L–Fuzzy $T$–Ideals of $\beta$–Algebras

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Abstract

In this paper, we introduce the notion of $L$– fuzzy $T$–ideals of $\beta$–algebras and investigate some of their properties.

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1 Introduction

J.Neggers and H.S. Kim introduced the notion of $\beta$–algebras [4]. An important point in the evaluation of the modern concept of uncertainty was the notion of fuzzy sets introduced by Lofti A. Zadeh[7]. L.Goguen [3] generalized the notion of fuzzy sets into the notion of $L$–fuzzy sets. For the general study of structures of $\beta$– algebras, the ideal theory and fuzzy ideal theory play an important role.

In [1] the authors have introduced the notion of fuzzy $\beta$– subalgebras of $\beta$–algebras. In [2] they have introduced the notion of fuzzy $\beta$–ideals of $\beta$–algebras. In [5], we introduced the notion of $L$– fuzzy $\beta$– subalgebras of...
$\beta$-algebras and investigated their properties. In [6], we introduced the notion of $L$-fuzzy $\beta$-ideals of $\beta$-algebras and investigated their properties. In this paper, we introduce the notion of $L$-fuzzy T-ideals of a $\beta$-algebra, and investigate some of their properties.

2 Preliminaries

In this section we recall some basic definitions that are required in the sequel.

**Definition 2.1** [4] A $\beta$-algebra is a non-empty set $X$ with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

1. $x - 0 = x$.
2. $(0 - x) + x = 0$.
3. $(x - y) - z = x - (z + y) \forall x, y, z \in X$.

**Example 2.2** Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with constant 0 and two binary operations $+$ and $-$ defined by the Cayley tables:

\[
\begin{array}{cccccc}
+ & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 0 & 4 & 5 & 2 & 3 \\
2 & 2 & 5 & 0 & 4 & 3 & 1 \\
3 & 3 & 4 & 5 & 0 & 1 & 2 \\
4 & 4 & 3 & 1 & 2 & 5 & 0 \\
5 & 5 & 2 & 3 & 1 & 0 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccc}
- & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 0 & 4 & 5 & 3 & 2 \\
2 & 2 & 5 & 0 & 4 & 1 & 3 \\
3 & 3 & 4 & 5 & 0 & 2 & 1 \\
4 & 4 & 3 & 1 & 2 & 0 & 5 \\
5 & 5 & 2 & 3 & 1 & 4 & 0 \\
\end{array}
\]

Then $(X, +, -, 0)$ is a $\beta$-algebra.

**Definition 2.3** Let $X$ be any non-empty set. A $L$-fuzzy set $\mu$ on $X$ is defined as a function $\mu : X \rightarrow L$, where $L$ is a complete lattice with glb 0 and lub 1.

**Definition 2.4** Let $\mu$ be a fuzzy set in a set $X$. For $t \in [0, 1]$, then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called a level subset of $\mu$.

**Definition 2.5** [1] Let $\mu$ be a fuzzy set in a $\beta$-algebra $X$. Then $\mu$ is called a $L$-fuzzy $\beta$-subalgebra of $X$ if

1. $\mu(x + y) \geq \mu(x) \land \mu(y) \forall x, y \in X$.
2. $\mu(x - y) \geq \mu(x) \land \mu(y) \forall x, y \in X$. 
3 \hspace{1em} \textit{L– Fuzzy T- ideals of }\beta\text{–algebras}

In this section we introduce the notion of \textit{L–fuzzy T–ideals of }\beta\text{–algebras} and prove some simple theorems.

\textbf{Definition 3.1} A non-empty subset \(I\) of a \(\beta\)-algebra of \((X,+,−,0)\) is called \(T\)- ideal of \(X\) if the following conditions are satisfied.

1. \(0 \in I\)
2. \((x + y) + z \in I\) and \(y \in I \Rightarrow (x + z) \in I\) and
3. \((x − y) − z \in I\) and \(y \in I \Rightarrow (x − z) \in I\) \(\forall x, y, z \in I\).

