Characterizations of Fuzzy Subalgebras in $BCK/BCI$-Algebras

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Abstract
The concepts of $(\in, \in \vee q_0^\delta)$-fuzzy subalgebras and $\in \vee q_0^\delta$-level sets
are introduced, and related properties are investigated. Relations between an \((\in, \in)\)-fuzzy subalgebra and an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra are discussed, and characterizations of \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebras are discussed. Homomorphic image and pre-image of an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra are considered.

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1 Introduction

Murali [7] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [8], played a vital role to generate some different types of fuzzy algebraic structures. It is worth pointing out that Bhakat and Das [1, 2] gave the concepts of \((\alpha, \beta)\)-fuzzy subgroups by using the “belongs to” relation \((\in)\) and “quasi-coincident with” relation \((q)\) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an \((\in, \in \lor q)\)-fuzzy subgroup. In particular, \((\in, \in \lor q)\)-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. Also, Jun [3, 4] considered the concepts of \((\alpha, \beta)\)-fuzzy subalgebras by using the “belongs to” relation \((\in)\) and “quasi-coincident with” relation \((q)\) between a fuzzy point and a fuzzy subalgebra, and introduced the concept of an \((\in, \in \lor q)\)-fuzzy subalgebra. As a general form of “quasi-coincident with” relation \((q)\), Jun et al. [5] introduced the concept of “\(\delta\)-quasi-coincident with” relation \((q_0^\delta)\), and apply it to fuzzy subgroups.

In this paper, we apply this new notion to \(BCK/BCI\)-algebras. We introduce the notion of \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebras, which is a generalization of \((\in, \in \lor q)\)-fuzzy subalgebras, and investigate related properties. We discuss relations between an \((\in, \in)\)-fuzzy subalgebra and an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra. We give a condition for an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra to be an \((\in, \in)\)-fuzzy subalgebra. We provide characterizations of an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra. We consider the homomorphic (pre) image of an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra.

2 Preliminaries

By a \(BCI\)-algebra we mean an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the axioms:

(i) \(\forall x, y, z \in X)((x * y) * (x * z)) * (z * y) = 0)\),
Characterizations of fuzzy subalgebras

(ii) \((\forall x, y \in X) ((x * (x * y)) * y = 0),\)

(iii) \((\forall x \in X) (x * x = 0),\)

(iv) \((\forall x, y \in X) (x * y = y * x = 0 \Rightarrow x = y).\)

We can define a partial ordering \(\leq\) by \(x \leq y\) if and only if \(x * y = 0\). If a BCI-algebra \(X\) satisfies \(0 * x = 0\) for all \(x \in X\), then we say that \(X\) is a BCK-algebra. A nonempty subset \(S\) of a BCK/BCI-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). We refer the reader to the books [6] for further information regarding BCK/BCI-algebras.

A fuzzy set \(\lambda\) in a set \(X\) of the form

\[
\lambda(y) := \begin{cases} 
  t \in (0, 1] & \text{if } y = x, \\
  0 & \text{if } y \neq x,
\end{cases}
\]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \(x_t\).

For a fuzzy point \(x_t\) and a fuzzy set \(\lambda\) in a set \(X\), Pu and Liu [8] gave meaning to the symbol \(x_t \alpha \lambda\), where \(\alpha \in \{\in, q, \in \vee q, \in \wedge q\}\).

To say that \(x_t \in \lambda\) (resp. \(x_t q \lambda\)) means that \(\lambda(x) \geq t\) (resp. \(\lambda(x) + t > 1\)), and in this case, \(x_t\) is said to belong to (resp. be quasi-coincident with) a fuzzy set \(\lambda\).

To say that \(x_t \in \in \vee q \lambda\) (resp. \(x_t \in \in \wedge q \lambda\)) means that \(x_t \in \lambda\) or \(x_t q \lambda\) (resp. \(x_t \in \lambda\) and \(x_t q \lambda\)).

