Some New Applications of Integrally Modified q-Szasz-Mirakyan Operators

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Abstract
This paper we introducing a new sequence of positive q-integral new Modified q-Szasz-Mirakyan Operators. Korovkin-type theorems for fuzzy continuous functions, an estimate for the rate of convergence and some properties are also obtained for these operators.

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1 Introduction
The approximation of functions by positive linear operators is an important research topic. q-Calculus is a generalization of any subjects, such as hyper geometric series, complex analysis and particle physics. Currently it continues being an important subject of study. It has been shown that linear positive operators constructed by q-numbers are quite effective as far as the rate of
convergence is concerned and we can have some unexpected results, which are not observed for classical case. In 1987, Lupas defined a q-analogue of Bernstein operators and studied some approximation properties of them. In 1997, Phillips introduced another generalization of Bernstein operators based on the q-integers called q-Bernstein operators. Aral [1] introduced the q-Szasz-Mirakyan operators. Aral and Gupta [1],[15] extended the study and established some approximation properties for q-Szasz Mirakyan operators. In the last decade some new generalizations of well known positive linear operators, based on q-integers were introduced and studied by several authors. For instance q-Meyer-König and Zeller operators studied by Trif. [16], and Gupta [2] etc. In 20011, Aral and Gupta [1],[15] introduced a q-generalization of the classical Baskakov operators. In 2012, Honey Sharma [4],[5] introduced the q-Durrmeyer type operators.

Very recently we published a paper based on q-Durrmeyer type operators [12]. In this paper motivated by H. Sharma we introduced a q-analogue of the Szasz-Mirakyan Operators and we study better rate of convergence.

2 Preliminary Notations

We mention some important definitions of q-Calculus.

Definition 2.1 For any fixed real number $q > 0$ and $k \in N$, the q-integers are defined by

$$[k]_q = \begin{cases} 
  k, & \text{if } q = 1, \\
  1 + q + q^2 + \ldots + q^{k-1}, & \text{if } q \neq 1.
\end{cases}$$

In this way for a real number $n$ we may write $[n]_q = \frac{1 - q^n}{1 - q}; q \neq 1$.

Definition 2.2 The q-factorial are defined by

$$[k]_q! = \begin{cases} 
  1, & \text{if } k = 0, \\
  [1]_q \cdot [2]_q \cdot \ldots \cdot [k]_q, & \text{if } k = 1, 2, \ldots
\end{cases}$$

Definition 2.3 For any integers $n,k$ satisfying $n \geq k \geq 0$, the q-binomial coefficient are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$
where,
\[ p_{n,k}(x) = \left( 1 + \frac{1}{n} \right)^n \binom{n}{k} x^k \left( \frac{n}{n+1} - x \right)^{n-k}, \]

and established some approximation results on it.

H. Sharma [4] introduced the following q-Durrmeyer type operators defined as: for \( f \in CI_{n,q} \) where \( I_{n,q} = \left[ 0, \frac{[n]_q}{[n+1]_q} \right] \)

\[ (M_{n,q}^* f)(x) = \frac{[n+1]^2_q}{[n]_q} \sum_{k=0}^{n} q^{-k} p_{n,k}^*(q; x) \int_0^{[n+1]_q} p_{n,k}(q; qt) f(t) d_q t \] (2)

where,
\[ p_{n,k}^*(q; x) = \binom{n}{k}_q \left( \frac{[n+1]_q x}{[n]_q} \right)^k \left( 1 - \frac{[n+1]_q x}{[n]_q} \right)^{n-k} \]

and established some approximation results on it.

In this year motivated by H. Sharma and N. Deo [8],[9],[10], we introduced a q-Szasz-Mirakyan type operators [13] defined as: for \( f \in CI_{n,q} \)

\[ (S_{n,q}^* f)(x) = \frac{[n+1]^2_q}{[n]_q E_q([n]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n]_q x)^k}{[k]_q!} \int_0^{[n+1]_q} p_{n,k}(q; qt) f(t) d_q t. \] (3)

Again we modified above equations for \( p \geq 0 \) so, we get

\[ (S_{n,q,p}^* f)(x) = \frac{[n+1]^2_q}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \int_0^{[n+p]_q} p_{n,k}(q; qt) f(t) d_q t. \] (4)

and established statistical approximation theorem on it [14].