\textbf{Example 3.2} In example 2.2 of the \(\beta\)-algebra of \(X\), \(I_1 = \{0, 2\}\) is a \(T\)-ideal of \(X\) while \(I_2 = \{0, 4\}\) is not a \(T\)-ideal of \(X\) - for, \((0+4)+4 = 4+4 = 5 \notin I_2\).

\textbf{Definition 3.3} Let \(\mu\) be an \(L\)-fuzzy set in a \(\beta\)-algebra of \(X\). Then \(\mu\) is called an \(L\)-fuzzy \(T\)- ideal of \(X\) if

1. \(\mu(0) \geq \mu(x)\).
2. \(\mu(x + z) \geq \mu((x + y) + z) \land \mu(y)\) and
3. \(\mu(x − z) \geq \mu((x − y) − z) \land \mu(y)\) \(\forall x, y, z \in X\).

\textbf{Example 3.4} In the \(\beta\)-algebra \(X\) of example 2.2, the fuzzy set \(\mu_1 : X \rightarrow [0,1]\) defined by

\[
\mu_1(x) = \begin{cases} 
    t_5 & \text{if} \quad x = 0 \\
    t_4 & \text{if} \quad x = 1 \\
    t_3 & \text{if} \quad x = 2 \\
    t_2 & \text{if} \quad x = 3 \\
    t_1 & \text{if} \quad x = 4, 5 
\end{cases}
\]

where \(0 \leq t_1 < t_2 < t_3 < t_4 < t_5 \leq 1, t_1, t_2, t_3, t_4, t_5 \in L\) is an \(L\)-fuzzy \(T\)-ideal of \(X\).

\textbf{Lemma 3.5} Let \(\mu\) be an \(L\)-fuzzy \(T\)-ideal of a \(\beta\)-algebra \(X\). If \(x \leq y\) then \(\mu(x) \geq \mu(y)\).

\textbf{Proof:} For \(x, y \in X, \; x \leq y \Rightarrow x − y = 0\). Then

\[
\mu(x) = \mu(x - 0) \geq \mu((x - y) - 0) \land \mu(y) = \mu(0 - 0) \land \mu(y) = \mu(0) \land \mu(y) = \mu(y).
\]

Hence \(\mu(x) \geq \mu(y)\).
Theorem 3.6 Let $A$ be a subset of $X$. Define an $L-$fuzzy set $\mu :X \rightarrow [0,1]$ such that

$$\mu(x) = \begin{cases} 
  t_0 & \text{if } x \in A \\
  t_1 & \text{if } x \notin A 
\end{cases}$$

where $t_0, t_1 \in [0,1]$ with $t_0 > t_1$. Then $\mu$ is an $L-$fuzzy $T-$ideal of a $\beta-$algebra $X$ if and only if $A$ is a $T-$ideal of $X$.

**Proof:** Assume that $\mu$ is an $L-$fuzzy $T-$ideal of $X$.
If $x \in A, \mu(0) \geq t_0.$
and if $x \notin A, \mu(0) \geq t_1$ since $t_0 > t_1$.
$\Rightarrow \mu(0) \geq t_0 > t_1$
$\Rightarrow \mu(0) = t_0$
$\Rightarrow 0 \in A.$
For $x, y, z \in A \Rightarrow \mu(x) = t_0, \mu(y) = t_0$ and $\mu(z) = t_0.$

$$\mu(x + z) \geq \mu((x + y) + z) \wedge \mu(y)$$
$$\geq (\mu(x + y) \wedge \mu(z)) \wedge \mu(y)$$
$$= (((\mu(x) \wedge \mu(y)) \wedge \mu(z))) \wedge \mu(y)$$
$$= t_0 \wedge t_0 \wedge t_0$$
$$= t_0$$

