \boldsymbol{\xi}_{n-1}, & \mathbf{e}_j \cdot \mathbf{T}_{n-1,\beta}^{(0)} = 0, \quad \boldsymbol{\xi}_{n-1} \in \Sigma'_{in-1,\beta}, \quad j \neq i, \\ \mathbf{n} \cdot \mathbf{N}_{n-1,\beta}^{(0)} = 0, & \mathbf{e}_j \cdot \mathbf{T}_{n-1,\beta}^{(0)} = 0, \quad \boldsymbol{\xi}_{n-1} \in \Sigma_{in-1,\beta}. \end{cases} \quad (16)$$

Here $\tilde{V}_{n-1,\beta}^\xi$ is the 1/8 part of PC $V_{n-1,\beta}^\xi$, $\mathbf{E}_{n-1,\beta}^{(0)}$ is the pseudostrain fourth-order tensor, $\mathbf{T}_{n-1,\beta}^{(0)}$ is the pseudostress tensor connected to $\boldsymbol{\sigma}_{n\gamma}^{(0)}$ by the relation $\langle \boldsymbol{\sigma}_{n\gamma}^{(0)} \rangle_{n\beta} = \bar{\mathbf{C}}_{n\beta} \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{n-1,\beta} = (\bar{\mathbf{C}}_{n\beta} \cdot \cdot \boldsymbol{\Delta}_{n-1,\beta}) \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{n-2,\alpha} = \mathbf{T}_{n-1,\beta}^{(0)} \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{n-2,\alpha}$. A solution of the problem is sought in domain $\tilde{V}_{n-1,\beta}^\xi$, and the solution is extended into the remaining part of PC $V_{n-1,\beta}^\xi$ in the symmetric or antisymmetric way described above. Integrating the last relation over PC $V_{n-1,\beta}^\xi$, with account (2) we obtain the constitutive relations of elasticity at the next $(n-1)$ th structural level: $\langle \boldsymbol{\sigma}_{n-1,\beta}^{(0)} \rangle_{n-1,\alpha} = \bar{\mathbf{C}}_{n-1,\beta} \cdot \cdot \bar{\boldsymbol{\varepsilon}}_{n-1,\alpha}$, $\bar{\mathbf{C}}_{n-1,\beta} = \langle \mathbf{T}_{n-1,\beta}^{(0)} \rangle_{n-1,\beta}$, where $\bar{\mathbf{C}}_{n-1,\beta}$ are the

effective elasticity modules at the n th level. After the problems' sequence (12) has been solved, these relations and (15) allow us to determine the tensor $\bar{\mathbf{C}}$ in the recurrent way. For $n = 2$, with account of $\langle \rangle_{1\beta} \equiv \langle \rangle_1$ and $\langle \sigma_{1\beta}^{(0)} \rangle_{1\alpha} = \langle \sigma_{1\alpha}^{(0)} \rangle_1$ we get the desired equation for $\bar{\mathbf{C}}$: $\bar{\mathbf{C}} = \bar{\mathbf{C}}_{1\beta} = \langle \mathbf{T}_{1\beta}^{(0)} \rangle_{1\beta}$. For linear elasticity problems over a MHS, it is convenient to use the stress concentration tensors, which on the base of once solved problems' sequence (12) at some macrostresses $\langle \sigma_{1\alpha}^{(0)} \rangle_1$, allow us to find microstresses $\sigma_{n\gamma}^{(0)}$ at any structural level for any other values $\langle \sigma_{1\alpha}^{(0)} \rangle_1$ without additional solving the problems' sequence (12). The tensors $\mathbf{B}_{n\beta}$ are determined by

$$\sigma_{n\beta}^{(0)}(\boldsymbol{\xi}_n) = \mathbf{B}_{n\beta}(\boldsymbol{\xi}_n) \cdot \sigma_{n-1,\beta}^{(0)}, \quad \text{where } \mathbf{B}_{n\beta}(\boldsymbol{\xi}_n) = \mathbf{T}_{n\beta}^{(0)}(\boldsymbol{\xi}_n) \cdot \bar{\mathbf{C}}_{n-1,\alpha}^{-1}. \quad (17)$$

The example of numerical simulation of a two-level textile composite.