H. S. Kasana et. al. [3] obtained a sequence of modified Szász operators for integrable function on \([0, \infty)\) defined as:

\[ (M_{n,x}^* f)(x) \equiv M_{n,x}(f(y); t) = \sum_{k=0}^{\infty} b_{n,k}(t) \int_0^{\infty} b_{n,k}(y) f(x+y) dy \] (5)

where, \( x \) and \( t \) belong to \([0, \infty)\) and \( x \) is fixed.

In this paper motivated by H. S. Kasana and H. Sharma, we introduce a q-Szasz-Mirakyan type operators defined as: for \( f \in CI_{n,q} \)

\[ (S_{n,q,x,p}^* f)(t) = \frac{[n+1]^2_q}{[n]_q E_q([n+p]_q t)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q t)^k}{[k]_q!} \int_0^{[n+p]_q} p_{n,k}(q; qt) f(x+y) d_q y. \] (6)
where, $x$ and $t$ belong to $I_{n,q}$ and $x$ is fixed.

The aim of this paper we study estimate moments for the approximation properties of a new generalization of the q-Szasz-Mirakyan operators based on q-integers. Finally, we give Korovkin-type theorems for fuzzy continuous functions and better error estimations for operators (4) and (6).

### 3 Estimation of moments

We use the lemma-1 [4] for $s = 1, 2, \ldots$ and by the definition of q-Beta function, we get $\int_0^{\frac{1}{[n+1]_q}} p^*_{n,k}(q;qt)t^s dt = \frac{[n]_q^{n+1}}{[n+1]_q} q^k [n]_q[k+s]_q! [k]_q[n+n+1]_q!$

**Theorem 3.1** Let the sequence of positive linear operators $(S^*_{n,q,x,p})_n$ defined by (6). For all $n \in N; q \in (0, 1), p \geq 0; f \in CI_{n,q}; x \in I_{n,q},$ we get

\[
(S^*_{n,q,x,p}1)(t) = 1
\]

\[
(S^*_{n,q,x,p}y)(t) = \frac{[n]_q([n+p]_qt+1)}{[n+2]_q[n+1]_q}
\]

\[
(S^*_{n,q,x,p}y^2)(t) = \frac{(1+q)[n]_q^2 + q(1+q^2)t[n+p]_q[n]_q^2 + q^3t^2[n+p]_q^2[n]_q^2}{[n+3]_q[n+2]_q[n+1]_q^2}
\]

Proof: We put $f(y) = 1$ in the operators $S^*_{n,q,x,p}$, we get

\[
(S^*_{n,q,x,p}1)(t) = \frac{[n+1]_q^2}{[n]_q E_q([n+p]_qt)} \sum_{k=0}^{\infty} q \frac{k^2_{-k-2}([n+p]_qt)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q; qy) 1 dy
\]

\[
= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_qt)} \sum_{k=0}^{\infty} \frac{k^2_{-k-2}([n+p]_qt)^k}{[k]_q!} \frac{[n]_q}{[n+1]_q} q^k [n]_q!
\]

\[
= \frac{1}{E_q([n+p]_qt)} \sum_{k=0}^{\infty} \frac{k^2_{(k-1)}([n+p]_qt)^k}{[k]_q!} = 1
\]

Again we put $f(y) = y$ in the operators $S^*_{n,q,x,p}$, we get

\[
(S^*_{n,q,x,p}y)(t) = \frac{[n+1]_q^2}{[n]_q E_q([n+p]_qt)} \sum_{k=0}^{\infty} \frac{k^2_{-k-2}([n+p]_qt)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q; qy) 1 dy
\]

\[
= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_qt)} \sum_{k=0}^{\infty} \frac{k^2_{-k-2}([n+p]_qt)^k}{[k]_q!} \left( \frac{[n]_q^2}{[n+1]_q^2} q^k [n]_q[k+1]_q! \right)
\]

\[
= \frac{[n]_q}{[n+2]_q[n+1]_q E_q([n+p]_qt)} \sum_{k=0}^{\infty} \frac{k^2_{(k-1)}([n+p]_qt)^k}{[k]_q!} [k+1]_q
\]

\[
= \frac{[n]_q([n+p]_qt+1)}{[n+2]_q[n+1]_q}
\]
Similarly, we put $f(y) = y^2$ in the operators $S^*_n,q,x,p$, we get