Therefore $\mu(x + z) \geq t_0 \Rightarrow x + z \in A.$
For $x, y, z \in A \Rightarrow \mu((x - y) - z) = t_0$ and $\mu(y) = t_0.$
Now $\mu(x - z) \geq \mu((x - y) - z) \wedge \mu(y) = t_0 \wedge t_0 = t_0$
$\Rightarrow \mu(x - z) = t_0 \Rightarrow x - z \in A.$
Hence $A$ is a $T-$ideal of $X$.
Conversely, Suppose $A$ is a $T-$ideal of $X$.
Now $0 \in A \Rightarrow \mu(0) = t_0.$
Also $\forall x \in X, Im(\mu) = \{t_0, t_1\}$ and $t_0 > t_1$
$\Rightarrow \mu(0) \geq \mu(x).$
Since $A$ is a $T-$ideal of $X$,
$\forall x, y, z \in S, (x + y) + z \in A$ and $y \in A \Rightarrow x + z, x - z \in A.$
Then $\mu(x + z) = t_0 \geq \mu((x + y) + z) \wedge \mu(y).$
Similarly we can prove that $\mu(x - z) \geq \mu((x - y) - z) \wedge \mu(y).$
Hence $\mu$ is an $L-$fuzzy $T-$ideal of $X$.

Theorem 3.7 An $L-$fuzzy set $\mu$ is a $T-$ideal of $X$ if and only if the non empty level subset $\mu_t$ is a $T-$ideal of $X, \forall t \in [0,1].$

**Proof:** Assume that $\mu$ is an $L-$fuzzy $T-$ideal of $X$.
Now $\mu(0) \geq \mu(x) \ \forall x \in X$
$\Rightarrow \mu(0) \geq t$ for any $t \in [0,1] \Rightarrow 0 \in \mu_t, \forall t \in [0,1].$
For any $t \in [0,1], \mu_t \neq \emptyset.$
For any $x, y, z \in \mu_t$, we have $\mu((x + y) + z) \geq t$ and $\mu(y) \geq t$.

Now $\mu(x + z) \geq \mu((x + y) + z) \wedge \mu(y) \geq t \Rightarrow x + z \in \mu_t$.

Also we have $\mu((x - y) - z) \geq t$ and $\mu(y) \geq t$.

Hence $\mu((x - y) - z) \geq \mu((x - y) - z) \wedge \mu(y) \geq t \wedge t = t \Rightarrow x - z \in \mu_t$.

Hence $\mu_t$ is a $T$-ideal of $X$.

Conversely assume that each non-empty level subset $\mu_t$ of a fuzzy subset $\mu$ of $X$ is a $T$-ideal of $X$.

Then we claim that $\mu$ is an $L$-fuzzy $T$-ideal of $X$.

For any $x \in X$, let $\mu(x) = t$. Since $\mu_t$ is a $T$-ideal of $X$, $0 \in \mu_t$.

$\Rightarrow \mu(0) \geq \mu(x)$, $\forall x \in X$.

Choose $x, y, z \in X$, such that $\mu((x + y) + z) = t_1$ and $\mu(y) = t_2$, where $t_1, t_2 \in [0, 1]$. Then $x + z \in \mu_{t_1}$ and $y \in \mu_{t_2}$.

Assume $t_1 \leq t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$, hence $y \in \mu_{t_1}$.

Since $\mu_t$ is a $T$-ideal of $X$, we have $x + z \in \mu_{t_1}$.

Thus $\mu(x + z) \geq t_1 = \mu((x + y) + z) \wedge \mu(y)$.

Similarly we can prove that $\mu((x - y) - z) \geq \mu((x - y) - z) \wedge \mu(y)$.

Therefore $\mu$ is an $L$-fuzzy $T$-ideal of $X$.

**Theorem 3.8** Let $\mu_1$ and $\mu_2$ be two $L$-fuzzy $T$-ideals in a $\beta$-algebra $X$.

Then the direct product $\mu_1 \times \mu_2$ is an $L$-fuzzy $T$-ideals in $X_1 \times X_2$.