Jun et al. [5] considered a general form of quasi-coincident fuzzy point. Let \(\delta \in (0, 1]\). For a fuzzy point \(x_t\) and a fuzzy set \(\lambda\) in a set \(X\), we say that \(x_t\) is a \(\delta\)-quasi-coincident with \(\lambda\), written \(x_t q_0^\delta \lambda\), (see [5]) if \(\lambda(x) + t > \delta\).

Obviously, \(x_t q \lambda\) implies \(x_t q_0^\delta \lambda\). If \(\delta = 1\), then the \(\delta\)-quasi-coincident with \(\lambda\) is the quasi-coincident with \(\lambda\), that is, \(x_t q_0^1 \lambda = x_t q \lambda\).

To say that \(x_t \in \in \vee q_0^\delta \lambda\) (resp. \(x_t \in \in \wedge q_0^\delta \lambda\)) means that \(x_t \in \lambda\) or \(x_t q_0^\delta \lambda\) (resp. \(x_t \in \lambda\) and \(x_t q_0^\delta \lambda\)).

3 Generalizations of \((\in, \in \vee q)\)-fuzzy subalgebras

In what follows let \(\delta\) and \(X\) denote an element of \((0, 1]\) and a BCK/BCI-algebra, respectively, unless otherwise specified.

**Definition 3.1** A fuzzy set \(\lambda\) in \(X\) is called an \((\alpha, \beta)\)-fuzzy subalgebra of \(X\) if for all \(x, y \in X\) and \(t, r \in (0, \delta]\),

\[
x_t \alpha \lambda, \ y_r \alpha \lambda \Rightarrow (x * y)_{\min\{t,r\}}^\beta \lambda, \tag{1}
\]

where \(\alpha \in \{\in, q, q_0^\delta\}\) and \(\beta \in \{\in, q_0^\delta, \in \vee q_0^\delta\}\).
We say that $x_t \overline{\alpha} \lambda$ if $x_t \alpha \lambda$ does not hold.

**Example 3.2** Let $X = \{0, a, b, c\}$ be a BCI-algebra in which the operation $*$ is described by Table 1.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
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<td>b</td>
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<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Cayley table of the operation $*$

Define a fuzzy set $\lambda$ in $X$ as follows:

$$\lambda : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.45 & \text{if } x = 0, \\ 0.74 & \text{if } x = b, \\ 0.35 & \text{if } x \in \{a, c\}. \end{cases}$$

Then $\lambda$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$ with $\delta \in (0, 0.9]$. If $\delta \in (0.9, 1]$, then $\lambda$ is not an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$.

It is obvious that if $\delta_1 \geq \delta_2$ in $(0, 1]$, then every $(\in, \in \vee q_0^{\delta_1})$-fuzzy subalgebra is an $(\in, \in \vee q_0^{\delta_2})$-fuzzy subalgebra, but the converse is not true as seen in Example 3.2.

**Proposition 3.3** If $\lambda$ is a nonzero $(q_0^\delta, \in)$ (or $(q_0^\delta, q_0^\delta)$)-fuzzy subalgebra of $X$, then the set

$$X_0 := \{x \in X \mid \lambda(x) > 0\}$$

is a subalgebra of $X$.

**Proof.** Let $x, y \in X_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Hence $\lambda(x) + \delta > \delta$ and $\lambda(y) + \delta > \delta$, that is, $x \delta q_0^\delta \lambda$ and $y \delta q_0^\delta \lambda$. It follows from (1) that $(x * y)_\delta \in \lambda$, i.e., $\lambda(x * y) \geq \delta > 0$. Thus $x * y \in X_0$. Thus $X_0$ is a subalgebra of $X$. For $(q_0^\delta, q_0^\delta)$-fuzzy subalgebra case, we can prove similarly.