$$(S^*_n,q,x,p y^2)(t)$$

$$= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q t)} \sum_{k=0}^\infty q^{k^2-2k-2} ([n+p]_q t)^k \int_0^{[n]_q} p_{n,k}(q;qt) y^2 dy$$

$$= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q t)} \sum_{k=0}^\infty q^{k^2-k-2} ([n+p]_q t)^k \left[ \frac{[n]_q^3}{[k]_q!} q^k [n]_q! [k+2]_q! \right]$$

$$= \frac{[n+3]_q [n+2]_q [n+1]_q^3 E_q([n+p]_q t)}{[n]_q} \sum_{k=0}^\infty q^{k(k-1)/2} ([n+p]_q t)^k \left[ (k+1)_q[k+2]_q \right]$$

$$= \frac{[n]_q^2 \left[ 1 + q + q(1+q^2)[n+p]_q t + q^4 \left( \frac{([n+1]_q[n+p]_q t^2}{[n]_q} + [n+p]_q t \right) - [n+p]_q t \right]}{[n+3]_q [n+2]_q [n+1]_q^2}$$

This completes the proof of the theorem.

**Lemma 3.2** For the special case $q = 1$ we have

$$(S^*_n,1,x,p 1)(t) = 1;$$

$$(S^*_n,1,x,p y)(t) = \frac{n(n+p)t + n}{(n+2)(n+1)};$$

$$(S^*_n,1,x,p y^2)(t) = \frac{n^2((n+p) t^2 + 4(n+p)t + 2)}{(n+3)(n+2)(n+1)^2}. $$

**Lemma 3.3** The sequence of positive linear operators $S^*_n,q,x,p$, we get following central moments: let $\phi^i = (y-t)^i, i = 1, 2, \ldots$

$$(S^*_n,q,x,p \phi^i)(t) = (S_{n,q,x,p} y)(t) - t(S_{n,q,x,p} 1)(t)$$

$$= \frac{[n]_q([n+p]_q t + 1)}{[n+2]_q[n+1]_q} - t \cdot 1$$

$$= \frac{[n]_q(1 + (p-3) t) - 2t}{[n+2]_q[n+1]_q};$$
In this section we mention some important definitions given by M. Burgin [6].

A number \(a\) is called an \(r\)-limit of a sequence \(S\) (it is denoted by \(a = r - \lim S\)) if for any \(\epsilon \in R\), the inequality \(|a - a_i| < r + \epsilon\) is valid for almost all \(a_i\), i.e. there is such \(n\) that for any \(i > n\), we have \(|a - a_i| < r + \epsilon\).

**Definition 4.1** A number \(a\) is called an \(r\)–limit of a sequence \(S\) (it is denoted by \(a = r - \lim S\)) if for any \(\epsilon \in R\), the inequality \(|a - a_i| < r + \epsilon\) is valid for almost all \(a_i\), i.e. there is such \(n\) that for any \(i > n\), we have \(|a - a_i| < r + \epsilon\).

**Definition 4.2** A sequence \(S\) that has an \(r\)–limit is called \(r\)-convergent and it is said that \(S\), \(r\) – converges to its \(r\)–limit \(a\). It is denoted by \(S \rightarrow ra\).

**Definition 4.3** A function \(f : R \rightarrow R\) is called \(r\)-continuous in \(X \subset R\) if \(\gamma(f, X) \leq r\) and is called fuzzy continuous in \(X\) if \(\gamma(f, X) \leq \infty\) where \(\gamma(f, X)\) defined as,

\[
\gamma(f, X) \geq \inf\{\sup\{|f(x) - g(x)| : x \in X\} : g(x) \in C(X)\}.
\]
For example the functions \( f(x) = x^n \) when \( x \in [n, n+1), n \in \mathbb{Z} \) and \( g(x) = [x]^n \) are fuzzy continuous in each finite interval of the real line \( \mathbb{R} \), but they are not continuous in any interval with the length larger than 1. To define the Riemann integral for a continuous function \( f(x) \), step functions are utilized. If the integral of \( f(x) \) exists, then any such step function is fuzzy continuous.

**Theorem 4.4** Let a sequence \((q_n)_n; q_n \in (0, 1)\) such that \( r - \lim_{n \to \infty} q_n = 1 \) and let the sequence of positive linear operators \( S_{n,q_n,x,p}^*; n \in \mathbb{N} \) be defined by (6). If \( r_i - \lim_{n \to \infty} |(S_{n,q_n,x,p}^* e_i)(t) - e_i| = 0 \) for \( i = 0, 1, 2 \). Then for non-decreasing function \( f \in C(I_n) \), we get

\[
r - \lim_{n \to \infty} |(S_{n,q_n,x,p}^* f)(t) - f| = 0
\]

where, \( r \) is any real number such that \( r \geq K_3(r_0 + r_1 + r_2) \) for some \( K_3 > 0 \).