To solve the problems $\hat{L}_{n\beta}$ and $\hat{A}_{n,\alpha}$ numerically, we apply the finite-element method [4, 5]. As an example of the method developed above, let us consider a two-level textile composite, fibers of which are curved in accordance with the sinusoidal law in planes Ox^1x^3 (the first fibers' system is a warp) and Ox^2x^3 (the second fibers' system is a weft) and are represented by bundles of a great number of fibers. For such composite material, PC of the first structural level V_1^ξ is formed by two sets of fibers V_{11} , V_{12} and matrix V_{13} surrounding them. The second level consists of three types of PCs $V_{2\alpha}^\xi$ ($\alpha = 1, 2, 3$), where V_{21}^ξ is the PC consisting of one unidirectional monofiber oriented along the direction Ox^1 and the matrix surrounding the fiber; the PC V_{22}^ξ consists of one unidirectional monofiber oriented along the direction Ox^2 and the matrix surrounding the fiber; and the PC V_{23}^ξ is the matrix itself without monofibers. Numbers $m_{n,\alpha}$ for this case are the following: $m_{1,1} = m_1 = 3$, $m_{2,1} = 2$, $m_{2,2} = 2$, $m_{2,3} = 1$. The recurrent sequence of problems (12) in this case has the form $L_{21}, L_{22}, L_{23} \rightarrow A_{11} \rightarrow A_0$.

The problem L_{23} is trivial, because its PC V_{23}^ξ consists only of one matrix, and there is no need to solve the problem. We should solve numerically two local problems of the second level L_{21} , L_{22} and one problem of the first level A_{11} . Since a geometric structure of monofibers' location in PCs V_{21}^ξ and V_{22}^ξ is the same, it is sufficient to solve only one problem, for example L_{21} , and a solution of the second problem L_{22} is found by simple renumbering indices at tensors: $1 \leftrightarrow 2$. The matrix and monofibers are assumed to be isotropic. Elastic modules' tensors of matrix \mathbf{C}_{23} and monofibers \mathbf{C}_{21} and \mathbf{C}_{22} are given, their components $C_{2\alpha}^{ijkl}$ are calculated with the help of elastic modules E_m , E_f and Poisson coefficients ν_m , ν_f of the matrix and monofibers in the standard way [4]. The effective technical elastic constants (elastic modules E_i , Poisson coefficients ν_{ij} and shear modules G_{ij} of a composite) are calculated with the help of components \bar{C}^{ijkl} of the effective elastic modules' tensor $\bar{\mathbf{C}}$.

In computations we used the constants: $\nu_m = 0.35$, $\nu_f = 0.2$, $E_m = 3$ GPa, $E_f = 70$ GPa, corresponding to the epoxy matrix and glass fibers. The volume