**Proposition 3.4** Let $S$ be a subalgebra of $X$ and let $\lambda$ be a fuzzy set in $X$ such that

(i) $\lambda(x) \geq \frac{\delta}{2}$ for all $x \in S$,

(ii) $\lambda(x) = 0$ for all $x \in X \setminus S$. 
Then \( \lambda \) is a \((q_0^\delta, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

**Proof.** Let \( x, y \in X \) and \( t_1, t_2 \in (0, \delta] \) such that \( x_{t_1} q_0^\delta \lambda \) and \( y_{t_2} q_0^\delta \lambda \). Then 
\[
\lambda(x) + t_1 > \delta \quad \text{and} \quad \lambda(y) + t_2 > \delta,
\]
which imply that \( x, y \in S \). Hence \( x \ast y \in S \), and so 
\[
\lambda(x \ast y) \geq \frac{\delta}{2}.
\]
If \( \min\{t_1, t_2\} > \frac{\delta}{2} \), then \( \lambda(x \ast y) + \min\{t_1, t_2\} > \delta \), i.e., 
\[
(x \ast y)_{\min\{t_1, t_2\}} q_0^\delta \lambda.
\]
If \( \min\{t_1, t_2\} \leq \frac{\delta}{2} \), then \( \lambda(x \ast y) \geq \frac{\delta}{2} \geq \min\{t_1, t_2\} \) and so 
\[
(x \ast y)_{\min\{t_1, t_2\}} \in \lambda.
\]
Therefore \( (x \ast y)_{\min\{t_1, t_2\}} \in \lor q_0^\delta \lambda \), and \( \lambda \) is a \((q_0^\delta, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

**Theorem 3.5** For any fuzzy set \( \lambda \) in \( X \), the following are equivalent.

(i) \( \lambda \) is an \((\in, \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

(ii) \( (\forall x, y \in X) \left( \lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} \right) \).

**Proof.** Assume that \( \lambda \) is an \((\in, \lor q_0^\delta)\)-fuzzy subalgebra of \( X \). For any \( x, y \in X \), we consider two cases:

1. \( \min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2} \) and 
2. \( \min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2} \).

For the first case, suppose that 
\[
\lambda(x \ast y) < \min\{\lambda(x), \lambda(y)\}
\]
and take \( t \in (0, \frac{\delta}{2}] \) such that 
\[
\lambda(x \ast y) < t \leq \min\{\lambda(x), \lambda(y)\}.
\]
Then \( x_t \in \lambda \) and \( y_t \in \lambda \) but 
\[
(x \ast y)_t = (x \ast y)_t \lor q_0^\delta \lambda
\]
since \( (x \ast y)_t \in \lambda \) and \( \lambda(x \ast y) + t < 2t < \delta \), 
that is, \( (x \ast y)_t \lor q_0^\delta \lambda \). This is a contradiction. Hence \( \lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y)\} \)
whenever \( \min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2} \). Now assume that the case (2) is valid. Then 
\[
x_{\frac{\delta}{2}} \in \lambda \quad \text{and} \quad y_{\frac{\delta}{2}} \in \lambda,
\]
which imply that 
\[
(x \ast y)_{\frac{\delta}{2}} = (x \ast y)_{\min\{\frac{\delta}{2}, \frac{\delta}{2}\}} \lor q_0^\delta \lambda.
\]
If 
\[
\lambda(x \ast y) < \frac{\delta}{2},
\]
then 
\[
\lambda(x \ast y) + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta
\]
and so \( \lambda(x \ast y) < \frac{\delta}{2} \) which shows that 
\[
(x \ast y)_{\frac{\delta}{2}} \lor q_0^\delta \lambda
\]
This is a contradiction, and thus \( \lambda(x \ast y) \geq \frac{\delta}{2} \).

Therefore \( \lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} \) for all \( x, y \in X \).