Proof: Let the functions \( e_i \) defined as; \( e_i(t) = y^i \) for all \( t \in I_n \). Now, for each \( \epsilon > 0 \), there corresponds \( \delta > 0 \) such that \( |\lambda(y - t)| \leq \epsilon \) whenever \( |y - t| \leq \delta \). Again for \( |y - t| > \delta \), then there exist a positive number \( M \) such that \( |\lambda(y - t)| \leq M \leq \frac{M(y-t)^2}{\delta^2} \). Thus for all \( y \) and \( t \in I_n \), we get

\[
|\lambda(y - t)| \leq \epsilon + M \frac{(y-t)^2}{\delta^2}.
\]

Applying \( S_{n,q_n,x,p}^* \) on (10), we get

\[
|(S_{n,q_n,x,p}^* f)(t) - f(x + t)| \leq \epsilon(S_{n,q_n,x,p}^* e_0)(x) + \frac{M}{\delta^2} (S_{n,q_n,x,p}^* (t-x)^2)(x)
\]

\[
|(S_{n,q_n,x,p}^* f)(t) - f(x + t)| \leq \epsilon + \epsilon |(S_{n,q_n,x,p}^* e_0)(x) - e_0(x)| + K_3 (|S_{n,q_n,x,p}^* e_i(t) - e_i(t)|) + \sum_{i=0}^{2} |(S_{n,q_n,x,p}^* e_i)(t) - e_i(t)|
\]

where, \( K_3 = \max\{\frac{M}{\delta^2}, \frac{2Mx}{\delta^2}, \frac{M|x|^2}{\delta^2}\} \). Then for every \( \epsilon > 0 \) there exist \( N = N(\epsilon) > 0 \) such that for all \( n \in N \), we get

\[
|(S_{n,q_n,x,p}^* f)(t) - f(x + t)| \leq \epsilon + \epsilon (r_0 + \epsilon) + K_3 (3\epsilon + r_0 + r_1 + r_2) \leq \epsilon + \epsilon_1
\]

here, \( \epsilon_1 = \epsilon(1 + r_0 + \epsilon + 3K_3) \). Since \( \epsilon \) is arbitrary and small, \( r - \lim_{n \to \infty} q_n = 1 \), we get \( r - \lim_{n \to \infty} |(S_{n,q_n,x,p}^* f)(t) - f| = 0 \). This completes the proof of the theorem.

**Theorem 4.5** Let a sequence \((q_n)_n; q_n \in (0, 1)\) such that \( r - \lim_{n \to \infty} q_n = 1 \) and let the sequence of positive linear operators \( S_{n,q_n,p}; n \in \mathbb{N} \) be defined by (4). If \( r_i - \lim_{n \to \infty} |(S_{n,q_n,p} e_i)(x) - e_i| = 0 \) for \( i = 0, 1, 2 \). Then for non-decreasing function \( f \in C(I_n) \), we get

\[
r - \lim_{n \to \infty} |(S_{n,q_n,p} f)(x) - f| = 0
\]

where, \( r \) is any real number such that \( r \geq K_4(r_0 + r_1 + r_2) \) for some \( K_4 > 0 \).
The proof of the theorem is analogous as theorem 4.4.

**Theorem 4.6** Let \( f \) be the integrable and bounded in the interval \( I_n \) and let \( f'' \) exists at a point \( x + t \in I_n \). Let a sequence \((q_n)_{n \in \mathbb{N}}\) such that \( \lim_{n \to \infty} q_n = 1 \) and let the sequence of positive linear operators \( S_{n,q_n,x,p}^* \); \( n \in \mathbb{N} \) be defined by (6). Then, one gets that

\[
\lim_{n \to \infty} [n]_{q_n} |(S_{n,q_n,x,p}^*)^*(f)(t) - f(x + t)| = \left(1 + (p - 3t)\right)f'(x + t) + (2t - t^2)\frac{f''(x + t)}{2}
\]

Proof: Let if \( f'' \) exists at a point \( x + t \in I_n \), then by using Taylor's expansion, we can write

\[
f(x + y) = f(x + t) + (y - t)f'(x + t) + \frac{(y - t)^2}{2!} f''(x + t) + (y - t)^2 A(y - t)
\]

where, \( A(y - t) \to 0 \) as \( y \to t \).