**Proof** For any $(x, y) \in X_1 \times X_2$ we have

$$(\mu_1 \times \mu_2)(0, 0) = \mu_1(0) \wedge \mu_2(0)$$

$$\geq \mu_1(x) \wedge \mu_2(y)$$

$$= (\mu_1 \times \mu_2)(x, y)$$

Let $(x_1, x_2), (y_1, y_2)$ and $(z_1, z_2) \in X_1 \times X_2$. Then

$$(\mu_1 \times \mu_2)((x_1, x_2) + (z_1, z_2))$$

$$= (\mu_1 \times \mu_2)((x_1 + z_1, x_2 + z_2))$$

$$= \mu_1(x_1 + z_1) \wedge \mu_2(x_2 + z_2)$$

$$\geq (\mu_1((x_1 + y_1) + z_1) \wedge \mu_1(y_1)) \wedge (\mu_2((x_2 + y_2) + z_2) \wedge \mu_2(y_2))$$

$$= (\mu_1((x_1 + y_1) + z_1) \wedge \mu_2((x_2 + y_2) + z_2)) \wedge (\mu_1(y_1) \wedge \mu_2(y_2))$$

$$= (\mu_1 \times \mu_2)((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$$

$$= (\mu_1 \times \mu_2)(((x_1 + y_1), (x_2 + y_2)) + (z_1, z_2)) \wedge (\mu_1 \times \mu_2)(y_1, y_2)$$

$$= (\mu_1 \times \mu_2)(((x_1, x_2), (y_1, y_2)) + (z_1, z_2)) \wedge (\mu_1 \times \mu_2)(y_1, y_2)$$

Similarly we can prove that

$$(\mu_1 \times \mu_2)((x_1, x_2) - (z_1, z_2)) \geq (\mu_1 \times \mu_2)(((x_1, x_2), (y_1, y_2)) + (z_1, z_2)) \wedge (\mu_1 \times \mu_2)(y_1, y_2)$$

Hence $\mu_1 \times \mu_2$ is an $L$-fuzzy $T$-ideal of a $\beta$-algebra in $X_1 \times X_2$. 

Theorem 3.9 Let $\mu_1$ and $\mu_2$ be two fuzzy sets in a $\beta-$ algebra $X$ such that $\mu_1 \times \mu_2$ is an $L-$ fuzzy $T$-ideal of $X_1 \times X_2$. Then

1. Either $\mu_1(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$ $\forall x \in X$.

2. If $\mu_1(0) \geq \mu_1(x)$, $\forall x \in X$ then either $\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$.

3. If $\mu_2(0) \geq \mu_2(x)$, $\forall x \in X$ then either $\mu_1(0) \geq \mu_1(x)$ or $\mu_1(0) \geq \mu_2(x)$.

4. Either $\mu_1$ or $\mu_2$ is a $L-$ fuzzy $T$-ideal of $X$.

Proof: Let $\mu_1 \times \mu_2$ is an $L-$ fuzzy $T$-ideal of $X_1 \times X_2$.

Suppose that $\mu_1(0) < \mu_1(x)$ and $\mu_2(0) < \mu_2(y)$ for some $x, y \in X$. Then

$$(\mu_1 \times \mu_2)(x, y) = \mu_1(x) \land \mu_2(y) \geq \mu_1(0) \land \mu_2(0) = (\mu_1 \times \mu_2)(0, 0)$$

This contradiction yields that either $\mu_1(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$ $\forall x \in X$. Given $\mu_1(0) \geq \mu_1(x)$, $\forall x \in X$ and assume that there exist $x, y \in X$ such that $\mu_2(0) < \mu_1(x)$ and $\mu_2(0) < \mu_2(y)$ $\forall x, y \in X$.

Now $\mu_2(0) < \mu_1(x) \leq \mu_1(0) \Rightarrow \mu_2(0) < \mu_1(0)$.

Then $(\mu_1 \times \mu_2)(0, 0) = \mu_1(0) \land \mu_2(0) = \mu_2(0)$.