Conversely assume that (ii) is valid. Let \( x, y \in X \) and \( t, r \in (0, \delta] \) such that 
\( x_t \in \lambda \) and \( y_r \in \lambda \). Then 
\( \lambda(x) \geq t \) and \( \lambda(y) \geq r \). Suppose that 
\( \lambda(x \ast y) < \min\{t, r\} \). If 
\( \min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2} \), then
\[
\lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, r\},
\]
a contradiction. Hence \( \min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2} \), and so
\[
\lambda(x \ast y) + \min\{t, r\} > 2\lambda(x \ast y) \geq 2\min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} = \delta.
\]
This shows that 
\( (x \ast y)_{\min\{t, r\}} q_0^\delta \lambda \). Therefore \( \lambda \) is an \((\in, \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

Obviously, every \((\in, \lor)\)-fuzzy subalgebra is an \((\in, \lor q_0^\delta)\)-fuzzy subalgebra, but the converse is not true in general. In fact, the \((\in, \lor q_0^\delta)\)-fuzzy subalgebra \( \lambda \) with \( \delta \in (0, 0.9] \) in Example 3.2 is not an \((\in, \lor)\)-fuzzy subalgebra of \( X \) since 
\( b_{0.6} \in \lambda \) and \( b_{0.7} \in \lambda \) but \( (b \ast b)_{\min\{0.6, 0.7\}} = 0_{0.6} \).
Proposition 3.6 Let $\lambda$ be an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$ such that $\lambda(x) < \frac{\delta}{2}$ for all $x \in X$. Then $\lambda$ is an $(\in, \in)$-fuzzy subalgebra of $X$.

Proof. Let $x, y \in X$ and $t, r \in (0, \delta]$ be such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. It follows from the hypothesis and Theorem 3.5 that

$$
\lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} = \min\{\lambda(x), \lambda(y)\} \geq \min\{t, r\}
$$

so that $(x \ast y)_{\min\{t, r\}} \in \lambda$. Hence $\lambda$ is an $(\in, \in)$-fuzzy subalgebra of $X$.

For a subset $S$ of $X$, a fuzzy set $\chi_S^\delta$ in $X$ defined by

$$
\chi_S^\delta : X \rightarrow [0, 1], \ x \mapsto \begin{cases} 
\delta & \text{if } x \in S, \\
0 & \text{otherwise,}
\end{cases}
$$

is called a $\delta$-characteristic fuzzy set of $S$ in $X$ (see [5]).

Theorem 3.7 For any subset $S$ of $X$ and the $\delta$-characteristic fuzzy set $\chi_S^\delta$ of $S$ in $X$, the following are equivalent:

(i) $S$ is a subalgebra of $X$.

(ii) $\chi_S^\delta$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$.

Proof. (i) $\Rightarrow$ (ii) is straightforward.

Assume that $\chi_S^\delta$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$. Let $x, y \in S$. Then $\chi_S^\delta(x) = \delta = \chi_S^\delta(y)$, and so $x_{\delta} \in \chi_S^\delta$ and $y_{\delta} \in \chi_S^\delta$. It follows that $(x \ast y)_{\delta} = (x \ast y)_{\min\{t, r\}} \in \vee q_0^\delta \lambda$, which yields $\chi_S^\delta(x \ast y) > 0$. Hence $x \ast y \in S$ and $S$ is a subalgebra of $X$.

Theorem 3.8 A fuzzy set $\lambda$ in $X$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$ if and only if the set

$$
U(\lambda; t) := \{x \in X \mid \lambda(x) \geq t\}
$$

is a subalgebra of $X$ for all $t \in (0, \frac{\delta}{2}]$.

Proof. Assume that $\lambda$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$. Let $t \in (0, \frac{\delta}{2}]$ and $x, y \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 3.5 that

$$
\lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t
$$

and so that $x \ast y \in U(\lambda; t)$. Therefore $U(\lambda; t)$ is a subalgebra of $X$.