Now applying operator \( S_{n,q_n,x,p}^* \), we get

\[
(S_{n,q_n,x,p}^* f)(t) = f(x + t)(S_{n,q_n,x,p}^* 1)(t) + f'(x + t)(S_{n,q_n,x,p}^* (t - x))(t)
\]

By using theorem 3.1 and Multiplying \([n]_{q_n}\) both sides, we get

\[
[n]_{q_n} |(S_{n,q_n,x,p}^*)^*(f)(t) - f(x + t)| = f'(x + t)[n]_{q_n} \left( \frac{[n]_q ([n + p]_q t + 1)}{[n + 2]_q [n + 1)_q} \right) \ldots
\]

\[
\ldots + \frac{f''(x + t)}{2!} S_{n,q_n,x,p}^* (\phi^2(t)) + [n]_{q_n} R[n]_{q_n} (y, t).
\]

Here we write,

\[
[n]_{q_n} R[n]_{q_n} (y, t) = \frac{[n + 1]^2_q}{[n]_q E_q([n + p]_q t)} \sum_{k=0}^{\infty} q^{\frac{k^2}{2} - \frac{k}{2}} \frac{([n + p]_q t)^k}{[k]_q!}
\]

\[
\times \int_0^{[n]_{q_n}} p_{n,k}(q_n; q_n y) \phi^2 \lambda \phi dq y.
\]

\[
|[n]_{q_n} R[n]_{q_n} (y, t)| \leq \frac{[n + 1]^2_q}{[n]_q E_q([n + p]_q t)} \sum_{k=0}^{\infty} q^{\frac{k^2}{2} - \frac{k}{2}} \frac{([n + p]_q t)^k}{[k]_q!} \int_0^{[n]_{q_n}} p_{n,k}(q_n; q_n y) \phi^2 \lambda \phi dq y
\]

\[
\leq [n]_{q_n} \epsilon (S_{n,q_n,x,p}^* (y - t)^2)(t) + \frac{[n]_{q_n} M}{\delta^2} (S_{n,q_n,x,p}^* (y - t)^4)(t)
\]

\[
\leq [n]_{q_n} \epsilon \left( \frac{1}{[n]_{q_n}} \right) + \frac{[n]_{q_n} M}{\delta^2} o \left( \frac{1}{[n]_{q_n}} \right)
\]

\[
\leq \epsilon + \frac{M}{([n]_{q_n})^2} o \left( \frac{1}{[n]_{q_n}} \right) \leq \epsilon + Mo \left( \frac{1}{\sqrt{[n]_{q_n}}} \right).
\]
here we choose $\delta = ([n]_{q_n})^{\frac{1}{4}}$.

Since $\varepsilon$ is arbitrary and small, $\lim_{n \to \infty} q_n = 1$ and whenever $n \to \infty$, we get

$$\|[n]_{q_n} R_{[n]_{q_n}}(y, t)\| \to 0.$$  

(13)

By using (12) in equation (13), we get

$$\lim_{n \to \infty} [n]_{q_n} |(S_{n, q_n, x, p} f)(t) - f(x + t)| = (1 + (p - 3t)) f'(x + t) + \frac{(2t - t^2)}{2} f''(x + t).$$

This completes the proof of the theorem.

**Theorem 4.7** Let $f$ be the integrable and bounded in the interval $I_n$ and let if $f''$ exists at a point $x \in I_n$. Let a sequence $(q_n)_n; q_n \in (0, 1)$ such that $\lim_{n \to \infty} q_n = 1$ and let the sequence of positive linear operators $S_{n, q_n, p}; n \in N$ be defined by (4). Then, one gets that

$$\lim_{n \to \infty} [n]_{q_n} |(S_{n, q_n, p} f)(x) - f(x)| = (1 + (p - 3x)) f'(x) + \frac{2x - x^2}{2} f''(x).$$

The proof of the theorem is analogous as theorem 4.6.

**5 Conclusion**

We conclude that q-Szasz-Mirakyan modified operators (4) and (6) improve the approximation process when the value of $n$ is very large i.e. when $n$ tends to infinity.

**References**


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