$(\mu_1 \times \mu_2)(x, y) = \mu_1(x) \land \mu_2(y) \geq \mu_2(0) \land \mu_2(0) = \mu_2(0) = (\mu_1 \times \mu_2)(0, 0)$

Thus $(\mu_1 \times \mu_2)(x, y) \geq (\mu_1 \times \mu_2)(0, 0)$ which is a contradiction.

Hence if $\mu_1(0) \geq \mu_1(x)$, $\forall x \in X$ then either $\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$.

Similarly we can prove that if $\mu_2(0) \geq \mu_2(x)$, $\forall x \in X$ then either $\mu_1(0) \geq \mu_1(x)$ or $\mu_1(0) \geq \mu_2(x)$.

First we prove that $\mu_2$ is a $L-$ fuzzy $T$-ideal of $X$.

Assume that $\mu_2(0) \geq \mu_2(x)$ $\forall x \in X$.

Then it follows that either $\mu_1(0) \geq \mu_1(x)$ or $\mu_1(0) \geq \mu_2(x)$.

If $\mu_1(0) \geq \mu_2(x)$ for any $x \in X$. Then

$$\begin{align*}
\mu_2(x) & \geq \mu_1(0) \land \mu_2(x) \\
& = (\mu_1 \times \mu_2)(0, x) \\
\mu_2(x + z) & \geq \mu_1(0) \land \mu_2(x + z) \\
& = (\mu_1 \times \mu_2)(0, x + z) \\
& = (\mu_1 \times \mu_2)(0 + 0, x + z) \\
& = (\mu_1 \times \mu_2)((0, x) + (0, z)) \\
& \geq (\mu_1 \times \mu_2)(((0, x) + (0, y)) + (0, z)) \land (\mu_1 \times \mu_2)(0, y) \\
& = (\mu_1 \times \mu_2)(((0 + 0), (x + y)) + (0, z)) \land (\mu_1 \times \mu_2)(0, y) \\
& = (\mu_1 \times \mu_2)(((0 + 0) + 0, (x + y) + z)) \land (\mu_1 \times \mu_2)(0, y) \\
& = (\mu_1 \times \mu_2)((0, (x + y) + z)) \land (\mu_1 \times \mu_2)(0, y) \\
& = \mu_2((x + y) + z) \land \mu_2(y)
\end{align*}$$
Similarly we can prove that \( \mu_2(x - z) \geq \mu_2((x - y) - z)) \land \mu_2(y) \).
Hence \( \mu_2 \) is a \( L^{-}\) fuzzy \( T\)-ideal of \( X \).
Similarly we can prove that \( \mu_1 \) is a \( L^{-}\) fuzzy \( T\)-ideal of \( X \).

**Theorem 3.10** Let \( f : X \rightarrow X \) be an endomorphism on a \( \beta^{-}\) algebra. Let \( \mu \) be an \( L^{-}\) fuzzy \( T\)-ideal of \( X \). Define a fuzzy set \( \mu_f : X \rightarrow [0,1] \) defined by \( \mu_f(x) = \mu(f(x)), \forall x \in X \). Then \( \mu_f \) is an \( L^{-}\) fuzzy \( T\)-ideal of \( X \).

**Proof:** Let \( x \in X \). Then \( \mu_f(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = \mu_f(0) \).
Let \( x, y \in X \).

\[
\begin{align*}
\mu_f(x + z) &= \mu(f(x + z)) = \mu(f(x) + f(z)) \\
&\geq \mu((f(x) + f(y)) + f(z)) \land \mu(f(y)) \\
&= \mu((f(x + y)) + f(z)) \land \mu(f(y)) \\
&= \mu(f((x + y) + z)) \land \mu(f(y)) \\
&= \mu_f((x + y) + z)) \land \mu_f(y) 
\end{align*}
\]
Similarly we can prove that \( \mu_f(x - z) \geq \mu_f((x - y) - z)) \land \mu_f(y) \).
Hence \( \mu_f \) is an \( L^{-}\) fuzzy \( T\) - ideal of \( X \).

**References**


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