Conversely, let $\lambda$ be a fuzzy set in $X$ such that $U(\lambda; t)$ is a subalgebra of $X$ for all $t \in (0, \frac{\delta}{2}]$. Suppose that there are elements $a$ and $b$ of $X$ such that
\[ \lambda(a * b) < \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}, \]

and take \( t \in (0, \delta] \) such that \( \lambda(a * b) < t \leq \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\} \). Then \( a, b \in U(\lambda; t) \) and \( t \leq \frac{\delta}{2} \), which implies that \( a * b \in U(\lambda; t) \) since \( U(\lambda; t) \) is a subalgebra of \( X \). This induces \( \lambda(a * b) \geq t \), and this is a contradiction. Hence \( \lambda(x * y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} \) for all \( x, y \in X \), and therefore \( \lambda \) is an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \) by Theorem 3.5.

We say that the subalgebra \( U(\lambda; t) \) in Theorem 3.8 is a level subalgebra of \( X \).

**Theorem 3.9** Let \( \{\lambda_i \mid i \in \Lambda\} \) be a family of \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebras of \( X \). Then \( \lambda := \bigcap_{i \in \Lambda} \lambda_i \) is an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

**Proof.** Suppose that \( x_i \in \lambda \) and \( y_i \in \lambda \) for all \( x, y \in X \) and \( t, r \in (0, \delta] \).

Assume that \( (x * y)_{\min\{t, r\}} \in q_0^\delta \lambda \). Then \( \lambda(x * y) < \min\{t, r\} \) and \( \lambda(x * y) + \min\{t, r\} \leq \delta \), which imply that

\[ \lambda(x * y) < \frac{\delta}{2} \]

(2)

Let \( \Omega_1 := \{i \in \Lambda \mid (x * y)_{\min\{t, r\}} \in \lambda_i\} \) and

\[ \Omega_2 := \{i \in \Lambda \mid (x * y)_{\min\{t, r\}} q_0^\delta \lambda_i \} \cap \{j \in \Lambda \mid (x * y)_{\min\{t, r\}} \in \lambda_j\}. \]

Then \( \Lambda = \Omega_1 \cup \Omega_2 \) and \( \Omega_1 \cap \Omega_2 = \emptyset \). If \( \Omega_2 = \emptyset \), then \( (x * y)_{\min\{t, r\}} \in \lambda_i \) for all \( i \in \Lambda \), that is, \( \lambda_i(x * y) \geq \min\{t, r\} \) for all \( i \in \Lambda \), which yields \( \lambda(x * y) \geq \min\{t, r\} \). This is a contradiction. Hence \( \Omega_2 \neq \emptyset \), and so for every \( i \in \Omega_2 \) we have \( \lambda_i(x * y) < \min\{t, r\} \) and \( \lambda_i(x * y) + \min\{t, r\} > \delta \). It follows that \( \min\{t, r\} > \frac{\delta}{2} \).

Now \( x_i \in \lambda \) implies \( \lambda_i(x) \geq t \) and thus \( \lambda_i(x) \geq \lambda(x) \geq t \geq \min\{t, r\} > \frac{\delta}{2} \) for all \( i \in \Lambda \). Similarly we get \( \lambda_i(y) > \frac{\delta}{2} \) for all \( i \in \Lambda \). Next suppose that \( t := \lambda_i(x * y) < \frac{\delta}{2} \).

Taking \( t < r < \frac{\delta}{2} \), we get \( x_i \in \lambda_i \) and \( y_i \in \lambda_i \), but \( (x * y)_{\min\{t, r\}} = (x * y)_r \in q_0^\delta \lambda_i \). This contradicts that \( \lambda_i \) is an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \). Hence \( \lambda_i(x * y) \geq \frac{\delta}{2} \) for all \( i \in \Lambda \), and so \( \lambda(x * y) \geq \frac{\delta}{2} \) which contradicts (2). Therefore \( (x * y)_{\min\{t, r\}} \in q_0^\delta \lambda \) and consequently \( \lambda \) is an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

The following example shows that the union of \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebras of \( X \) may not be an \((\in, \in \lor q_0^\delta)\)-fuzzy subalgebra of \( X \).