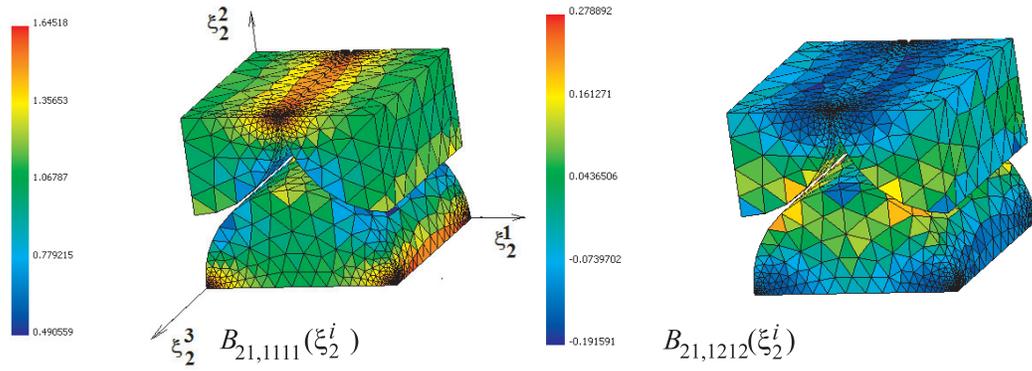


Fig.2. Distributions of components $B_{21,ijkl}(\xi_1^i)$ in PC V_{21}^ξ

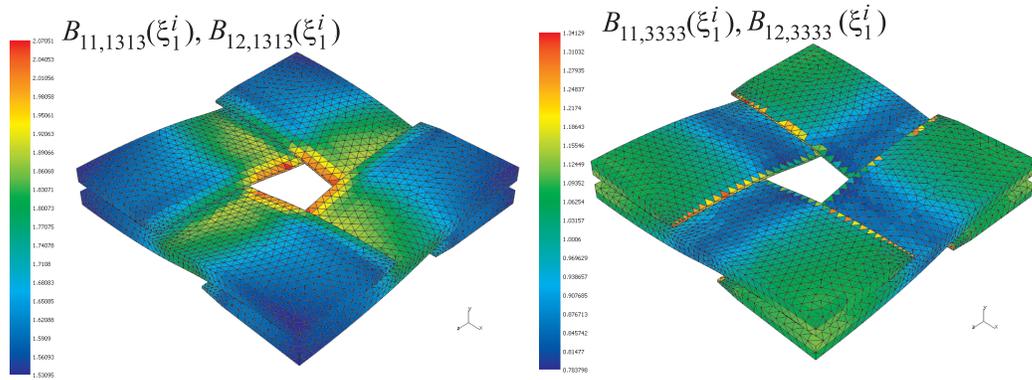


Fig.3. Distributions of components $B_{11,ijkl}(\xi_1^i)$ and $B_{12,ijkl}(\xi_1^i)$ of stress concentration tensors in domains V_{11} and V_{12} (weaved threads) of PC V_1^ξ

content of monofibers in the fibers is 0.61 (hexagonal monofibers' laying in PC2 is considered), the fibers' content in PC1 is 0.52. Computations gave the following effective technical elastic constants: $E_1 = 20.1$ GPa, $E_3 = 4.3$ GPa, $\nu_{12} = 0.26$, $\nu_{13} = 0.28$, $G_{13} = 1.94$ GPa. To solve the local elasticity problems of the n th structural level L_{21} and A_{11} we applied the modified finite-element method with tetrahedral finite elements (FE) and linear approximation of displacements in the FE [4, 5]. For finite-element solving the problems and visualization of computed results we developed the specific software [5].

Figure 2 shows examples of solutions of the problem L_{21} in the form of distributions of components $B_{21,ijkl}(\xi_2^i)$ of the stress concentration tensor $\mathbf{B}_{21}(\xi_2)$ (17). The figure presents values of some components, which have the greatest characteristic magnitudes. Most important components $B_{21,1111}(\xi_2^i)$ and $B_{21,1313}(\xi_2^i)$ corresponding to stresses of tension and shear reach their maxima in the matrix within zones of the closest approach of neighboring monofibers. Just in these zones there appears a microfracture of the composite material.

Examples of solutions of the problem A_{11} in the form of distributions of components $B_{11,ijkl}(\xi_1^i)$ of the stress concentration tensor $\mathbf{B}_{11}(\xi_1)$ (17) in two intersecting threads are shown in Fig.3. Components $B_{11,1111}(\xi_1^i)$ and $B_{12,3333}(\xi_1^i)$ reach their maxima in threads oriented along $O\xi_1^1$ and $O\xi_1^3$, respectively. Stress concentration components $B_{11,1313}(\xi_1^i)$ reach their maxima in the middle of threads in both the directions.

Conclusions.

The paper develops the asymptotic averaging method for periodic structures in the case of MHSs with scales of different levels. Recurrent sequences of local and averaged elasticity problems are stated over PCs of different structural levels. As the example of applying the method developed, a stress state was simulated for the two-level structure of a textile composite material by the finite-element analysis. Thus, the character of stresses' distribution in composite components was found, and maxima of stress concentration tensor components are shown to play a major role in possible fracture of a composite, they occur at the matrix-thread interface within the zone of most sharpened edges of threads (for the first structural level) and in zones of the closest approach of monofibers (for the second structural level).

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