**Example 3.10** Let \( X = \{0, 1, a, b\} \) be a BCI-algebra in which the operation \( * \) is described by Table 2.

Define fuzzy sets \( \lambda_1 \) and \( \lambda_2 \) in \( X_1 \) as follows:

\[
\lambda_1 : X \to [0, 1], \quad x \mapsto \begin{cases} 
0.48 & \text{if } x \in \{0, 1\}, \\
0.20 & \text{if } x \in \{a, b\}, 
\end{cases}
\]

\[
\lambda_2 : X \to [0, 1], \quad x \mapsto \begin{cases} 
0.50 & \text{if } x \in \{0, 1\}, \\
0.30 & \text{if } x \in \{a, b\}, 
\end{cases}
\]

\[
\lambda_1(x * y) = \min\{\lambda_1(x), \lambda_1(y), \frac{\delta}{2}\} = \min\{0.48, 0.50, \frac{\delta}{2}\} = \min\{0.48, \frac{\delta}{2}\} = \frac{\delta}{2} < \delta.
\]

This shows that \( \lambda_1 \) and \( \lambda_2 \) do not satisfy the condition for being a fuzzy subalgebra of \( X \).
Table 2: Cayley table of the operation $\ast$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>a</th>
<th>b</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

and

$$\lambda_2 : X \to [0,1], \; x \mapsto \begin{cases} 
0.5 & \text{if } x \in \{0,a\}, \\
0.3 & \text{if } x \in \{1,b\}.
\end{cases}$$

Then $\lambda_1$ and $\lambda_2$ are $(\in, \in \cup \triangledown q_0^{0.9})$-fuzzy subalgebras of $X$. The union $\lambda_1 \cup \lambda_2$ of $\lambda_1$ and $\lambda_2$ is described as follows:

$$\lambda_1 \cup \lambda_2 : X \to [0,1], \; x \mapsto \begin{cases} 
0.5 & \text{if } x \in \{0,a\}, \\
0.48 & \text{if } x = 1, \\
0.3 & \text{if } x = b,
\end{cases}$$

and it is not an $(\in, \in \cup \triangledown q_0^{0.9})$-fuzzy subalgebra of $X$ since

$$(\lambda_1 \cup \lambda_2)(1 \ast a) = (\lambda_1 \cup \lambda_2)(b) = 0.3 \not\geq 0.45 = \min\{(\lambda_1 \cup \lambda_2)(1), (\lambda_1 \cup \lambda_2)(a), 0.45\}.$$ 

**Theorem 3.11** Let $\{\lambda_i \mid i \in \Lambda\}$ be a family of $(\in, \in \cup q_0^\delta)$-fuzzy subalgebras of $X$ such that $\lambda_i \subseteq \lambda_j$ or $\lambda_j \subseteq \lambda_i$ for all $i, j \in \Lambda$ Then $\lambda := \bigcup_{i \in \Lambda} \lambda_i$ is an $(\in, \in \cup q_0^\delta)$-fuzzy subalgebra of $X$.

**Proof.** For all $x, y \in X$, we have

$$\lambda(x \ast y) = \left(\bigcup_{i \in \Lambda} \lambda_i\right)(x \ast y) \geq \bigvee_{i \in \Lambda} \lambda_i(x \ast y) \geq \bigvee_{i \in \Lambda} \min\{\lambda_i(x), \lambda_i(y), \frac{\delta}{2}\}$$

$$= \min\left\{\bigvee_{i \in \Lambda} \lambda_i(x), \bigvee_{i \in \Lambda} \lambda_i(y), \frac{\delta}{2}\right\} = \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}.$$ 

Therefore $\lambda$ is an $(\in, \in \cup q_0^\delta)$-fuzzy subalgebra of $X$.

**Definition 3.12** ([5]) Let $\lambda$ be a fuzzy set in $X$ and $t \in (0,1]$. Then the set

$$\Omega(\lambda; t) := \{x \in X \mid x \in \bigvee q_0^\delta \lambda\}$$

is called an $(\in \cup q_0^\delta)$-level set in $X$ determined by $\lambda$ and $t$. 

Using Theorem 3.11, we can obtain the following theorem.

**Theorem 3.13** A fuzzy set $\lambda$ in $X$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$ if and only if the set $\Omega(\lambda; t)$ is a subalgebra of $X$ for all $t \in (0, \delta]$.

**Proof.** We omit the proof.

**Proposition 3.14** Let $S$ be a subalgebra of $X$. For any $t \in (0, \frac{\delta}{2}]$, there exists an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra $\lambda$ of $X$ such that $U(\lambda; t) = S$.

**Proof.** Let $\lambda$ be a fuzzy set in $X$ defined by

$$\lambda(x) = \begin{cases} t & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$ where $t \in (0, \frac{\delta}{2}]$. Obviously, $U(\lambda; t) = S$. Assume that there exist $a, b \in X$ such that $\lambda(a \ast b) < \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$. Since $|\text{Im}(\lambda)| = 2$, it follows that $\lambda(a \ast b) = 0$ and $\min\{\lambda(a), \lambda(b), \frac{\delta}{2}\} = t$, and so $\lambda(a) = t = \lambda(b)$, so that $a, b \in S$ but $a \ast b \notin S$. This is a contradiction, and so $\lambda(x \ast y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in X$. Using Theorem 3.5, we know that $\lambda$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$.

**Theorem 3.15** Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras and let $\lambda$ and $\nu$ be $(\in, \in \vee q_0^\delta)$-fuzzy subalgebras of $X$ and $Y$, respectively. Then

(i) $f^{-1}(\nu)$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$.

(ii) If, for any subset $T$ of $X$, there exists $x_0 \in T$ such that

$$\lambda(x_0) = \bigvee\{\lambda(x) \mid x \in T\},$$

then $f(\lambda)$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $Y$ when $f$ is onto.

**Proof.**

(i) Let $x, y \in X$ and $t, r \in (0, \delta]$ be such that $x_t \in f^{-1}(\nu)$ and $y_r \in f^{-1}(\nu)$. Then $(f(x))_t \in \nu$ and $(f(y))_r \in \nu$. Since $\nu$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $Y$, it follows that

$$(f(x \ast y))_{\min\{t, r\}} = (f(x) \ast f(y))_{\min\{t, r\}} \in \bigvee q_0^\delta \nu$$

so that $(x \ast y)_{\min\{t, r\}} \in \bigvee q_0^\delta f^{-1}(\nu)$. Therefore $f^{-1}(\nu)$ is an $(\in, \in \vee q_0^\delta)$-fuzzy subalgebra of $X$.

(ii) Let $a, b \in Y$ and $t, r \in (0, \delta]$ be such that $a_t \in f(\lambda)$ and $b_r \in f(\lambda)$. Then $(f(\lambda))(a) \geq t$ and $(f(\lambda))(b) \geq r$. By assumption, there exists $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\lambda(x) = \bigvee\{\lambda(z) \mid z \in f^{-1}(a)\}$$

and

$$\lambda(y) = \bigvee\{\lambda(z) \mid z \in f^{-1}(b)\}.$$
and
\[ \lambda(y) = \bigvee \{ \lambda(w) \mid w \in f^{-1}(b) \} . \]
Then \( x_t \in \lambda \) and \( y_r \in \lambda \). Since \( \lambda \) is an \((\in, \in \lor q^0)\)-fuzzy subalgebra of \( X \), we have \( (x \ast y)_{\min\{t, r\}} \in \lor q^0 \lambda \). Now \( x \ast y \in f^{-1}(a \ast b) \) and so \( (f(\lambda))(a \ast b) \geq \lambda(x \ast y) \).

Thus
\[ (f(\lambda))(a \ast b) \geq \min\{t, r\} \text{ or } (f(\lambda))(a \ast b) + \min\{t, r\} > \delta \]
which means that \( (a \ast b)_{\min\{t, r\}} \in \lor q^0 f(\lambda) \). Consequently, \( f(\lambda) \) is an \((\in, \in \lor q^0)\)-fuzzy subalgebra of \( Y \).

References


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